PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 101(115) (2017), 183–190 https://doi.org/10.2298/PIM1715183T

ON MATSUMOTO CHANGE OF m-th ROOT FINSLER METRICS

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Abstract. We consider Matsumoto change of Finsler metrics. First, we find a condition under which the Matsumoto change of a Finsler metric is projectively related to it. Then, considering the subspace of m-th root Finsler metrics, if $\bar{F}$ is the Matsumoto change of F, we prove that $\bar{F}$ is locally projectively flat if and only if it is locally dually flat. In this case, F and $\bar{F}$ reduce to locally Minkowskian metrics.

1. Introduction

Let (M, F) be a Finsler space. In 1984, C. Shibata studied the properties of Finsler space $(M, \bar{F})$ whose fundamental metric function $\bar{F}$ is obtained from F by the relation $F(x, y) \to \bar{F}(x, y) = f(F, \beta)$, where $\beta(x, y) = b_i(x)y^i$ is a 1-form on M and $f = f(F, \beta)$ is a positively homogeneous function of F and β. This change of Finsler metric function has been called a β-change. He studied some geometrical properties of tensors being invariant by β-change of the metric [**9**]. If $||\beta||_F := \sup_{F(x,y)=1} |\beta| < 1$, then $\bar{F}$ is again a Finsler metric.

There is a special case of β-change, namely

$$\bar{F}(x, y) = \frac{F^2}{F - \beta} \tag{1.1}$$

which is called the Matsumoto change of F. If F reduces to a Riemannian metric α, then $\bar{F}$ reduces to the Matsumoto metric $F = \frac{\alpha^2}{\alpha - \beta}$. Due to this reason, transformation (1.1) is called the Matsumoto change of Finsler metrics. The Matsumoto metric is an important metric in the Finsler geometry which is the Matsumoto's slope-of-a-mountain metric. This metric was introduced by Matsumoto as a realization of Finsler's idea "a slope measure of a mountain with respect to a time measure".

Two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if any geodesic of the first is also geodesic for the second and vice versa. In this case,

2010 *Mathematics Subject Classification:* Primary 53B40; Secondary 53C60.
Key words and phrases: Matsumoto change, locally dually flat metric, projectively flat metric, m-th root metric.
Communicated by Stevan Pilipović.

there is a scalar function $P = P(x, y)$ defined on TM_0 such that $\bar{G}^i = G^i + Py^i$, where $\bar{G}^i$ and G^i are the geodesic spray coefficients of $\bar{F}$ and F, respectively. In this paper, we find a condition under which the Matsumoto change of a Finsler metric is projectively related to it. Let (M, F) be a Finsler manifold and $\beta = b_i(x)y^i$ a 1-form on M. Put $r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i})$, $s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i})$, $r_{00} = r_{ij}y^iy^j$, where $|$ denotes the horizontal derivation with respect to the Berwald connection of F. Then, we have the following.

THEOREM 1.1. *Let (M, F) be a Finsler manifold. Suppose that $\bar{F} = F^2/(F-\beta)$ be the Matsumoto change of F. Then $\bar{F}$ is projectively related F if only if β satisfies*

$$s_{ij} = \big[\mathcal{A}_j r_{ik} - \mathcal{A}_i r_{jk}\big]y^k, \quad \textit{where} \quad \mathcal{A}_i := \frac{2(F_i - b_i)F - 2F_i(F-\beta)}{F(F-\beta)}.$$

In this case, the projective factor is given by $P = \frac{1}{2(F-\beta)}r_{00}$.

Let M be an n-dimensional C^∞ manifold, TM its tangent bundle. Let $F = \sqrt[m]{A}$ be a Finsler metric on M, where $A := a_{i_1\dots i_m}(x)y^{i_1}y^{i_2}\dots y^{i_m}$ with $a_{i_1\dots i_m}$ symmetric in all its indices. Then F is called an m-th root Finsler metric [**10**–**18**]. The special m-th root metric in the form $F = \sqrt[m]{y^1y^2\dots y^m}$ is called the Berwald–Moór metric [**1**–**3**, **5**, **6**].

A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. It is known that a Finsler metric $F(x, y)$ on an open domain $U \subset \mathbb{R}^n$ is locally projectively flat if and only if $G^i = Py^i$, where $P = P(x, y)$ is called the projective factor and is a C^∞ scalar function on TM_0 satisfying $P(x, \lambda y) = \lambda P(x, y)$ for all $\lambda > 0$.

THEOREM 1.2. *Let $F = \sqrt[m]{A}$ $(m > 2)$, be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Suppose that $\bar{F} = F^2/(F-\beta)$ be the Matsumoto change of F. Then $\bar{F}$ is locally projectively flat if and only if $A_{x^l} = 0$ and $b_i =$ constant.*

A Finsler metric F on a manifold M is said to be locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients are in the form $G^i = -\frac{1}{2}g^{ij}H_{y^j}$, where $H = H(x, y)$ is a positively homogeneous scalar function on $TM_0 = TM \setminus \{0\}$ [**8**].

THEOREM 1.3. *Let $F = \sqrt[m]{A}$ $(m > 2)$, be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Suppose that $\bar{F} = F^2/(F-\beta)$ be the Matsumoto change of F. Then $\bar{F}$ is a locally dually flat Finsler metric if and only if $A_{x^l} = 0$ and $b_i =$ constant.*

2. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. To this aim we first prove the following result.

LEMMA 2.1 (Rapcsák [**7**]). *Let F and $\bar{F}$ be two Finsler metrics on a manifold M. Then $\bar{F}$ is projectively related to F if and only if $\bar{F}$ satisfies $\bar{F}_{|k,l}y^k - \bar{F}_{|l} = 0$,* *where* $|$ *denotes the horizontal derivation with respect to the Berwald connection of* F. *In this case, the spray coefficients are related by* $\bar{G}^i = G^i + Py^i$, *where*

$$P = \frac{\bar{F}_{|k} y^k}{2\bar{F}}. \tag{2.1}$$

The $P = P(x,y)$ *is called the* projective factor *of* $F(x,y)$.

Throughout this paper, we use the Berwald connection and the h- and v- covariant derivatives of a Finsler tensor field are denoted by " $|$ " and " $,$ " respectively. Now, let (M,F) be a Finsler manifold and $\beta = b_i(x)y^i$ a 1-form on M. Put

$$\begin{aligned} r_{ij} &:= \tfrac{1}{2}(b_{i|j} + b_{j|i}), \qquad r_{i0} := r_{ij}y^j, \qquad r_{00} := r_{ij}y^iy^j,\\ s_{ij} &:= \tfrac{1}{2}(b_{i|j} - b_{j|i}), \qquad s_{i0} := s_{ij}y^j,\\ R_{ij} &:= \frac{1}{2}\Big(\frac{\partial b_i}{\partial x^j} + \frac{\partial b_j}{\partial x^i}\Big), \quad R_{0i} := R_{ji}y^j, \quad R_{00} := R_{ij}y^iy^j. \end{aligned}$$

Proof of Theorem 1.1. The following relations hold

$$b_{i|j} = \frac{\partial b_i}{\partial x^l} - \Gamma^s_{il} b_s, \quad \Gamma^s_{il} = \Gamma^s_{li},$$

where $\Gamma^i_{jk} = \Gamma^i_{jk}(x,y)$ is the Christoffel symbols of the Berwald connection of F. Then we have

$$s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}) = \frac{1}{2}\Big(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\Big), \tag{2.2}$$

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}) = \frac{1}{2}\Big(\frac{\partial b_i}{\partial x^j} + \frac{\partial b_j}{\partial x^i} - 2b_s\Gamma^s_{ij}\Big) = R_{ij} - b_s\Gamma^s_{ij}. \tag{2.3}$$

By (2.2) and (2.3), we get

$$\beta_{|l} = b_{i|l}y^i = \Big(\frac{\partial b_i}{\partial x^l} - \Gamma^s_{il}b_s\Big)y^i, \tag{2.4}$$

$$\beta_{|l}y^l = \Big(\frac{\partial b_i}{\partial x^l} - \Gamma^s_{il}b_s\Big)y^iy^l = R_{00} - 2b_sG^s = r_{00}, \tag{2.5}$$

$$\beta_{|k,l}y^k = \Big(\frac{\partial b_l}{\partial x^k} - b_s\Gamma^s_{lk}\Big)y^k, \tag{2.6}$$

whence (2.4) and (2.6) imply $\beta_{|k,l}y^k - \beta_{|l} = s_{lk}y^k = s_{l0}$. For $\bar{F} = F^2/(F-\beta)$, we have

$$\begin{aligned} \bar{F}_{|l} &= \frac{\beta_{|l}F^2}{(F-\beta)^2}, \quad \bar{F}_{|k,l} = \Big[\frac{(2F_l\beta_{|k} + F\beta_{|k,l})(F-\beta) - 2(F_l - b_l)F\beta_{|k}}{(F-\beta)^3}\Big]F,\\ \bar{F}_{|k,l}y^k &= \Big[\frac{(2F_lr_{00} + F\beta_{|k,l}y^k)(F-\beta) - 2(F_l-b_l)Fr_{00}}{(F-\beta)^3}\Big]F. \end{aligned} \tag{2.7}$$

Then

$$\begin{aligned} \bar{F}_{|k,l}y^k - \bar{F}_{|l} &= \Big[\frac{(2F_lr_{00} + F\beta_{|k,l}y^k - F^2\beta_{|l})(F-\beta) - 2(F_l-b_l)Fr_{00}}{(F-\beta)^3}\Big]F\\ &= \Big[\frac{(2F_lr_{00} + Fs_{l0})(F-\beta) - 2(F_l-b_l)Fr_{00}}{(F-\beta)^3}\Big]F. \end{aligned}$$

Put

$$\mathcal{A}_i := \frac{2(F_i - b_i)F - 2F_i(F-\beta)}{F(F-\beta)}.$$

Then by Lemma 2.1, $\bar{F}$ is projectively related to F if and only if $s_{l0} = \mathcal{A}_l r_{00}$, which, taking a vertical derivation, yields

$$s_{li} = \mathcal{A}_{li} r_{00} + 2\mathcal{A}_l r_{0i}. \tag{2.8}$$

Since $s_{li} = -s_{il}$, then by (2.8) we get $\mathcal{A}_{li} r_{00} + 2\mathcal{A}_l r_{0i} = -\mathcal{A}_{il} r_{00} - 2\mathcal{A}_i r_{0l}$ or

$$\mathcal{A}_{li} r_{00} = -\mathcal{A}_l r_{0i} - \mathcal{A}_i r_{0l}. \tag{2.9}$$

By (2.8) and (2.9), we get $s_{ij} = \mathcal{A}_i r_{0j} - \mathcal{A}_j r_{0i}$. Now, by (2.5) and (2.7), we have

$$\bar{F}_{|y} y^k = \frac{\beta_{|k} y^k F^2}{(F-\beta)^2} = \frac{F^2 r_{00}}{(F-\beta)^2}.$$

By (2.1), it follows that

$$P = \frac{\bar{F}_{|k} y^k}{2\bar{F}} = \frac{r_{00}}{2(F-\beta)}. \qquad \square$$

3. Proof of Theorem 1.2

It is known that a Finsler metric $F(x,y)$ on $U \subset R^n$ is projective if and only if its geodesic coefficients G^i are of the form $G^i(x,y) = P(x,y)y^i$, where $P\colon TU = U \times R^n \to R$ is positively homogeneous of degree one with respect to y. In [**4**], Hamel showed that a Finsler metric F on $U \subset R^n$ is projectively flat if and only if it satisfies $F_{x^k y^l} y^k = F_{x^l}$.

Lemma 3.1. *Let $F = \sqrt[m]{A}$ $(m > 2)$, be an m-th root Finsler metric on an open subset $U \subset R^n$. Suppose that the equation*

$$\Psi A^{\frac{2}{m}-1} + \Xi A^{\frac{2}{m}} + \Phi A^{\frac{1}{m}} + \Theta A^{\frac{1}{m}+1} + \Upsilon A^{\frac{1}{m}-1} + A^{\frac{1}{m}+2}\Omega + \Gamma = 0$$

holds, where $\Phi, \Psi, \Theta, \Upsilon, \Omega, \Xi$ are homogeneous polynomials in y. Then $\Psi = \Xi = \Phi = \Theta = \Upsilon = \Omega = \Gamma = 0$.

For an m-th root metric $F = \sqrt[m]{A}$, put

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^j \partial y^j}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i} y^i, \quad A_{0l} = A_{x^k y^l} y^k = \frac{\partial^2 A}{\partial x^i \partial y^l} y^k.$$

Then we have the following.

Proof of Theorem 1.2. For $\bar{F} = \frac{\sqrt[m]{A^2}}{\sqrt[m]{A} - \beta}$, we infer

$$[\bar{F}]_{x^l} = \frac{1}{m(\sqrt[m]{A} - \beta)^2}\Big[A^{\frac{3}{m}-2} A_{x^l} - 2A^{\frac{2}{m}-1} A_{x^l}\beta + mA^{\frac{2}{m}}\beta_{x^l}\Big], \tag{3.1}$$

$$[\bar{F}]_{x^k y^l} y^k = \frac{1}{m}\Bigg[\frac{(\frac{1}{m}-2)A_0 A_l A^{\frac{4}{m}-3} + A_{0l}A^{\frac{4}{m}-2} + (2\beta_l A_0 - A_{0l}\beta + 2A_0 A_l \beta)A^{\frac{3}{m}-2}}{(A^{\frac{1}{m}} - \beta)^3}$$
$$+ \frac{(-2\beta A_{0l} - 2A_0\beta_l)A^{\frac{3}{m}-1} + (2 - \frac{3}{m})\beta A_0 A_l A^{\frac{3}{m}-3} + m\beta_{0l}A^{\frac{3}{m}}}{(A^{\frac{1}{m}} - \beta)^3}$$

$$+\frac{(2\beta^2 A_{0l}-2\beta A_l\beta_0-6\beta A_0\beta_l)A^{\frac{2}{m}-1}+(\frac{4}{m}-2)\beta^2A_0A_lA^{\frac{2}{m}-2}}{(A^{\frac{1}{m}}-\beta)^3} \tag{3.2}$$
$$+\frac{(2m\beta_0\beta_l-m\beta\beta_{0l})A^{\frac{2}{m}}}{(A^{\frac{1}{m}}-\beta)^3}\Big],$$

where
$$\beta_{x^i}:=\frac{\partial\beta}{\partial x^i},\quad \beta_i:=\frac{\partial\beta}{\partial y^i}=b_i,\quad \beta_0:=\beta_{x^i}y^i,\quad \beta_{0l}:=\beta_{x^il}y^i.$$

Since $\bar{F}$ is locally projectively flat metric, we have $[\bar{F}]_{x^ky^l}y^k-[\bar{F}]_{x^l}=0$. By substituting (3.1) and (3.2) into this, we get

$$\begin{aligned}\big(\tfrac{1}{m}-2\big)&A_0A_lA^{\frac{4}{m}-3}+(A_{0l}-A_{x^l})A^{\frac{4}{m}-2}\\ &+\big[2\beta_lA_0-\beta(A_{0l}-A_{x^l})+2A_0A_l\beta\big]A^{\frac{3}{m}-2}\\ &+\big[-2\beta(A_{0l}-A_{x^l})-2A_0\beta_l\big]A^{\frac{3}{m}-1}+\big(2-\tfrac{3}{m}\big)\beta A_0A_lA^{\frac{3}{m}-3}\\ &+m(\beta_{0l}-\beta_{x^l})A^{\frac{3}{m}}+\big[2\beta^2(A_{0l}-A_{x^l})-2\beta A_l\beta_0-6\beta A_0\beta_l\big]A^{\frac{2}{m}-1}\\ &+\big(\tfrac{4}{m}-2\big)\beta^2A_0A_lA^{\frac{2}{m}-2}+\big[2m\beta_0\beta_l-m\beta(\beta_{0l}-\beta_{x^l})\big]A^{\frac{2}{m}}=0.\end{aligned} \tag{3.3}$$

Simplifying (3.3), it results that

$$\begin{aligned}\big(\tfrac{1}{m}-2\big)&A_0A_lA^{\frac{2}{m}-1}+(A_{0l}-A_{x^l})A^{\frac{2}{m}}\\ &+\big[2\beta_lA_0-\beta(A_{0l}-A_{x^l})+2A_0A_l\beta\big]A^{\frac{1}{m}}\\ &-\big[2\beta(A_{0l}-A_{x^l})+2A_0\beta_l\big]A^{\frac{1}{m}+1}+\big(2-\tfrac{3}{m}\big)\beta A_0A_lA^{\frac{1}{m}-1}\\ &+m(\beta_{0l}-\beta_{x^l})A^{\frac{1}{m}+2}+m\big[2\beta_0\beta_l-\beta(\beta_{0l}-\beta_{x^l})\big]A^2\\ &+2\beta\big[\beta(A_{0l}-A_{x^l})-A_l\beta_0-3A_0\beta_l\big]A+\big(\tfrac{4}{m}-2\big)\beta^2A_0A_l=0.\end{aligned} \tag{3.4}$$

According to Lemma 3.1, (3.4) reduces to the following

$$A_0A_l=0, \tag{3.5}$$
$$A_{0l}-A_{x^l}=0, \tag{3.6}$$
$$2\beta_lA_0-\beta(A_{0l}-A_{x^l})+2A_0A_l\beta=0,$$
$$\beta(A_{0l}-A_{x^l})+A_0\beta_l=0,$$
$$\beta_{0l}-\beta_{x^l}=0, \tag{3.7}$$
$$2\beta_0\beta_l-\beta(\beta_{0l}-\beta_{x^l})=0,$$
$$\beta(A_{0l}-A_{x^l})-A_l\beta_0-3A_0\beta_l=0. \tag{3.8}$$

By (3.6), we have

$$A_{x^l}-A_{0l}=0. \tag{3.9}$$

The relations (3.5), $A_l\neq 0$ and $\beta\neq 0$ imply that

$$A_0=0. \tag{3.10}$$

Taking a vertical derivation of (3.10) yields

$$A_{x^l}+A_{0l}=0. \tag{3.11}$$

By (3.9) and (3.11), we get $A_{x^l}=0$. On the other hand, by substituting (3.9) and (3.10) in (3.8), we have $\beta_0=0$. Taking a vertical derivation of it implies that $\beta_{0l}+\beta_{x^l}=0$. By considering (3.7), we get $\beta_{x^l}=0$, which means that b_i are constants. □

4. Proof of Theorem 1.3

In [**8**], Shen proved that the Finsler metric F on an open subset $U\subset R^n$ is dually flat if and only if it satisfies $(F^2)_{x^ky^l}y^k=2(F^2)_{x^l}$. Now, we are going to characterize locally dually flat Finsler metrics which is obtained by a Matsumoto change of m-th root metrics. First, we remark the following.

LEMMA 4.1. *Let $F=\sqrt[m]{A}$ $(m>2)$, be an m-th root Finsler metric on an open subset $U\subset\mathbb{R}^n$. Suppose that the equation*

$$\Psi A^{\frac{2}{m}+1}+\Xi A^{\frac{2}{m}}+\Phi A^{\frac{1}{m}}+\Theta A^{\frac{1}{m}+1}+\Upsilon A^{\frac{1}{m}+2}+\Omega=0$$

holds, where $\Phi,\Psi,\Theta,\Upsilon,\Omega,\Xi$ are homogeneous polynomials in y. Then $\Psi=\Xi=\Phi=\Theta=\Upsilon=\Omega=0$.

PROOF OF THEOREM 1.3. The following holds

$$[\bar F^2]_{x^k}=\frac{2A^{\frac{4}{m}-1}\big[A^{\frac{1}{m}}A_{x^l}-2A_{x^l}\beta+mA\beta_{x^l}\big]}{m(A^{\frac{1}{m}}-\beta)^3}, \tag{4.1}$$

$$\begin{aligned}[\bar F^2]_{x^ky^l}y^k=\frac{2}{m}\Bigg[&\frac{A_{0l}A^{\frac{6}{m}-1}+(\beta_lA_0-3A_{0l}\beta+A_l\beta_0)A^{\frac{5}{m}-1}+(\frac{2}{m}-1)A_0A_lA^{\frac{6}{m}-2}}{(A^{\frac{1}{m}}-\beta)^4}\\
&+\frac{(\frac{3}{m}-1)\beta A_0A_lA^{\frac{5}{m}-2}+m\beta_{0l}A^{\frac{5}{m}}+(\frac{8}{m}-2)\beta^2A_0A_lA^{\frac{4}{m}-2}}{(A^{\frac{1}{m}}-\beta)^4}\\
&+\frac{(2\beta^2A_{0l}-4\beta A_0\beta_l+4\beta A_l\beta_0)A^{\frac{4}{m}-1}+m\beta\beta_{0l}A^{\frac{4}{m}}}{(A^{\frac{1}{m}}-\beta)^4}\Bigg].\end{aligned} \tag{4.2}$$

Since $\bar F$ is a locally dually flat metric, then

$$[\bar F^2]_{x^ky^l}y^k-2[\bar F^2]_{x^l}=0. \tag{4.3}$$

By substituting (4.1) and (4.2) in (4.3), we infer:

$$\begin{aligned}&\Big(A_{0l}-2A_{x^l}\Big)A^{\frac{6}{m}-1}+\big[\beta_lA_0-3(A_{0l}-2A_{x^l})\beta+A_l\beta_0\big]A^{\frac{5}{m}-1}\\
&+\Big(\frac{2}{m}-1\Big)A_0A_lA^{\frac{6}{m}-2}+\Big(\frac{3}{m}-1\Big)\beta A_0A_lA^{\frac{5}{m}-2}+m(\beta_{0l}-2\beta_{x^l})A^{\frac{5}{m}}\\
&+2\beta\big[\beta(A_{0l}-2A_{x^l})-2A_0\beta_l+2A_l\beta_0\big]A^{\frac{4}{m}-1}+\Big(\frac{8}{m}-2\Big)\beta^2A_0A_lA^{\frac{4}{m}-2}\\
&+m\beta(\beta_{0l}-2\beta_{x^l})A^{\frac{4}{m}}=0.\end{aligned} \tag{4.4}$$

Simplifying (4.4) implies that

$$\begin{aligned}(A_{0l}-2A_{x^l})A^{\frac{2}{m}+1}+\big[\beta_l A_0-3(A_{0l}-2A_{x^l})\beta+A_l\beta_0\big]A^{\frac{1}{m}+1}\\+\Big(\frac{2}{m}-1\Big)A_0A_lA^{\frac{2}{m}}+\Big(\frac{3}{m}-1\Big)\beta A_0A_lA^{\frac{1}{m}}+m\Big(\beta_{0l}-2\beta_{x^l}\Big)A^{\frac{1}{m}+2}\\+m\beta(\beta_{0l}-2\beta_{x^l})A^2+2\beta\big[\beta(A_{0l}-2A_{x^l})-2A_0\beta_l+2A_l\beta_0\big]A\\+\Big(\frac{8}{m}-2\Big)\beta^2A_0A_l=0.\end{aligned}\tag{4.5}$$

By Lemma 4.1, (4.5) reduces to

$$A_0A_l=0,\tag{4.6}$$

$$A_{0l}-2A_{x^l}=0,\tag{4.7}$$

$$2\beta(\beta_{0l}-2\beta_{x^l})=0,$$

$$3\beta(A_{0l}-2A_{x^l})-A_0\beta_l-A_l\beta_0=0,\tag{4.8}$$

$$\beta_{0l}-2\beta_{x^l}=0,\tag{4.9}$$

$$\beta(A_{0l}-2A_{x^l})+2(\beta_lA_0+\beta_0A_l)=0.$$

By (4.6), we have $A_0=0$. Taking a vertical derivation of it implies that $A_{x^l}+A_{0l}=0$. Then by (4.7), it follows that $A_{x^l}=0$. In this case, (4.8) reduces to $\beta_0=0$. Taking a vertical derivation of it implies that $\beta_{0l}+\beta_{x^l}=0$. Then by (4.9), we get $\beta_{x^l}=0$ which means that b_i are constants. This completes the proof. □

COROLLARY 4.1. *Let $F=\sqrt[m]{A}$ $(m>2)$ be an m-th root Finsler metric on an open subset $U\subset\mathbb{R}^n$. Suppose that $\bar{F}=F^2/(F-\beta)$ be the Matsumoto change of F. Then $\bar{F}$ is locally projectively flat if and only if it is locally dually flat. In this case, F and $\bar{F}$ are Berwald–Moór metrics.*

PROOF. By Theorem 1.2 and 1.3, $\bar{F}$ is locally projectively flat if and only if it is locally dually flat. Since $A_{x^i}=0$, then $a_{i_1\dots i_m}(x)=c$ is a constant. In this case, we get

$$F=\sqrt[m]{c\,y^1y^2\dots y^m}$$

which is a locally Minkowskian metric. Since $b_i=$ *constant* and $\bar{F}=F^2/(F-\beta)$, then $\bar{F}$ is locally Minkowskian, too. □

5. Conclusion

Every locally Minkowskian metric is locally projectively flat and locally dually flat metric. In this paper, we study the Matsumoto change of a Finsler metric and prove that the Matsumoto change of an m-th root metric is locally projectively flat if and only if it is locally dually flat if and only if it is locally Minkowskian. The study of this Finslerian change on an m-th root metric will enhance our understanding of the geometric meaning of the class of m-th root metrics.

References

1. V. Balan, *Notable submanifolds in Berwald–Moór spaces*, BSG Proc. **17** (2010), 21–30.
2. V. Balan, N. Brinzei, *Einstein equations for (h, v)-Berwald–Moór relativistic models*, Balkan. J. Geom. Appl. **11**(1) (2006), 20–26.
3. V. Balan, S. Lebedev, *On the Legendre transform and Hamiltonian formalism in Berwald–Moór geometry*, Differ. Geom. Dyn. Syst. **12** (2010), 4–11.
4. G. Hamel, *Über die Geometrien, in denen die Geraden die Kürtzesten sind*, Math. Ann. **57** (1903), 231–264.
5. M. Matsumoto, H. Shimada, *On Finsler spaces with 1-form metric. II. Berwald–Moór's metric* $L = (y^1 y^2 \dots y^n)^{1/n}$, Tensor, New Ser. **32** (1978), 275–278.
6. D. G. Pavlov, *Four-dimensional time*, Hypercomplex Numbers in Geometry and Physics **1** (2004), 31–39.
7. Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer, Dordrecht, 2001.
8. ———, *Riemann–Finsler geometry with applications to information geometry*, Chin. Ann. Math. **27** (2006), 73–94.
9. C. Shibata, *On invariant tensors of β-changes of Finsler metrics*, J. Math. Kyoto Univ. **24** (1984), 163–188.
10. H. Shimada, *On Finsler spaces with metric* $L = \sqrt[m]{a_{i_1 i_2 \dots i_m} y^{i_1} y^{i_2} \dots y^{i_m}}$, Tensor, New Ser. **33** (1979), 365–372.
11. A. Srivastava, P. Arora, *Randers change of mth root metric*, Indian J. Pure Appl. Math., **43**(1) (2012), 37–48.
12. A. Tayebi, B. Najafi, *On m-th root Finsler metrics*, J. Geom. Phys. **61** (2011), 1479–1484.
13. ———, *On m-th root metrics with special curvature properties*, C. R. Acad. Sci., Paris, Sér. I **349** (2011), 691–693.
14. A. Tayebi, A. Nankali, E. Peyghan, *Some curvature properties of Cartan spaces with m-th root metrics*, Lith. Math. J. **54**(1) (2014), 106–114.
15. ———, *Some properties of m-th root Finsler metrics*, J. Contemp. Math. Anal., Armen. Acad. Sci. **49**(4) (2014), 157–166.
16. A. Tayebi, E. Peyghan, M. Shahbazi Nia, *On generalized m-th root Finsler metrics*, Linear Algebra Appl. **437** (2012), 675–683.
17. A. Tayebi, M. Shahbazi Nia, *A new class of projectively flat Finsler metrics with constant flag curvature* $\mathbf{K} = 1$, Differ. Geom. Appl. **41** (2015) 123–133.
18. A. Tayebi, T. Tabatabaeifar, E. Peyghan, *On Kropina-change of m-th root Finsler metrics*, Ukr. Math. J. **66**(1) (2014), 1027–3190.

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(Received 18 08 2014)
(Revised 13 06 2016 and 27 07 2016)