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ON THE MIKUSIŃSKI–ANTOSIK DIAGONAL THEOREM AND THE EQUIVALENCE OF TWO TYPES OF CONVERGENCE IN KÖTHE SPACES

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In memory of Professors Jan Mikusiński and Piotr Antosik

Abstract. We present a simple proof of the Mikusiński–Antosik diagonal theorem and apply this result to prove, in an extended form, the theorem on the equivalence of the strong and weak boundedness of sets and, consequently, of the strong and weak convergence of sequences in Köthe spaces.

1. Introduction

The known sliding-hump method, used in functional analysis in its early period, was expressed by some authors in the form of abstract theorems of various types. Such a theorem, the first version of the so-called diagonal theorem, was given in [**20**] by Jan Mikusiński and applied by him in proofs of several theorems of measure theory and functional analysis. The theorem was slightly reformulated in [**4**] and this version of the theorem, presented then in the book [**7**, pp. 217–219], is called after [**15**] the Mikusiński–Antosik diagonal theorem.

Various diagonal type theorems were proved later by Piotr Antosik in several papers (see e.g. [**6**]) and, in common with other authors, in [**9**] and [**8**]. These and other abstract forms of the sliding-hump method, including Rosenthal's lemma [**25**], the Antosik–Swartz basic matrix theorem [**9**, **33**], Antosik's lemma [**6**, **37**], Weber's lemma [**36**], were studied by many authors; see e.g. [**1**–**3**, **15**, **23**, **35**, **37**–**40**] and the references in [**9**] and [**33**]. It should be also noticed that the Mikusiński–Antosik diagonal theorem is related to the famous Ramsey theorem (see [**24**] and [**23**]) whose various versions and generalizations have been investigated and applied in varied areas for many years by numerous authors (see e.g. [**11**, **12**, **14**, **16**, **19**, **22**, **26**–**32**]). In particular, the theorems of Nash-Williams [**21**] and Ellentuck [**13**] originated the infinite Ramsey theory (see e.g., [**34**]). Lately, Solecki [**32**] has discovered that

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the Nash–Williams theorem is a particular case of a special general form of the induction principle.

In the first part of this article we discuss the proof of the Mikusiński–Antosik diagonal theorem. The proof given in [**4**] and [**7**, pp. 218–219] is based on a clever idea and its beauty justifies our wish to clarify all details and to resolve any doubts concerning completeness of the reasoning. In this note we show a precise and simple proof (simpler than that given in [**18**]) of this elegant theorem, explaining the role of implication $(*)$ (see Section 2). We present the theorem in a more general form than in [**7**] and its proof in a concise but clear way.

The diagonal type theorems stand for a very useful tool in proving numerous theorems in measure theory and functional analysis formulated in a more general way than their previous versions proved by means of the Baire category method; see the articles [**4**, **6**, **20**, **35**–**38**, **40**], the monographs [**9**, **33**] and references there.

The theorems have important applications in the theory of generalized functions. Lately, diagonal methods appeared to be very efficient in the theory of product of tempered distributions (see [**17**]). Earlier, they played an important role in an elementary proof of the equivalence of the functional and sequential approaches to theory of distributions presented in [**7**, **18**]. The main idea of this proof consists in using Hermite expansions of tempered distributions and replacing two types of convergence in the space $\mathcal{S}'$ by the corresponding types of convergence of matrices of Hermite coefficients of tempered distributions. This allows one to reduce the problem to the equivalence of strong and weak boundedness of sets in Köthe spaces and the proof of the equivalence given in [**7**] is based on the Mikusiński–Antosik diagonal theorem.

However the proof of this equivalence given in [**7**] contains, in our opinion, a subtle gap. We discuss certain nuances concerning the proof in Remark 3.1 at the beginning of Section 3 and present in Section 3 an essential modification of the original proof which allows us to fill the gap. We extend our proofs of the equivalences of the strong and weak boundedness of sets and of the strong and weak convergence of sequences in Köthe spaces to some more general cases (see Theorems 3.1, 3.2 and Remarks 3.1, 3.2).

2. Mikusiński–Antosik diagonal theorem

The symbols $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}$ denote the sets of all positive integers, all nonnegative integers and all real numbers, respectively. The symbol $\mathcal{F}$ denotes the family of all finite subsets of $\mathbb{N}$ and the symbol $i > J$ for $i \in \mathbb{N}$ and $J \in \mathcal{F}$ means that $i > j$ for all $j \in J$. By a *quasi-normed group* $(X, |\cdot|)$ we mean an Abelian group $(X, +)$ endowed with a functional $|\cdot| \colon X \to \mathbb{R}$, called a *quasi-norm*, satisfying the conditions

$$1^\circ\ \ |0| = 0; \quad 2^\circ\ \ |-x| = |x|; \quad 3^\circ\ \ |x+y| \leqslant \big|x| + |y|, \quad x, y \in X.$$

Let $(X, |\cdot|)$ be a quasi-normed group (not necessarily complete) and J be a subset (finite or not) of $\mathbb{N}$. For every sequence (x_n) in X such that

$$\sum_{i \in J} |x_i| < \infty, \tag{2.1}$$

denote

$$\Big|\sum_{i\in J} x_i\Big| := \lim_{n\to\infty}\Big|\sum_{i\in J_n} x_i\Big|, \tag{2.2}$$

where (J_n) is a nondecreasing sequence of finite subsets of J such that $\bigcup_{n=1}^{\infty} J_n = J$. The notation makes sense, because (2.1) implies that the limit in (2.2) exists and does not depend on the selection of (J_n).

We present the diagonal theorem in the form given in [**4**] (cf. [**7**, p. 217]); see comments in Introduction. The version of the theorem given in [**5**] for topological groups easily follows from that formulated below, due to the result obtained in [**10**].

Theorem 2.1 (Mikusiński–Antosik diagonal theorem). *Let $(X, |\cdot|)$ be a quasi-normed group and $x_{i,j} \in X$ for $i, j \in \mathbb{N}$. Assume that*

$$\lim_{j\to\infty} |x_{i,j}| = 0 \quad \textit{for} \quad i \in \mathbb{N}. \tag{2.3}$$

Then there is an infinite $I \subseteq \mathbb{N}$ and a subset J (finite or not) of I such that

$$\sum_{j\in J} |x_{i,j}| < \infty \quad \textit{and} \quad \Big|\sum_{j\in J} x_{i,j}\Big| \geqslant \frac{1}{2}|x_{i,i}| \qquad \textit{for} \quad i \in I. \tag{2.4}$$

The starting point of the proof of Theorem 2.1 shown in [**4**] (and repeated in [**7**]) is the following observation: to show the assertion, one may additionally assume that the following implication holds for all $J \in \mathcal{F}$:

$$\forall i \in J \ \Big|\sum_{j\in J} x_{i,j}\Big| \geqslant \frac{1}{2}|x_{i,i}| \Rightarrow \exists i_0 > J \ \ \forall i \geqslant i_0 \ \Big|\sum_{j\in J} x_{i,j}\Big| < \frac{1}{2}|x_{i,i}|. \tag{$*$}$$

In fact, if implication $(*)$ does not hold for some set $J \in \mathcal{F}$, say $J =: J_0$, then one can select an increasing sequence (ι_n) of positive integers such that

$$\Big|\sum_{j\in J_0} x_{\iota_n,j}\Big| \geqslant \frac{1}{2}|x_{\iota_n,\iota_n}| \qquad \text{for} \quad n \in \mathbb{N}$$

and the assertion is then true for $J := J_0$ and $I := J_0 \cup \{\iota_n : \ n \in \mathbb{N}\}$. Thus it remains to consider the opposite case, assuming $(*)$ for all $J \in \mathcal{F}$.

Under assumption $(*)$, one can construct an increasing sequence (i_n) of positive integers in such a way that the inequalities in (2.4) are satisfied for $J = I := \{i_n : n \in \mathbb{N}\}$. In the inductive construction, to select an index $i_{n+1} > J_n := \{i_1, \ldots, i_n\}$ for a given $n \in \mathbb{N}$ the right-hand side of implication $(*)$ is used, so it is necessary to verify that the left-hand side of $(*)$ is satisfied for $i \in J := J_n$. The verification, omitted in [**4**] and [**7**], can be carried out exactly as in formula (2.9) below; the same idea is used in formula (8) in [**4**] and [**7**, p. 219], placed at the end of the proof to conclude the second part of (2.4). This means that the same reasoning is used twice in such a form of the proof, even if the first use is not marked explicitly.

To avoid repeating the reasoning and clarify the proof we impose on the sequence (i_n), inductively constructed, beside conditions (2.7) and (2.6) (cf. (5) and (6) in [**4**] and [**7**]) an additional condition (2.5) which directly guarantees that the left hand side of $(*)$ is satisfied for the indices $i \in J = J_n$ selected prior to i_{n+1}.

Obviously, we need not repeat the reasoning used in (2.9) at the end of our proof, because the second part of (2.4) follows due to property (2.5) of the sequence (i_n) proved earlier by induction. On the other hand, according to remarks in the preceding paragraph, the proof given below contains another form of the proof given in [**4**] and [**7**].

Proof of Theorem 2.1. Due to the above comment we assume that implication $(*)$ is true.

Since the left hand side of implication $(*)$ holds for $J := \{1\}$, there is an index $i_1 \in \mathbb{N} \setminus \{1\}$ such that $|x_{i,i}| > 0$ for $i \geqslant i_1$. Starting from this index i_1 and $\varepsilon_1 := 1/2$, we will construct inductively an increasing sequence of indices $i_n \in \mathbb{N}$, denoting $J_n := \{i_1, \dots, i_n\}$, and a sequence of $\varepsilon_n \in (0, 1/2]$ for $n \in \mathbb{N}$ such that the following conditions are satisfied:

$$\Big|\sum_{j \in J_m} x_{i,j}\Big| > \frac{1}{2}|x_{i,i}| \quad \text{for} \quad m \in \mathbb{N}, \quad i \in J_m; \tag{2.5}$$

$$\sum_{j \in J_m \setminus J_l} |x_{i_l,j}| < \varepsilon_l |x_{i_l,i_l}| \quad \text{for} \quad l, m \in \mathbb{N}, \quad l < m, \tag{2.6}$$

where

$$\varepsilon_m := \bigg(\frac{1}{2}|x_{i_m,i_m}| - \Big|\sum_{j \in J_{m-1}} x_{i_m,j}\Big|\bigg)|x_{i_m,i_m}|^{-1}, \quad m \in \mathbb{N}, \tag{2.7}$$

with $J_{m-1} := \emptyset$ for $m = 1$ in (2.7) and the convention (used also later) that any sum over the empty set of indices is 0.

Clearly, (2.5) is true for $m = 1$ and the fixed i_1, ε_1. Condition (2.6) makes sense for $m \geqslant 2$ and it will be satisfied for $m = 2$ after a proper choice of the index i_2 made in the course of the induction construction. Suppose that indices $i_1 < \cdots < i_{n-1}$ and $\varepsilon_1, \dots, \varepsilon_{n-1} \in (0, 1/2]$ are selected so that (2.5) holds for $m < n$ in case $n \geqslant 2$ and (2.6) is true for $l < m < n$ in case $n \geqslant 3$. Due to (2.5) for $m = n-1$, the left and so the right hand side of implication $(*)$ hold for $J = J_{n-1}$. Thus there is an index $i'_n > i_{n-1}$ such that

$$\Big|\sum_{j \in J_{n-1}} x_{i,j}\Big| < \frac{1}{2}|x_{i,i}|, \quad i \geqslant i'_n. \tag{2.8}$$

Applying for $m = n-1$ inequality (2.3) and, in case $n \geqslant 3$, inequality (2.6), we find an index $i_n > i'_n$ such that

$$\sum_{j \in J_n \setminus J_l} |x_{i_l,j}| < \varepsilon_l |x_{i_l,i_l}|, \quad l < n,$$

i.e., condition (2.6) holds for $m = n$. By (2.7) and (2.8), $\varepsilon_n \in (0, 1/2]$. Moreover,

$$\Big|\sum_{j \in J_n} x_{i_l,j}\Big| \geqslant |x_{i_l,i_l}| - \Big|\sum_{j \in J_{l-1}} x_{i_l,j}\Big| - \sum_{j \in J_n \setminus J_l} |x_{i_l,j}| > \frac{1}{2}|x_{i_l,i_l}|, \tag{2.9}$$

whenever $l \leqslant n$, due to (2.6) and (2.7), i.e., (2.5) holds for $m = n$. By induction, conditions (2.5) and (2.6) hold true for all $l, m \in \mathbb{N}$, $l < m$.

Now put $I = J := \{i_n : n \in \mathbb{N}\}$. It follows from (2.6) that

$$\sum_{k=l+1}^{\infty} |x_{i_l,i_k}| \leqslant \varepsilon_l |x_{i_l,i_l}| < \infty, \quad l \in \mathbb{N}$$

and this implies the first part of (2.4). But from (2.5) we obtain

$$\left|\sum_{j\in J} x_{i_l,j}\right| = \lim_{n\to\infty}\left|\sum_{k=1}^{n} x_{i_l,i_k}\right| \geqslant \frac{1}{2}|x_{i_l,i_l}|, \quad l \in \mathbb{N},$$

i.e., the second part of (2.4) is also true. □

3. Köthe spaces

Let K be a fixed countable set of indices. We can order it e.g. in the form $K := \{k_i : i \in \mathbb{N}\}$. Let $X = (X, |\cdot|)$ be a *semi-normed space*, i.e., a linear space over $\mathbb{R}$ with a semi-norm $|\cdot|$ meant as a functional $|\cdot|\colon X \to \mathbb{R}$, satisfying the conditions:

$$1^\circ \ \ |\alpha x| = |\alpha||x|, \quad 2^\circ \ \ |x+y| \leqslant |x| + |y|, \qquad \alpha \in \mathbb{R}, \quad x, y \in X.$$

A mapping $A\colon K \to X$, i.e., $A \in X^K =: \mathfrak{X}$, is denoted by $A =: [a_k]$ and called a *vector* and the values a_k are called its *coordinates*. The space of all vectors with coordinates belonging to X (to $\mathbb{R}$) is denoted by $\mathfrak{X}$ (by $\mathfrak{R}$). A vector in $\mathfrak{R}$ is called *positive* if all its coordinates are positive. By e_k we denote the vector whose k-th coordinate is 1 and the remaining ones are 0.

If $A = [a_k] \in \mathfrak{X}$, $B = [b_k] \in \mathfrak{X}$ and $\alpha \in \mathbb{R}$, then we define $A+B := [a_k+b_k] \in \mathfrak{X}$, $\alpha A := [\alpha a_k] \in \mathfrak{X}$. We will use the notation $AB = [a_k b_k]$, under the assumption, adopted always in the sequel, that $A \in \mathfrak{R}$ and $B \in \mathfrak{X}$ or, vice versa, $A \in \mathfrak{X}$ and $B \in \mathfrak{R}$.

Define the seminorms

$$|A|_1 := \sum_{k\in K} |a_k|, \quad |A|_\infty := \sup_{k\in K} |a_k|, \quad A = [a_k] \in \mathfrak{X}.$$

If X is complete and $|AB|_1 = \sum_{k\in K} |a_k b_k| < \infty$, then $|(A,B)|$ is the value of the semi-norm $|\cdot|$ on the element $(A,B) \in X$ uniquely defined by

$$(A,B) := \sum_{k\in K} a_k b_k := \lim_{n\to\infty} \sum_{i=1}^{n} a_{k_i} b_{k_i}.$$

In general, if X is a semi-normed space (not necessarily complete), the number $|(A,B)|$ is meant as follows:

$$|(A,B)| := \lim_{n\to\infty} \left|\sum_{i=1}^{n} a_{k_i} b_{k_i}\right|.$$

Definition 3.1. Let $(W_i)_{i\in\mathbb{N}}$ be a sequence of positive vectors on T satisfying the following condition:

$$(3.1) \qquad \left|W_i W_j^{-1}\right|_\infty < 2^{i-j}, \qquad i, j \in \mathbb{N}, \quad i < j.$$

A vector $S \in \mathfrak{R}$ is *rapidly decreasing*, if $|W_i S|_1 < \infty$ for $i \in \mathbb{N}$. The set of all rapidly decreasing vectors will be called *Köthe echelon space* and denoted by $\mathfrak{S}$. A vector $A \in \mathfrak{X}$ is *tempered* if there is an index $i_0 \in \mathbb{N}$ such that $|W_{i_0}^{-1} A|_\infty < \infty$. The set of all tempered vectors in $\mathfrak{X}$ will be called *Köthe co-echelon space* and denoted by $\mathfrak{T}$.

DEFINITION 3.2. A set $\mathcal{A} \subseteq \mathfrak{T}$ of tempered vectors in $\mathfrak{X}$ is said to be

(a) *strongly bounded* if there exist an $i_0 \in \mathbb{N}$ and a positive number β such that $|W_{i_0}^{-1} A|_\infty < \beta$ for all $A \in \mathcal{A}$;
(b) *weakly bounded* if the numerical set $\{|(A, S)| : A \in \mathcal{A}\}$ is bounded for all $S \in \mathfrak{S}$.

DEFINITION 3.3. Let $A_n = [a_{n,k}] \in \mathfrak{T}$ for $n \in \mathbb{N}_0$. We say that A_n is

(a) *strongly convergent* to A_0 (in symbols: $A_n \xrightarrow{s} A_0$) if $A_n \to A_0$ coordinatewise, i.e., $|a_{n,k} - a_{0,k}| \to 0$ as $n \to \infty$ for each $k \in K$ and, moreover, is strongly bounded;
(b) *weakly convergent* to A_0 (in symbols: $A_n \xrightarrow{w} A_0$) if $(A_n, S) \to (A_0, S)$ as $n \to \infty$ for every $S \in \mathfrak{S}$.

THEOREM 3.1. *Every countable set $\mathcal{T} \subseteq \mathfrak{T}$ is weakly bounded if and only if it is strongly bounded.*

REMARK 3.1. In the original proof of Theorem 3.1, given in [**7**, pp. 220–221], two sequences (n_i) and (s_i) of positive integers are inductively constructed, of which only the sequence (n_i) is evidently strictly increasing. However nothing is known about the constructed sequence (s_i); it can be bounded, for instance (because of the "a contrario" method used in this part of the proof in [**7**]). However, the definition of the vector S given in (3.7), which is crucial for the proof, requires that the constructed sequence (s_i) does not contain any constant subsequence.

To overcome this hindrance and construct in our proof below a strictly increasing sequence (s_i) we select indices s_i more specifically imposing on the inductively constructed sequences certain stronger conditions. More exactly, we define inductively in (3.4) an increasing sequence of positive numbers β_i which satisfy inequalities in (3.3). As a consequence, inequalities in (3.5) are satisfied and they force the increase of the constructed sequence (s_i).

PROOF OF THEOREM 3.1. Let $\mathcal{T} = \{T_n : n \in \mathbb{N}\}$, where $T_n \in \mathfrak{T}$. Assume that $\mathcal{T}$ is strongly bounded, i.e., there are an index $i_0 \in \mathbb{N}$ and a constant $\beta > 0$ such that $|W_{i_0}^{-1} T_n|_\infty \leqslant \beta$ for all $n \in \mathbb{N}$. Let $S \in \mathfrak{S}$, i.e., $|W_i S|_1 < \infty$ for all $i \in \mathbb{N}$. The set $\mathcal{T}$ is weakly bounded, because

$$|(T_n, S)| \leqslant |T_n S|_1 \leqslant |W_{i_0}^{-1} T_n|_\infty \cdot |W_{i_0} S|_1 \leqslant \beta |W_{i_0} S|_1, \quad n \in \mathbb{N}.$$

Suppose now that $\mathcal{T}$ is weakly but not strongly bounded, i.e., the following two assertions hold:

(I) the sequence $(|W_i^{-1} T_n|_\infty)$ is unbounded for each $i \in \mathbb{N}$;
(II) for each pair of indices $i, s \in \mathbb{N}$ there exists a $\beta_{i,s} \geqslant 1$ such that

$$|(T_n, R_{i,s})| = |R_{i,s} T_n|_\infty \leqslant \beta_{i,s}, \quad n, i, s \in \mathbb{N}, \tag{3.2}$$

where $R_{i,s} := W_i^{-1} e_s \in \mathfrak{S}$ for $i, s \in \mathbb{N}$.

We will construct inductively increasing sequences (n_i) and (s_i) of positive integers such that

$$\beta_i < |W_i^{-1}T_{n_i}|_\infty - 1 < |R_{i,s_i}T_{n_i}|_\infty, \quad i \in \mathbb{N}, \tag{3.3}$$

where

$$\beta_1 := \beta_{1,1}; \quad \beta_i := \max\{\beta_{i,s} : s \leqslant s_{i-1}\} + \beta_{i-1}, \quad i > 1. \tag{3.4}$$

Obviously, $\beta_n \uparrow \infty$ as $n \uparrow \infty$.

Applying (I), we can find an $n_1 \in \mathbb{N}$ fulfilling the first inequality and then an $s_1 \in \mathbb{N}$ satisfying the second inequality in (3.3) for $i = 1$. Assume that indices $n_1 < \cdots < n_p$ and $s_1 < \cdots < s_p$, satisfying (3.3) for $i = 1, \ldots, p$, are already chosen. We apply (I) for $i = p+1$ to find an index $n_{p+1} > n_p$ such that $|W_{p+1}^{-1}T_{n_{p+1}}|_\infty > \beta_{p+1} + 1$. Now we can select, again by (I), an index $s_{p+1} \in \mathbb{N}$ such that the second inequality in (3.3) for $i = p+1$ holds, i.e., $|R_{p+1,s_{p+1}}T_{n_{p+1}}|_\infty > \beta_{p+1}$. On the other hand, by (3.2) and (3.4), we have

$$|R_{p+1,s}T_{n_{p+1}}|_\infty \leqslant \beta_{p+1,s} \leqslant \beta_{p+1} \quad \text{for all } s \leqslant s_p, \tag{3.5}$$

i.e., the index s_{p+1} just selected cannot be among indices $s \leqslant s_p$. Consequently, it must be $s_{p+1} > s_p$. Thus the inductive construction of increasing sequences (n_i) and (s_i) satisfying (3.3) is completed.

Put $x_{i,j} := (T_{n_i}, R_{j,s_j}) \in X$ for $i, j \in \mathbb{N}$. Since $T_{n_i} \in \mathfrak{T}$, there are $q_i \in \mathbb{N}$ and $\lambda_i > 0$ such that $|W_{q_i}^{-1}T_{n_i}|_\infty \leqslant \lambda_i$ for all $i \in \mathbb{N}$. Hence, due to (3.1),

$$|x_{i,j}| \leqslant |W_{q_i}^{-1}T_{n_i}|_\infty \cdot |W_{q_i}W_j^{-1}|_\infty \leqslant \lambda_i \cdot 2^{q_i - j}, \quad i, j \in \mathbb{N}$$

and thus $\lim_{j\to\infty} |x_{i,j}| = 0$ for every $i \in \mathbb{N}$. It follows from Theorem 2.1 that there exists an infinite set $I \subseteq \mathbb{N}$ and its subset J (finite or infinite) such that $\sum_{j\in J} |(T_{n_i}, R_{j,s_j})| < \infty$ and we have, for all $i \in I$,

$$\frac{1}{2}|(T_{n_i}, R_{i,s_i})| \leqslant \Big|\sum_{j\in J}(T_{n_i}, R_{j,s_j})\Big| = \lim_{n\to\infty}\Big|\sum_{j\in J_n}(T_{n_i}, R_{j,s_j})\Big|, \tag{3.6}$$

where finite J_n form an nondecreasing sequence such that $\bigcup_{n=1}^\infty J_n = J$.

We are now in a position to define the vector S whose s_j-th coordinate coincides with the s_j-th coordinate of W_j^{-1} for all $j \in J$ and the remaining coordinates are equal to 0, i.e.,

$$S := \sum_{j\in J} R_{j,s_j} = \lim_{n\to\infty} S_n, \tag{3.7}$$

where $S_n := \sum_{j\in J_n} R_{j,s_j}$ and the convergence is coordinatewise. Since the constructed sequence (s_j) of indices is strictly increasing, the above definition of S is correct also in case J is infinite. Fix $i, q, r \in \mathbb{N}$ such that $i < q < r$. We have

$$\Big|W_i \sum_{j=q+1}^{r} R_{j,s_j}\Big|_1 \leqslant \sum_{j=q+1}^{r} |W_iW_j^{-1}|_\infty \leqslant \sum_{j=q+1}^{r} 2^{i-j},$$

which implies that $|W_iS|_1 < \infty$ for all $i \in \mathbb{N}$, i.e., $S \in \mathfrak{S}$ and

$$\lim_{n\to\infty} |W_i(S - S_n)|_1 = 0 \quad \text{for } i \in \mathbb{N},$$

i.e., (3.7) holds also in the sense of the convergence in $\mathfrak{S}$. Hence, by (3.6), we have

$$|(T_{n_i}, S)| = \lim_{n\to\infty} |(T_{n_i}, S_n)| = \Big|\sum_{j\in J}(T_{n_i}, R_{j,s_j})\Big| \geqslant \frac{1}{2}|(T_{n_i}, R_{i,s_i})|,$$

which means, by (3.3), that $|(T_{n_i}, S)| \to \infty$ and this contradicts the assumption that the set $\mathcal{T}$ is weakly bounded. $\square$

THEOREM 3.2. *Let $T_n \in \mathfrak{T}$ for $n \in \mathbb{N}_0$. Then $T_n \xrightarrow{w} T_0$ if and only if $T_n \xrightarrow{s} T_0$.*

PROOF. Let $T_n = [t_{n,k}] \in \mathfrak{T}$ for $n \in \mathbb{N}_0$. If $T_n \xrightarrow{w} T_0$, then $(T_n, S) \to (T_0, S)$, so a sequence (T_n, S) is bounded for all $S \in \mathfrak{S}$, i.e., (T_n) is weakly bounded. By Theorem 3.1, (T_n) is strongly bounded. Moreover,

$$t_{n,k} = (T_n, e_k) \to (T_0, e_k) = a_{0,k}, \quad k \in K,$$

because $e_k \in \mathfrak{S}$ for $k \in K$. Consequently, $T_n \xrightarrow{s} T_0$.

Assume now that $T_n \xrightarrow{s} T_0$. Fix $S := [s_j] \in \mathfrak{S}$ and $\varepsilon > 0$. By strong boundedness of the sequence (T_n) and by (3.1), there are $i_0 \in \mathbb{N}$ and $\beta > 0$ such that

$$|W_{i_0}^{-1} T_n|_\infty \leqslant \beta \quad \text{for } n \in \mathbb{N}_0. \tag{3.8}$$

Since

$$|W_{i_0} S|_1 = \sum_{j=1}^{\infty} w_{i_0,j}|s_j| < \infty,$$

there is an index $j_0 \in \mathbb{N}$ such that

$$\sum_{j=j_0+1}^{\infty} w_{i_0,j}|s_j| < \frac{\varepsilon}{4\beta}. \tag{3.9}$$

Define $\tilde{s}_j := s_j$ if $j \leqslant j_0$ and $\tilde{s}_j := 0$ if $j > j_0$. Clearly, $\tilde{S} := [\tilde{s}_j] \in \mathfrak{S}$. By the assumption, $T_n \to T_0$ coordinate wise. Hence

$$(T_n, \tilde{S}) = \sum_{j=0}^{j_0} t_{n,j} s_j \to \sum_{j=0}^{j_0} t_{0,j} s_j = (T_0, \tilde{S}).$$

Due to (3.8) and (3.9), we get

$$\begin{aligned} |(T_n - T_0, S)| &\leqslant |(T_n - T_0, \tilde{S})| + |(T_n - T_0, S - \tilde{S})| \\ &< \frac{\varepsilon}{2} + |W_{i_0}^{-1}(T_n - T_0)|_\infty \cdot |W_{i_0}(S - \tilde{S})|_1 < \varepsilon \end{aligned}$$

for sufficiently large $n \in \mathbb{N}$. This completes the proof. $\square$

REMARK 3.2. The assertions of Theorems 3.1 and 3.2 are true if X is a locally convex spaces, because the reasoning used in the proofs of Theorems 3.1 and 3.2 can be applied to each semi-norm of the family describing the topology of such a space.

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