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A NOTE ON THE FEKETE–SZEGÖ PROBLEM FOR CLOSE-TO-CONVEX FUNCTIONS WITH RESPECT TO CONVEX FUNCTIONS

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Abstract. We discuss the sharpness of the bound of the Fekete–Szegö functional for close-to-convex functions with respect to convex functions. We also briefly consider other related developments involving the Fekete–Szegö functional $|a_3 - \lambda a_2^2|$ $(0 \leqslant \lambda \leqslant 1)$ as well as the corresponding Hankel determinant for the Taylor–Maclaurin coefficients $\{a_n\}_{n \in \mathbb{N} \setminus \{1\}}$ of normalized univalent functions in the open unit disk $\mathbb{D}$, $\mathbb{N}$ being the set of positive integers.

1. Introduction

A classical problem in geometric function theory of complex analysis, which was settled by Fekete and Szegö [**4**], is to find for each $\lambda \in [0,1]$ the maximum value of the coefficient functional $\Phi_\lambda(f)$ given by

$$\Phi_\lambda(f) := \left|a_3 - \lambda a_2^2\right| \tag{1.1}$$

over the class $\mathcal{S}$ of univalent functions f in the open unit disk

$$\mathbb{D} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

of the following normalized form (see, for details, [**5, 22, 24**]):

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{D}). \tag{1.2}$$

By applying the Loewner method, Fekete and Szegö [**4**] proved that

$$\max_{f \in \mathcal{S}} \Phi_\lambda(f) = \begin{cases} 1 + 2\exp\left(-\frac{2\lambda}{1-\lambda}\right) & (0 \leqslant \lambda < 1) \\ 1 & (\lambda = 1). \end{cases}$$

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For various compact subclasses $\mathcal{F}$ of the class $\mathcal{A}$ of all analytic functions f in $\mathbb{D}$ of the form (1.2), as well as with λ being an arbitrary real or complex number, many authors computed

$$\max_{f\in\mathcal{F}} \Phi_\lambda(f) \tag{1.3}$$

or calculated the upper bound of (1.3) (see, e.g., [**2**, **8**, **11**, **21**]).

Let $\mathcal{S}^*$ denote the class of *starlike* functions, that is, $f \in \mathcal{S}^*$ if

$$f \in \mathcal{A} \quad \text{and} \quad \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

Given $\delta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $g \in \mathcal{S}^*$, let $\mathcal{C}_\delta(g)$ denote the class of functions called *close-to-convex with argument δ with respect to g*, that is, the class of all functions $f \in \mathcal{A}$ such that

$$\operatorname{Re}\left(e^{i\delta}\frac{zf'(z)}{g(z)}\right) > 0 \quad (z \in \mathbb{D}). \tag{1.4}$$

We also suppose that, given $g \in \mathcal{S}^*$, $\mathcal{C}(g) := \bigcup_{g\in\mathcal{S}^*} \mathcal{C}_\delta(g)$ and that, given $\delta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\mathcal{C}_\delta := \bigcup_{g\in\mathcal{S}^*} \mathcal{C}_\delta(g)$. Let

$$\mathcal{C} := \bigcup_{\delta\in(-\frac{\pi}{2},\frac{\pi}{2})} \bigcup_{g\in\mathcal{S}^*} \mathcal{C}_\delta(g)$$

denote the class of *close-to-convex* functions (see, for details, [**20**, pp. 184–185], [**6**, **10**]).

For the whole class $\mathcal{C}$, the sharp bound of the Fekete–Szegö coefficient functional Φ_λ for $\lambda \in [0,1]$, given by (1.1), was calculated by Koepf [**13**] who extended the earlier result for the class $\mathcal{C}_0$ and for $\lambda \in \mathbb{R}$ due to Keogh and Merkes [**11**], namely, it holds

$$\max_{f\in\mathcal{C}} \Phi_\lambda(f) = \max_{f\in\mathcal{C}_0} \Phi_\lambda(f) = \begin{cases} |3-4\lambda| & \big(\lambda \in \big(-\infty, \frac{1}{3}\big] \cup [1,\infty)\big) \\ \frac{1}{3} + \frac{4}{9\lambda} & \big(\lambda \in \big[\frac{1}{3}, \frac{2}{3}\big]\big) \\ 1 & \big(\lambda \in \big[\frac{2}{3}, 1\big]\big). \end{cases}$$

For various subclasses of the class of close-to-convex functions, the problem to estimate the coefficient functional Φ_λ is continued in several subsequent works (see, for details, [**9**, **12**, **14**–**16**]). Some interesting and important subclasses of the class $\mathcal{C}$ are the classes $\mathcal{C}_\delta^c$ and $\mathcal{C}^c$, which are defined below.

Let $\mathcal{S}^c$ denote the class of *convex* functions, that is, $f \in \mathcal{S}^c$ if

$$f \in \mathcal{A} \quad \text{and} \quad \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

Since $\mathcal{S}^c \subsetneq \mathcal{S}^*$, the class $\mathcal{C}_\delta^c := \bigcup_{g\in\mathcal{S}^c} \mathcal{C}_\delta(g)$ is a proper subclass of the class $\mathcal{C}_\delta$ and the class

$$\mathcal{C}^c := \bigcup_{\delta\in(-\frac{\pi}{2},\frac{\pi}{2})} \bigcup_{g\in\mathcal{S}^c} \mathcal{C}_\delta(g)$$

is a proper subclass of the class $\mathcal{C}$.

The class $\mathcal{C}_0^c$ was defined by Abdel-Gawad and Thomas [**1**]. The class $\mathcal{C}^c$ of *close-to-convex functions with respect to convex functions* was introduced by Srivastava, Mishra and Das [**23**]. In both of these cited papers, the authors (Abdel-Gawad and Thomas [**1**] *and* Srivastava, Mishra and Das [**23**]) considered the coefficient functional Φ_λ with $\lambda \in [0,1]$ also. In fact, in Srivastava, Mishra and Das [**23**] extended, for the class $\mathcal{C}^c$, the earlier result of Abdel-Gawad and Thomas [**1**] for the class $\mathcal{C}_0^c$. However, in each of the above-cited papers, the proof for the sharpness of the bound in (1.3) for $\lambda \in \left(\frac{2}{3}, 1\right]$ was proposed incorrectly as $5/6$.

This note is motivated essentially by the earlier papers [**1**] and [**23**]. The main purpose of our investigation here is to discuss such sharpness results for the bound in (1.3). We also provide a rather brief consideration of other related developments involving the Fekete–Szegö functional $\left|a_3 - \lambda a_2^2\right|$ $(0 \leqslant \lambda \leqslant 1)$ in (1.1) as well as the corresponding Hankel determinant for the Taylor–Maclaurin coefficients $\{a_n\}_{n\in\mathbb{N}\setminus\{1\}}$ of normalized univalent functions of the form (1.2).

2. Main Observation

As we remarked in Section 1, in both of the afore cited papers [**1**,**23**], the upper bounds of the Fekete–Szegö coefficient functional Φ_λ $(0 \leqslant \lambda \leqslant 1)$ for the classes $\mathcal{C}_0^c$ and $\mathcal{C}^c$, were computed. In fact, Theorems 5 and 6 of Srivastava, Mishra and Das [**23**] state that the following sharp inequality

$$\max_{f\in\mathcal{C}^c} \Phi_\lambda(f) \leqslant \tfrac{5}{6} \quad \left(\lambda \in \left[\tfrac{2}{3}, 1\right]\right) \tag{2.1}$$

holds true and that this result is the same as in [**1**] for the class $\mathcal{C}_0^c$ (a part of Theorem 3). However, the assertion that the extremal function, for which the equality in (2.1) is satisfied when $\lambda \in (\frac{2}{3}, 1]$, belongs to $\mathcal{C}^c$ is incorrect. Indeed, here in this section, we note that the above-cited papers [**1**, **23**] contain a statement to the effect that the equality in (2.1) is attained by a function $f \in \mathcal{A}$ given by

$$\frac{zf'(z)}{h(z)} = \frac{1+\omega(z)}{1-\omega(z)} \quad (z \in \mathbb{D}), \tag{2.2}$$

where $h \in \mathcal{S}^c$ is of the form

$$h(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in \mathbb{D};\ b_2 = b_3 := 1) \tag{2.3}$$

and ω is a function of the form

$$\omega(z) = \sum_{n=1}^{\infty} \beta_n z^n \quad (z \in \mathbb{D}) \tag{2.4}$$

with

$$\beta_1 := \frac{2-3\lambda}{6\lambda} \pm i\frac{\sqrt{6\lambda-4}}{6\lambda} \qquad \text{and} \qquad \beta_2 := 1 - \beta_1^2. \tag{2.5}$$

Unfortunately, however, ω is not a Schwarz function for $\lambda \in (\frac{2}{3}, 1]$. We recall here that a Schwarz function means an analytic self-mapping of $\mathbb{D}$ with $\omega(0) := 0$. Let us denote the class of Schwarz functions by $\mathcal{B}_0$. In order to see that $\omega \notin \mathcal{B}_0$, we verify (by straightforward computation) that, for $\lambda \in (\frac{2}{3}, 1]$, the following inequality:

$$|\beta_2| \leqslant 1 - |\beta_1|^2 \tag{2.6}$$

is false, so a necessary condition for ω to be in $\mathcal{B}_0$ (see, for example, [**5**, Vol. II, p. 78]) does not hold true. Alternatively, in order to get a contradiction, we suppose that ω with its coefficients in (2.5) is a Schwarz function. Thus, clearly, (2.6) holds true. Hence we find from (2.5) that $1 - |\beta_1|^2 \geqslant |\beta_2| = |1 - \beta_1^2| \geqslant 1 - |\beta_1|^2$. Thus we have $|1 - \beta_1^2| = 1 - |\beta_1|^2$ and, therefore, $\beta_1 = |\beta_1|$ or $\beta_1 = -|\beta_1|$. This means that β_1 is a real number, which by (2.5) is possible only for $\lambda = \frac{2}{3}$. Consequently, for $\lambda \in (\frac{2}{3}, 1]$, the function ω with its coefficients in (2.5) does not belong to $\mathcal{B}_0$. So, in light of (2.2), it does not follow that f is in $\mathcal{C}^c$ or in $\mathcal{C}_0^c$.

Equivalently, let

$$p(z) := \frac{1 + \omega(z)}{1 - \omega(z)} \quad (z \in \mathbb{D}), \tag{2.7}$$

where ω is as given above. Then

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{D}), \tag{2.8}$$

where, in view of (2.7), (2.4) and (2.5), we have $c_1 = 2\beta_1$ and $c_2 = 2(\beta_2 + \beta_1^2) = 2$. We observe further that, for $\lambda \in (\frac{2}{3}, 1]$, the function p does not belong to the Carathéodory class. We recall here that the Carathéodory class, denoted as $\mathcal{P}$, contains analytic functions p of the form (2.8) with a positive real part. In order to see that $p \notin \mathbb{P}$, we verify for $\lambda \in (\frac{2}{3}, 1]$ that the inequality $\left|c_2 - c_1^2/2\right| \leqslant 2 - |c_1|^2/2$, is false, which happens to be a necessary condition for p to be in the class $\mathbb{P}$ (see, for example, [**22**, p. 166]).

3. Concluding remarks and further developments

By means of Theorem 3 of Abdel-Gawad and Thomas [**1**], Theorems 1 to 4 of Srivastava, Mishra and Das [**23**], and in light of our observation in Section 2, we arrive at the following result.

Theorem 1. *Each of the following assertions holds true:*

$$\max_{f \in \mathcal{C}^c} \Phi_\lambda(f) = \max_{f \in \mathcal{C}_0^c} \Phi_\lambda(f) = \begin{cases} \frac{5}{3} - \frac{9\lambda}{4} & \left(\lambda \in \left[0, \frac{2}{9}\right]\right) \\ \frac{2}{3} + \frac{1}{9\lambda} & \left(\lambda \in \left[\frac{2}{9}, \frac{2}{3}\right]\right) \end{cases} \tag{3.1}$$

$$\max_{f \in \mathcal{C}^c} \Phi_\lambda(f) \leqslant \frac{5}{6} \qquad \left(\lambda \in \left(\frac{2}{3}, 1\right]\right). \tag{3.2}$$

Remark 1. The sharpness of the inequality in (3.2) for the classes $\mathcal{C}^c$ and $\mathcal{C}_0^c$ is an *open problem.*

We now note that, by Loewner Theorem (see, for example, [**5**, Vol. I, p. 1127]), the function $h \in \mathcal{S}^c$ of the form (2.3) (with $b_2 = b_3 := 1$) is uniquely determined, that is, $h(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n$ $(z \in \mathbb{D})$. Then (1.4) with $g := h$ is of the form

$$\operatorname{Re}(e^{i\delta}(1 - z)f'(z)) > 0 \qquad (z \in \mathbb{D}) \tag{3.3}$$

and defines the class $\mathcal{C}_\delta(h)$, and further the class $\mathcal{C}(h)$. For the first time, the inequality in (3.3), treated as the univalence criterion, was distinguished explicitly in [**20**, p. 185]. For the class $\mathcal{C}(h)$, the upper bound of the Fekete–Szegö coefficient functional Φ_λ for $\lambda \in \mathbb{R}$ was recently obtained in [**14**], where the following result was proven.

Theorem 2. *It is asserted that*

$$\max_{f\in\mathcal{C}(h)} \Phi_\lambda(f) \leqslant \begin{cases} \left|\frac{1}{3}-\frac{1}{4}\lambda\right| + \frac{2}{3}\,|2-3\lambda| & \left(\lambda \in \left(-\infty, \frac{2}{9}\right] \cup \left[\frac{10}{9}, \infty\right)\right) \\ \frac{1}{12}\cdot\frac{(2-3\lambda)^2}{2-|2-3\lambda|} + \left|\frac{1}{3}-\frac{1}{4}\lambda\right| + \frac{2}{3} & \left(\lambda \in \left[\frac{2}{9}, \frac{10}{9}\right]\right). \end{cases} \tag{3.4}$$

For each $\lambda \in \left(-\infty, \frac{2}{3}\right] \cup \left[\frac{4}{3}, \infty\right)$*, the inequality is sharp and the equality in* (2) *is attained by a function in* $\mathcal{C}_0(h)$*.*

Remark 2. For $\lambda \in \left(-\infty, \frac{2}{3}\right] \cup \left[\frac{4}{3}, \infty\right)$, we can rewrite (3.4) as the following corollary.

Corollary 1. *The following assertion holds true:*

$$\max_{f\in\mathcal{C}(h)} \Phi_\lambda(f) = \begin{cases} \left|\frac{5}{3}-\frac{9\lambda}{4}\right| & \left(\lambda \in \left(-\infty, \frac{2}{9}\right] \cup \left[\frac{4}{3}, \infty\right)\right) \\ \frac{2}{3}+\frac{1}{9\lambda} & \left(\lambda \in \left[\frac{2}{9}, \frac{2}{3}\right]\right). \end{cases} \tag{3.5}$$

Remark 3. For $\lambda \in \left[0, \frac{2}{3}\right]$, the result (3.5) asserted by Corollary 3.5 coincides with (3.1). Thus, naturally, Theorem 1 and Theorem 2 yield Corollary 2 below.

Corollary 2. *Each of the following assertions holds true*:

$$\max_{f\in\mathcal{C}(h)} \Phi_\lambda(f) = \max_{f\in\mathcal{C}_0^c} \Phi_\lambda(f) = \max_{f\in\mathcal{C}^c} \Phi_\lambda(f) \quad \left(\lambda \in \left[0, \tfrac{2}{3}\right]\right),$$
$$\max_{f\in\mathcal{C}(h)} \Phi_\lambda(f) \leqslant \frac{9\lambda^2-30\lambda+26}{6(4-3\lambda)} \leqslant \frac{5}{6} \qquad \left(\lambda \in \left(\tfrac{2}{3}, 1\right]\right).$$

Remark 4. The maximum of Φ_λ for $\lambda \in \left[0, \frac{2}{3}\right]$, over the class $\mathcal{C}^c$ of close-to-convex functions with respect to convex functions and over its subclass $\mathcal{C}(h)$ of close-to-convex functions with respect to convex function h, are identical.

Remark 5. The sharpness of the inequality in (3.4) for $\lambda \in \left(\frac{2}{3}, \frac{4}{3}\right)$ is an *open problem.*

Remark 6. We reiterate the fact that the Fekete–Szegö coefficient functional $\left|a_3-\lambda a_2^2\right|$ is well known for its rich history in geometric function theory. Its origin was in the disproof by Fekete and Szegö [**4**] of the 1933 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity (see, for details, [**4**]). The λ-generalized Fekete–Szegö coefficient functional $\left|a_3-\lambda a_2^2\right|$ has since received great attention, particularly in connection with many subclasses of the class $\mathcal{S}$ of normalized analytic and univalent functions. On the other hand, in the year 1976, Noonan and Thomas [**17**] defined the $\mathfrak{q}$th Hankel determinant of the function f in (1.2) by

$$H_{\mathfrak{q}}(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+\mathfrak{q}-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+\mathfrak{q}} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+\mathfrak{q}-1} & a_{n+\mathfrak{q}} & \cdots & a_{n+2\mathfrak{q}-2} \end{vmatrix} \qquad (n, \mathfrak{q} \in \mathbb{N};\ a_1 := 1).$$

The determinant $H_{\mathfrak{q}}(n)$ has also been considered by several other authors. For example, Noor [**18**] determined the rate of growth of $H_{\mathfrak{q}}(n)$ as $n \to \infty$ for functions f given by (1.1) with bounded boundary. In particular, sharp upper bounds on $H_2(2)$ were obtained in the recent works [**7**, **18**] for different classes of functions. We note, in particular, that

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2 \quad \text{and} \quad H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

The Hankel determinant $H_2(1) = a_3 - a_2^2$ is the *classical* Fekete–Szegö coefficient functional. The upper bounds of $H_2(2)$ for some specific analytic function classes were discussed quite recently by Deniz et al. [**3**] (see also [**19**]).

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