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ON REDUCTION OF AUTOMATA IN LABYRINTHS

Goran Kilibarda

Abstract. It is shown that every automaton acceptable for rectangular labyrinths can be reduced to an automaton that behaves according to either the left-hand rule or the right-hand rule, or does not move at all, in every plane rectangular labyrinth without leaves. This enables us to approach certain fundamental problems of the theory of automata in labyrinths in a quite different way.

1. Introduction

In the past fifty years much attention has been devoted to research dealing with automata analysis of geometric environment, images, graphs, formal languages and other discrete structures. The obtained results enabled the formation of a new direction in automata theory, namely, the behavior of automata in labyrinths.

Shannon's paper on maze-solving machine [**1**] played an important role in the formation of the direction and outlined the range of research problems for the coming years. There he considers a model of a mouse, presented as an automaton, which should find a certain target in a maze.

One of the fundamental problems in the theory of automata in labyrinths is the question of existence of a perfect trap for an arbitrary finite automaton acceptable for rectangular labyrinths (the difference between mazes and mosaic labyrinth is negligible here). Intuitively it was clear that the answer to the question is positive, but the problem turned out to be far from simple and easy. The proof of the corresponding theorem was first given in [**2**], then it was considerably shortened in [**6**] chiefly by passing from the clear algebraic language in [**2**] to the language of the theory of automata in [**6**]. An altogether different solution of the problem was presented in [**5**].

Developing the ideas from [**5**], in this paper, we approach this and similar problems in a new way: while [**2, 6**] focused on the construction of the trap, here we focus on automata and prove that every automaton can be reduced to an automaton whose behavior in every plane rectangular labyrinth without leaves

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follows either the left-hand or the right-hand rule (when an automaton finds itself in a vertex of such a labyrinth, it always chooses the direction of its further movement which is the first either to the left or to the right, respectively, of the direction by which it reached the vertex), or does not move in it at all. By reduction we mean simplification of the automaton behavior. This reduction is done in such a way that for every given automaton $\mathfrak{A}$ we can construct the corresponding two basic "blocks" (also in the form of plane rectangular labyrinths) with which we can replace all the edges in an arbitrary plane rectangular labyrinth L so that the behavior of $\mathfrak{A}$ in thus obtained labyrinth, in respect to the vertices of L, is equivalent to the behavior of the corresponding reduced automaton in L. The above mentioned equivalence of behaviors allows us to approach some fundamental problems in the theory of automata in labyrinths in a considerably simpler way, because actually we reduce some of these problems to the corresponding problems for thus obtained reduced automata. The advantage of this method is illustrated with the proof of a theorem which is similar to the above mentioned theorem from [**2, 6**], but for plane rectangular labyrinths.

The paper is self-contained, but one can find the basic notions and the basic results in the theory of automata in labyrinths in [**3, 4**], where there is also a more or less complete list of literature on this theory.

2. Automata and labyrinths

Denote the power set of a set X by $\mathcal{P}(X)$, and let $\mathcal{P}_0(X) = \mathcal{P}(X) \smallsetminus \{\emptyset\}$. Let $X_1, \dots, X_n$ be arbitrary sets. For every $1 \leqslant i \leqslant n$, by pr_i denote the projection map of the Cartesian product $X_1 \times \cdots \times X_n$ onto X_i. Denote the set of all words over an alphabet A by A^*; by Λ denote the empty word.

Let $\mathfrak{D} = \{\mathbf{e}, \mathbf{n}, \mathbf{w}, \mathbf{s}\}$. Take that $\mathbf{e}^{-1} = \overline{\mathbf{e}} = \mathbf{w}$, $\mathbf{n}^{-1} = \overline{\mathbf{n}} = \mathbf{s}$, $\mathbf{w}^{-1} = \overline{\mathbf{w}} = \mathbf{e}$, and $\mathbf{s}^{-1} = \overline{\mathbf{s}} = \mathbf{n}$. The elements of the set $\mathfrak{D}$ can be interpreted as the cardinal points: east, north, west, and south. If $\alpha = \omega_1 \dots \omega_n \in \mathfrak{D}^*$, then $\alpha^{-1} = \omega_n^{-1} \dots \omega_1^{-1}$; certainly, $\Lambda^{-1} = \Lambda$.

A connected edge-labeled symmetric simple digraph (L, f), $L = (V, E)$, where V is the set of vertices, E is the set of edges and $f \colon E \to \mathfrak{D}$ is an edge labeling of L, is a *rectangular labyrinth* (or simply a *labyrinth*) if $f[(y,x)] = (f[(x,y)])^{-1}$ for every $(x,y) \in E$, and if $f(u) \neq f(v)$ for every $u, v \in E$ such that $u \neq v$ and $\mathrm{pr}_1(u) = \mathrm{pr}_1(v)$.

Let $|u|_L = f(u)$ for each $u \in E$. Also, let $[x]_L = \{|u|_L \mid \mathrm{pr}_1(u) = x, u \in E\}$ for every $x \in V$. If it is clear from the context what labyrinth L is meant, then instead of $|u|_L$ and $[x]_L$ we write $|u|$ and $[x]$ respectively. Adding to $\mathfrak{D}$ the element which we denote by $\mathbf{0}$ (the corresponding interpretation of this element will be given in the sequel), extend the definition of f on the pairs (x,x), $x \in V$, taking that $|(x,x)| = \mathbf{0}$.

Further on, we shall omit f in the designation of a labyrinth (L, f) considering that in every concrete case f is determined. Sometimes, the set of all vertices and the set of all edges of a labyrinth L are labeled by $V(L)$ and $E(L)$ respectively.

A labyrinth L is *finite* if $V(L)$ is a finite set; otherwise L is *infinite*. All labyrinths in the sequel will be finite if it is not stated otherwise.

Let L be a labyrinth. Instead of L we write $(L; x')$ (or $(V, E; x')$) $[(L; x', x'')$ (or $(V, E; x', x'')$)] if in L a vertex x' [two different vertices x' and x''] is marked [are marked] as the *entrance* [the *entrance* and the *exit*]; such vertices x' and x'' are sometimes denoted by $x_{\mathrm{s}}(L)$ and $x_{\mathrm{f}}(L)$ respectively. If L is a labyrinth with an entrance x' and an exit x'', then by L^{-1} we denote the same labyrinth, but with the entrance x'' and the exit x'.

Suppose that L is a labyrinth and $\rho = x_0, u_1, x_1, \ldots, u_n, x_n$ is a walk in L. Then by $|\rho|$ we denote the word $|u_1| \ldots |u_n|$. Let x be a vertex of the labyrinth L and let $\alpha \in \mathfrak{D}^*$. If in L there exists a walk ρ starting at x such that $|\rho| = \alpha$, then by $(x\alpha)_L$ we denote the end vertex of ρ. Let us take that $x\Lambda = x$ for every $x \in V(L)$. If it is clear from the context what labyrinth L is meant, then in the sequel we often write $x\alpha$ instead of $(x\alpha)_L$.

Let L be a labyrinth. Replacing each pair of opposite edges in L with the corresponding non-labeled undirected edge, we obtain an undirected graph $G(L)$. A labyrinth L is a *tree* if the graph $G(L)$ is a tree. A labyrinth $(L; x', x'')$ is an $\omega_1\omega_2$-*tree*, $\omega_1, \omega_2 \in \mathfrak{D}$, if L is a tree, $[x'] = \{\omega_1\}$, and $[x''] = \{\overline{\omega_2}\}$; if $\omega_1 = \omega_2 = \omega$, then an $\omega_1\omega_2$-tree L is called an ω-*tree*. A labyrinth L is a *labyrinth without leaves* if $G(L)$ has no leaves.

Let M and N, $M \neq N$, be some points of the plane. By $\overline{MN}$ denote the line segment which is defined by the given points, and by $|\overline{MN}|$ its length. Let $\mathbf{i}$ and $\mathbf{j}$ be the unit vectors in the direction of the x-axis and y-axis of the rectangular coordinate system respectively. The vector $\overrightarrow{MN} = \alpha_1\mathbf{i} + \alpha_2\mathbf{j}$ *goes in the direction*: 1) $\mathbf{e}$ if $\alpha_1 > 0$ and $\alpha_2 = 0$; 2) $\mathbf{n}$ if $\alpha_1 = 0$ and $\alpha_2 > 0$; 3) $\mathbf{w}$ if $\alpha_1 < 0$ and $\alpha_2 = 0$; and 4) $\mathbf{s}$ if $\alpha_1 = 0$ and $\alpha_2 < 0$.

A set T of line segments in the plane $\mathbf{R}^2$ is called a *configuration* (*of line segments*) if any two different line segments of the set T can have not more than one common point, and if such a point exists, it must be an end point for both the line segments. A labyrinth $L = (V, E)$, $V \subseteq \mathbf{R}^2$, is *plane* if the set of line segments $T = \{\overline{xy} \mid (x, y) \in E\}$ is a configuration and the vector $\overrightarrow{xy}$ goes in the direction $|(x, y)|$ for every $(x, y) \in E$. If L is a plane, and, in addition, it holds that $|\overline{xy}| = 1$ for every $(x, y) \in E$, then we say that L is a *mosaic* labyrinth. Moreover, a mosaic labyrinth M is a *maze* if it satisfies that for every $x, y \in V(M)$ from $|\overline{xy}| = 1$, it follows that $(x, y) \in E(M)$.

For every plane labyrinth L, the set $\overline{L} = \bigcup_{(x,y)\in E(L)} \overline{xy}$ is the (*geometric*) *realization* of a L. A plane labyrinth L is *bounded* if diam $\overline{L} < \infty$; otherwise it is *unbounded*.

Labyrinths L_1 and L_2 are called *isomorphic*, $L_1 \cong L_2$, if there exists a bijective function $g\colon V(L_1) \to V(L_2)$ such that:

(1) g is an isomorphism of edge-labeled digraphs L_1 and L_2, i.e., $(x, y) \in E(L_1)$ iff $(g(x), g(y)) \in E(L_2)$ for every $x, y \in V(L_1)$, and $|(x, y)|_{L_1} = |(g(x), g(y))|_{L_2}$ for every $(x, y) \in E(L_1)$;
(2) if one of the labyrinths has an entrance [an entrance and an exit], then the other of them has an entrance [an entrance and an exit], too, and $x_{\mathrm{s}}(L_2) = g[x_{\mathrm{s}}(L_1)]$ [$x_{\mathrm{s}}(L_2) = g[x_{\mathrm{s}}(L_1)]$ and $x_{\mathrm{f}}(L_2) = g[x_{\mathrm{f}}(L_1)]$].

Such a function g is called an *isomorphism* from L_1 to L_2. The set of all labyrinths isomorphic to a labyrinth L is denoted by $[L]$.

A plane [mosaic] labyrinth $(L; x', x'')$ is *regular* [*perfect*] if there exists an unbounded plane [mosaic] labyrinth L_1 such that $\overline{L} \cap \overline{L_1} = \{x''\}$ and $x'' \in V(L_1)$.

By automaton $\mathfrak{A}$ we mean a quintuple (A, Q, B, φ, ψ), where the finite non-empty sets A, Q and B are the input alphabet, the set of states and the output alphabet of the automaton respectively, $\psi\colon Q \times A \to B$ is its output function and $\varphi\colon Q \times A \to Q$ is its state-transition function. If a state q_0 is marked in Q, we get an initial automaton $\mathfrak{A}_{q_0} = (A, Q, B, \varphi, \psi, q_0)$ (in other words, $\mathfrak{A}_{q_0}$ is a Mealy machine). For the given initial or non-initial automaton $\mathfrak{A}$, we sometimes denote the input alphabet, the set of states, the output function, the output function, and the transition function by $A_{\mathfrak{A}}$, $Q_{\mathfrak{A}}$, $B_{\mathfrak{A}}$, $\varphi_{\mathfrak{A}}$, and $\psi_{\mathfrak{A}}$ respectively.

An automaton (initial or non-initial) $\mathfrak{A}$ is said to be *acceptable* if $A_{\mathfrak{A}} = \mathcal{P}(\mathfrak{D})$, $B_{\mathfrak{A}} = \mathfrak{D} \cup \{\mathbf{0}\}$ and $\psi_{\mathfrak{A}}(q, a) \in a \cup \{\mathbf{0}\}$ for all $q \in Q_{\mathfrak{A}}$ and $a \in A_{\mathfrak{A}}$. In the sequel, all automata will be acceptable, and because of that we just say 'automaton' instead of 'acceptable automaton' for the sake of brevity. An automaton $\mathfrak{A}$ is *trivial* if $\psi_{\mathfrak{A}}(q, a) = \mathbf{0}$ for every $q \in Q_{\mathfrak{A}}$ and $a \in \mathcal{P}(\mathfrak{D})$.

Let $L = (V, E; x_0)$ be a labyrinth and $\mathfrak{A}_{q_0} = (A, Q, B, \varphi, \psi, q_0)$ be an initial automaton.

A sequence $(q_0, x_0), (q_1, x_1), \ldots$ in $Q \times V$ is called the *behavior* of the automaton $\mathfrak{A}_{q_0}$ in the labyrinth $(L; x_0)$, and it is denoted by $\pi(\mathfrak{A}_{q_0}; L)$, if for every $i \geqslant 0$ it holds that $(x_i, x_{i+1}) \in E$ or $x_i = x_{i+1}$, $q_{i+1} = \varphi(q_i, [x_i])$ and $\psi(q_i, [x_i]) = |(x_i, x_{i+1})|$; the sequence $\tau(\mathfrak{A}_{q_0}; L) = x_0, x_1, \ldots$ is the *trajectory* of $\mathfrak{A}_{q_0}$ in L. Let $\pi_i(\mathfrak{A}_{q_0}; L) = (q_i, x_i)$ for each $i \geqslant 0$. If L has also an exit y_0 and $x_i = y_0$ for some $i \geqslant 1$, then we say that $\mathfrak{A}_{q_0}$ *goes out of* the labyrinth $(L; x_0, y_0)$; otherwise we say that $(L; x_0, y_0)$ is a *trap* for $\mathfrak{A}_{q_0}$. If $\mathfrak{A}_{q_0}$ goes out of $(L; x_0, y)$ for every $y \in V \setminus \{x_0\}$, we say that $\mathfrak{A}_{q_0}$ *searches* $(L; x_0)$.

Let $V' \subseteq V$. If all the pairs (q_i, x_i) for which $x_i \notin V'$ are thrown out of $\pi(\mathfrak{A}_{q_0}; L)$, we get ether a finite (empty or non-empty), or infinite sequence $(q_{i_0}, x_{i_0}), (q_{i_1}, x_{i_1}), \ldots$ which is called the V'-*behavior* of $\mathfrak{A}_{q_0}$ in $(L; x_0)$. The sequence $x_{i_0}, x_{i_1}, \ldots$ is the V'-*trajectory* of $\mathfrak{A}_{q_0}$ in $(L; x_0)$. It is clear that $\pi(\mathfrak{A}_{q_0}; L)$ $[\tau(\mathfrak{A}_{q_0}; L)]$ is the V-behavior [V-trajectory] of $\mathfrak{A}_{q_0}$ in $(L; x_0)$.

For every $V_1 \subseteq V$, determine the values $\mathrm{st}(\pi, V_1)$, $\mathrm{pl}(\pi, V_1)$, $\mathrm{dr}(\pi, V_1)$, $\mathrm{tm}(\pi, V_1)$, $\mathrm{dr}_0(\pi, V_1)$ and $\mathrm{st}_0(\pi, V_1)$, where $\pi = \pi(\mathfrak{A}_{q_0}; L)$, in the following way. If there exists $t > 0$ such that $x_t \in V_1$ and $x_{t'} \notin V_1$ for every $0 < t' < t$, then $\mathrm{st}(\pi, V_1) = q_t$, $\mathrm{pl}(\pi, V_1) = x_t$, $\mathrm{dr}(\pi, V_1) = \psi(q_t, [x_t]_L)$ and $\mathrm{tm}(\pi, V_1) = t$; otherwise $\mathrm{st}(\pi, V_1)$, $\mathrm{pl}(\pi, V_1)$, $\mathrm{dr}(\pi, V_1)$ are not determined and $\mathrm{tm}(\pi, V_1) = +\infty$. If $\mathrm{st}(\pi, V_1)$ is determined and if there exists the number

$$i_0 = \min\{i \in \mathbf{N} \mid i \geqslant \mathrm{tm}(\pi, V_1) \wedge |(x_i, x_{i+1})| \neq \mathbf{0}\},$$

then let $\mathrm{dr}_0(\pi, V_1) = |(x_{i_0}, x_{i_0+1})|$ and $\mathrm{st}_0(\pi, V_1) = q_{i_0}$. By $\overline{\mathrm{dr}}(\pi, t)$ denote the superword $|(x_t, x_{t+1})||x_{t+1}, x_{t+2})| \ldots$. Also, by $\overline{\mathrm{dr}}_0(\pi, t)$ denote the superword which results from the superword $\overline{\mathrm{dr}}(\pi, t)$ by replacing all the appearances of the one-letter subword $\mathbf{0}$ by the empty word.

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be initial automata, and let $(L_1; x_1')$ and $(L_2; x_2')$ be labyrinths. For a $V_1 \subseteq V(L_1)$, let $(q_{i_0}^{(1)}, x_{i_0}^{(1)}), (q_{i_1}^{(1)}, x_{i_1}^{(1)}), \ldots$ be the V_1-behavior of $\mathfrak{A}_1$ in L_1, and for a $V_2 \subseteq V(L_2)$, let $(q_{j_0}^{(2)}, x_{j_0}^{(2)}), (q_{j_1}^{(2)}, x_{j_1}^{(2)}), \ldots$ be the V_2-behavior of $\mathfrak{A}_2$ in L_2. We say that the V_1-behavior of $\mathfrak{A}_1$ in L_1 and the V_2-behavior of $\mathfrak{A}_2$ in L_2 are *isomorphic* if: 1) for every $k \geqslant 0$, $(q_{i_k}^{(1)}), x_{i_k}^{(1)})$ exists iff $(q_{j_k}^{(2)}, x_{j_k}^{(2)})$ exists; and 2) there exist bijections $g\colon Q_{\mathfrak{A}_1} \to Q_{\mathfrak{A}_2}$ and $h\colon V_1 \to V_2$ such that $(g(q_{i_m}^{(1)}), h(x_{i_m}^{(1)})) = (q_{j_m}^{(2)}, x_{j_m}^{(2)})$ for every $m \geqslant 0$ satisfying that $(q_{i_m}^{(1)}), x_{i_m}^{(1)})$ exists. For example, for every initial automaton $\mathfrak{A}_{q_0}$, if $(L_1; x_1') \cong (L_2; x_2')$, then $\pi(\mathfrak{A}_{q_0}; L_1)$ and $\pi(\mathfrak{A}_{q_0}; L_2)$ are isomorphic.

As the behaviors of an automaton in isomorphic labyrinths are isomorphic and, consequently, as it is not important for the problems, we investigate here which of isomorphic labyrinths is taken, we do not differentiate isomorphic labyrinths and we adopt the following convention. In the sequel, we introduce some binary operations on labyrinths, which are partially defined and which satisfy the following condition: if $*$ is one of such operations, and L_1 and L_2 some labyrinths, then the labyrinth [edge-labeled digraph] $L * L'$ belongs to the same class of isomorphic labyrinths [edge-labeled digraphs] $[L_1 * L_2]$ for every $L \in [L_1]$ and $L' \in [L_2]$ for which it is defined. In fact, we will consider these operations as operations on the corresponding classes of isomorphic labyrinths, and when we say 'given a labyrinth [edge-labeled digraph] $L_1 * L_2$' we mean, in fact, that is given a labyrinth [edge-labeled digraph] from the class $[L_1 * L_2]$, and, consequently, the result of the application of operation $*$ may exist even in the case when $L_1 * L_2$ does not exist.

If by applying some operation on some labyrinths the new edges do not appear and we do not change the original labels of the remaining edges which they had in given labyrinths, we do not describe the edge labeling function of the resulting labyrinth [edge-labeled digraph] for the sake of shortness.

Let $L_1 = (V_1, E_1)$ and $L_2 = (V_2, E_2)$ be arbitrary labyrinths such that $V_1 \cap V_2 = \emptyset$. By $L_1 \dot\cup L_2$ denote the disjoint union of labyrinths L_1 and L_2, i.e., $L_1 \dot\cup L_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

Let L be a labyrinth, and let x and y, $x \neq y$, be its vertices (not obviously adjacent). Denote the labyrinth $(V(L), E(L) \setminus \{(x,y),(y,x)\})$ by $L - \langle x, y\rangle$. If x and y are not adjacent, and $[x] \cap [y] = \emptyset$, then by $\mathrm{vi}(L, x, y)$ denote the labyrinth $(V \setminus \{y\}, [E \setminus ((\{y\} \times V) \cup (V \times \{y\}))] \cup \overleftarrow{E}(x,y) \cup \overrightarrow{E}(x,y))$, where $\overleftarrow{E}(x,y) = \{(y\omega, x) \mid \omega \in [y]\}$, $\overrightarrow{E}(x,y) = \{(x, y\omega) \mid \omega \in [y]\}$, and $|(x, y\omega)| = \omega$ and $|(y\omega, x)| = \overline{\omega}$ for every $\omega \in [y]$ (we do not change the labels of the other edges).

Let L and $(L_1; x_1', x_1'')$ be labyrinths such that $V(L) \cap V(L_1) = \emptyset$, and let x and y be different vertices of L. Suppose that $[x]_{L - \langle x,y\rangle} \cap [x_1']_{L_1} = [y]_{L - \langle x,y\rangle} \cap [x_1'']_{L_1} = \emptyset$. By $L\, {}_x{+}_y\, L_1$ denote the labyrinth $\mathrm{vi}(\mathrm{vi}((L - \langle x,y\rangle) \dot\cup L_1, x, x_1'), y, x_1'')$. The idea of the described operation is the following: labyrinth L_1 is "put" in L between the vertices x and y.

Let $(L_1; x_1', x_1'')$ and $(L_2; x_2', x_2'')$ be labyrinths such that $V(L_1) \cap V(L_2) = \emptyset$ and $[x_1'']_{L_1} \cap [x_2']_{L_2} = \emptyset$. Denote the labyrinth $(\mathrm{vi}(L_1 \dot\cup L_2, x_1'', x_2'); x_1', x_2'')$ by $L_1 L_2$. For given labyrinths $(L_i; x_i', x_i'')$, $1 \leqslant i \leqslant n$, by $L_1 \ldots L_n$ denote the expression

$(\dots((L_1L_2)L_3)\dots L_{n-1})L_n$; denote the entrance x_1' [the exit x_n''] of this labyrinth by $(L_1\dots L_n;0)$ $[(L_1\dots L_n;n)]$, and for every $1\leqslant i\leqslant n-1$, by $(L_1\dots L_n;i)$ denote, now in $L_1\dots L_n$, the vertex x_i''. If $L_1\cong\cdots\cong L_n\cong L$, we can write L^n instead of $L_1\dots L_n$ (see the above convention).

For every $a\in\mathcal{P}_0(\mathfrak{D})$, let $V'(a)=\{x_\omega\mid\omega\in a\}$, $V(a)=\{x_{\mathbf{0}}\}\cup V'(a)$ and $E(a)=(a\times\{x_{\mathbf{0}}\})\cup(\{x_{\mathbf{0}}\}\times a)$. By $L(a)$ denote the labyrinth $(V(a),E(a);x_{\mathbf{0}})$ for which it holds that $(|(x_{\mathbf{0}},x_\omega)|,|(x_\omega,x_{\mathbf{0}})|)=(\omega,\overline{\omega})$ for all $\omega\in a$. For brevity, let $\langle\omega\rangle=L(\{\omega\})$ for every $\omega\in\mathfrak{D}$.

For every word $\alpha\in\mathfrak{D}^*$ by $\nu(\alpha)$ denote the word obtained from α by replacing in it, until it is possible, each subword of the form $\omega\omega^{-1}$, $\omega\in\mathfrak{D}$, with the empty word (for example, if $\alpha=\mathbf{wwnsesnn}$, then $\nu(\alpha)=\mathbf{wn}$). A nonempty word $\alpha\in\mathfrak{D}^*$ is a *simple word* over $\mathfrak{D}$ if $\alpha=\nu(\alpha)$; by $\mathrm{Sim}(\mathfrak{D})$ denote the set of all simple words over $\mathfrak{D}$. For every $\alpha=\omega_1\dots\omega_n\in\mathrm{Sim}(\mathfrak{D})$ by $\langle\alpha\rangle$ denote a labyrinth $\langle\omega_1\rangle\dots\langle\omega_n\rangle$. A labyrinth L is *snakelike* if $L\cong\langle\alpha\rangle$ for an $\alpha\in\mathrm{Sim}(\mathfrak{D})$; for a given snakelike labyrinth L the corresponding simple word α is unique and we denote it by $\alpha(L)$.

PROPOSITION 2.1. *If $(L;x_0,x_1)$ is a regular labyrinth, U is an open disk which contains L, then there exists a plane snakelike labyrinth $(L_1;x_1,x_2)$ such that $V(L_1)\smallsetminus U=\{x_2\}$ and $\overline{L}\cap\overline{L_1}=\{x_1\}$.*

Let $L=(V,E;x',x'')$ be a labyrinth. If there exists an injective mapping $\mu\colon V\to\mathbf{R}^2$ such that the labyrinth $\mu(L)=(\mu(V),\mu(E);\mu(x'),\mu(x''))$, where $\mu(E)=\{(\mu(x),\mu(y))\mid(x,y)\in E\}$ and $|(\mu(x),\mu(y))|_{\mu(L)}=|(x,y)|_L$ for every $(x,y)\in E$, is plane, then L is *embeddable* and μ is an *embedding* of L. Obviously, if μ is an *embedding* of L, then L and $\mu(L)$ are isomorphic. If, in addition, $\mu(L)$ is regular, then L is said to be *perfectly embeddable* and μ is a *perfect embedding* of L. In the sequel, by an embedding [a perfect embedding] we sometimes mean $\mu(L)$ or even $\overline{\mu(L)}$. For example, if $(L;x',x'')$ is a tree, then it is embeddable; moreover, if $[x'']_L\neq\mathfrak{D}$, then L is perfectly embeddable.

A labyrinth $(L;x',x'')$ is called a *regular trap* for an initial automaton $\mathfrak{A}$ if $(L;x',x'')$ is a trap for $\mathfrak{A}$ and if it is perfectly embeddable. A labyrinth $(L;x')$ is a *regular trap* for an initial automaton $\mathfrak{A}$ if there exists $x''\in V(L)$ such that $(L;x',x'')$ is a regular trap for $\mathfrak{A}$.

A labyrinth $L=(V,E;x',x'')$ is an *ω-labyrinth*, $\omega\in\{\mathbf{e},\mathbf{n}\}$, if $[x']=\{\omega\}$, $[x'']=\{\overline{\omega}\}$, and if there exists an embedding μ of L such that $\mathrm{pr}_{k_1}(\mu(x'))=\mathrm{pr}_{k_1}(\mu(x''))$, $\mathrm{pr}_{k_2}(\mu(x''))-\mathrm{pr}_{k_2}(\mu(x'))=r>0$ and

$$|\,\mathrm{pr}_{k_2}(\mu(z))-\mathrm{pr}_{k_2}(\mu(x'))|+|\,\mathrm{pr}_{k_2}(\mu(x''))-\mathrm{pr}_{k_2}(\mu(z))|=r$$

for every $z\in V$, where $(k_1,k_2)=(2,1)$ or $(k_1,k_2)=(1,2)$ if $\omega=\mathbf{e}$ or $\omega=\mathbf{n}$ respectively; such embedding μ of L is called a *standard embedding* of ω-labyrinth L. It follows from the definition that for every positive reals r_1 and r_2 there exists a standard embedding μ of L such that $|\overline{\mu(x')\mu(x'\omega)}|>r_1$, $|\overline{\mu(x'')\mu(x''\overline{\omega})}|>r_1$ and $\mathrm{diam}(V\smallsetminus\{x',x''\})<r_2$.

Let $L=(V,E)$ be a labyrinth and $V_1\subseteq V$. For every $x\in V$, let $V_x=\{x\}\times(\mathfrak{D}\smallsetminus[x]_L)$ and $E_x=\{(x,y)\mid y\in V_x\}\cup\{(y,x)\mid y\in V_x\}$. By $\mathrm{Cross}(L,V_1)$ denote the labyrinth $(V\cup(\bigcup_{x\in V_1}V_x),E\cup(\bigcup_{x\in V_1}E_x))$ for which $|(x,(x,\omega))|=\omega$

and $|((x,\omega),x)| = \overline{\omega}$ for every $x \in V_1$ and $\omega \in \mathfrak{D} \setminus [x]_L$, and $|(x,y)| = |(x,y)|_L$ for all $(x,y) \in E$. If $V_1 = V$, then instead of $\mathrm{Cross}(L,V)$ write $\mathrm{Cross}(L)$.

Suppose that $\alpha = \omega_1 \ldots \omega_n \in \mathrm{Sim}(\mathfrak{D})$ and $x_i = (\langle\alpha\rangle; i)$ for every $0 \leqslant i \leqslant n$. Let

$$\dashv\alpha\vdash = \mathrm{Cross}(L,\{x_i \mid 1 \leqslant i \leqslant n-1\}), \qquad \dashv\alpha\dashv = \mathrm{Cross}(L,\{x_i \mid 1 \leqslant i \leqslant n\}),$$
$$\vdash\alpha\vdash = \mathrm{Cross}(L,\{x_i \mid 0 \leqslant i \leqslant n-1\}), \qquad \vdash\alpha\dashv = \mathrm{Cross}(L,\{x_i \mid 0 \leqslant i \leqslant n\}).$$

Suppose that L is a labyrinth, L_1 is an $\mathbf{e}$-labyrinth and L_2 is an $\mathbf{n}$-labyrinth. Arrange in a sequence $(x_1,y_1),\ldots,(x_m,y_m)$ all edges $(x,y) \in E(L)$ for which $|(x,y)| \in \{\mathbf{e},\mathbf{n}\}$. By $\Delta(L;L_1,L_2)$ denote the labyrinth

$$(\cdots(L\,{}_{x_1}{+}_{y_1} L_{\kappa(x_1,y_1)})_{x_2}{+}_{y_2} \cdots {}_{x_{m-1}}{+}_{y_{m-1}} L_{\kappa(x_{m-1},y_{m-1})})_{x_m}{+}_{y_m} L_{\kappa(x_m,y_m)},$$

and by $\Sigma(L;L_1,L_2)$ the labyrinth $\Delta(\mathrm{Cross}(L);L_1,L_2)$; here, if $|(x,y)| = \mathbf{e}$, then $\kappa(x,y) = 1$, and if $|(x,y)| = \mathbf{n}$, then $\kappa(x,y) = 2$.

Let L be a labyrinth, L_1 be an $\mathbf{e}$-labyrinth, L_2 be an $\mathbf{n}$-labyrinth, and let $V_1 \subseteq V(L)$. Suppose that L' is one of the labyrinths $\mathrm{Cross}(L,V_1)$, $\Delta(L;L_1,L_2)$ and $\Sigma(L;L_1,L_2)$. Let us agree that if $x \in V(L)$ is the entrance or the exit of L, then x is the entrance or the exit of L' respectively, unless otherwise stated. It is obvious that the following assertion holds.

Proposition 2.2. *Let L_1 be an $\mathbf{e}$-labyrinth and L_2 be an $\mathbf{n}$-labyrinth. If L is an embeddable [a perfectly embeddable] labyrinth, then $\Sigma(L;L_1,L_2)$ $[\Delta(L;L_1,L_2)]$ is an embeddable [a perfectly embeddable] labyrinth, too.*

3. The reduction of automata in a plane

Suppose that $(L;x',x'')$ is a labyrinth such that $[x']_L = \{\omega\}$ for an $\omega \in \mathfrak{D}$. If there exists a perfect embedding μ of L such that the ray going out from $\mu(x')$ in the direction $\overline{\omega}$ does not intersect with this embedding, then μ is said to be an *extraperfect embedding* for L, and L is an *extraperfectly* (or ω-*extraperfectly*) *embeddable* labyrinth.

Let $\mathfrak{A} = (A,Q,B,\varphi,\psi)$ be an automaton, $(L;x',x'')$ be an ω-extraperfectly embeddable labyrinth for an $\omega \in \mathfrak{D}$ and $L_1 = \mathrm{Cross}(L,\{x'\})$. If there exists $q \in Q$ such that $\psi(q,\mathfrak{D}) \in \{\omega,\mathbf{0}\}$ and $\mathrm{pl}(\pi(\mathfrak{A}_q;L_1),\{x',x''\}) \neq x''$, then $\mathfrak{A}$ is an L-*reducible* automaton, and L *reduces* $\mathfrak{A}$. An automaton $\mathfrak{A}$ is *reducible* if there exists a labyrinth L such that $\mathfrak{A}$ is L-reducible; otherwise it is *irreducible*. Directly from the definition we get the following proposition.

Proposition 3.1. *Let $\mathfrak{A}$ be an irreducible automaton and $(L;x')$ be a tree such that $[x']_L = \{\omega\}$ for an $\omega \in \mathfrak{D}$. Then, for every $q \in Q_{\mathfrak{A}}$ and every $x \in V(L) \setminus \{x'\}$ which satisfy that $\psi_{\mathfrak{A}}(q,\mathfrak{D}) = \omega$ and $[x]_L \neq \mathfrak{D}$, it holds that* $\mathrm{tm}(\pi(\mathfrak{A}_q;L_1),\{x'\}) > \mathrm{tm}(\pi(\mathfrak{A}_q;L_1),\{x\})$, *where* $L_1 = \mathrm{Cross}(L,\{x'\})$.

Proposition 3.2. *If $\mathfrak{A}$ is an irreducible automaton, then $|Q_{\mathfrak{A}}| > 1$ and for every $q \in Q_{\mathfrak{A}}$ it holds that $\psi_{\mathfrak{A}}(q,\mathfrak{D}) \neq \mathbf{0}$.*

Proof. We have that $\psi_{\mathfrak{A}}(q,\mathfrak{D}) \neq \mathbf{0}$ for every $q \in Q_{\mathfrak{A}}$ directly from the definition. Now suppose that q_0 is the unique state of $\mathfrak{A}$. Without loss of generality we can suppose that $\psi_{\mathfrak{A}}(q_0, \mathfrak{D}) = \mathbf{e}$. But then the labyrinth $\dashv \mathbf{en} \vdash$ reduces $\mathfrak{A}$. From the obtained contradiction we get that $|Q_{\mathfrak{A}}| > 1$. $\square$

If $\mathfrak{A}$ is L-reducible and L or L^{-1} is an $\mathbf{e}$-labyrinth or an $\mathbf{n}$-labyrinth, then L is an *absorbing labyrinth* for $\mathfrak{A}$. From Proposition 2.1 we get that the following assertion holds.

Proposition 3.3. *If an automaton $\mathfrak{A}$ is reducible, then there exists an absorbing labyrinth for $\mathfrak{A}$.*

Assume that $\mathfrak{A}_{q_0} = (A, Q, B, \varphi, \psi, q_0)$ is an initial automaton such that $\omega_1 = \psi(q_0, \mathfrak{D}) \neq \mathbf{0}$. Take an $\alpha = \omega_1 \dots \omega_n \in \mathrm{Sim}(\mathfrak{D})$, $n \geqslant 2$. Let $L = \vdash \alpha \vdash$, $\tau(\mathfrak{A}_{q_0}; L) = x_0, x_1, \dots$ and $z_i = (\langle \alpha \rangle; i)$ for every $0 \leqslant i \leqslant n$. We say that $\mathfrak{A}_{q_0}$ *returns on* α if for some i and j, $0 \leqslant i < j < n$, there exist m_1 and m_2 such that $m_2 < m_1$, $x_{m_1} = z_i$, $x_{m_2} = z_j$ and $x_k \neq z_n$ for every $k < m_1$.

Proposition 3.4. *If $\mathfrak{A}$ is an irreducible automaton, then $\mathfrak{A}_q$ does not return on every $\psi_{\mathfrak{A}}(q, \mathfrak{D})\omega_2 \dots \omega_n \in \mathrm{Sim}(\mathfrak{D})$, $n \geqslant 2$, for each $q \in Q_{\mathfrak{A}}$.*

Proof. Use the above designations. Suppose, on the contrary, that $\mathfrak{A}_q$ returns on an $\alpha = \omega_1 \dots \omega_n \in \mathrm{Sim}(\mathfrak{D})$, where $\omega_1 = \psi_{\mathfrak{A}}(q, \mathfrak{D})$ and $n \geqslant 2$, for a $q \in Q_{\mathfrak{A}}$. So there exists a trajectory segment $x_{m_3}, \dots, x_{m_2}, \dots, x_{m_1}$, $m_3 < m_2 < m_1$, such that for some i and j, $0 \leqslant i < j < n$, it holds that $x_{m_1} = x_{m_3} = z_i$ and $x_{m_2} = z_j$. Hence $\dashv \omega_{i+1} \dots \omega_n \vdash$ reduces $\mathfrak{A}$. Contradiction. $\square$

Suppose that $\mathfrak{A} = (A, Q, B, \varphi, \psi)$ is an automaton, L_1 is an $\mathbf{e}$-labyrinth and L_2 is an $\mathbf{n}$-labyrinth. Let $K = \Delta(L(\mathfrak{D}); L_1, L_2)$ and $\pi(q) = \pi(\mathfrak{A}_q; K)$ for every $q \in Q$. Suppose that for some $q \in Q$ the value $q' = \mathrm{st}(\pi(q), V(\mathfrak{D}))$ exists. Then, if $\mathrm{pl}(\pi(q), V(\mathfrak{D})) = x_{\mathbf{0}}$, write $q \simeq^* q'$, and if $\mathrm{pl}(\pi(q), V(\mathfrak{D})) = x_\omega$, $\omega \in \mathfrak{D}$, write $q \Rightarrow^* q'$. By $\simeq$ denote the smallest equivalence relation on Q that contains $\simeq^*$, and by $\Rightarrow$ the composition of relations $\simeq$ and $\Rightarrow^*$.

Let $Q(L_1, L_2) = \{q \in Q \mid (\exists q')\, q \Rightarrow q'\}$ and $q_0 \in Q$. For every $q \in Q$, if $q \in Q(L_1, L_2)$ and $q \not\simeq q_0$, let $[q] = \{q' \in Q \mid q' \simeq q\}$; otherwise $[q] = (Q \smallsetminus Q(L_1, L_2)) \cup \{q' \in Q \mid q' \simeq q_0\}$. Let $Q' = \{[q] \mid q \in Q\}$ (generally, $Q' \neq Q/{\simeq}$). Construct the automaton $\mathfrak{A}[L_1, L_2, q_0] = (A, Q', B, \varphi', \psi')$ in the following way. Given an arbitrary $a \in \mathcal{P}_0(\mathfrak{D})$. For every $q \in Q(L_1, L_2)$, if the value $q' = \mathrm{st}(\pi(q), V'(a))$ exists, let $\varphi'([q], a) = [q']$ and $\psi'([q], a) = \omega(q)$, where $\omega(q)$ such that $x_{\omega(q)} = \mathrm{pl}(\pi(q), V'(a))$, and if $\mathrm{st}(\pi(q), V'(a))$ is not defined, let $\varphi'([q], a) = [q]$ and $\psi'([q], a) = \mathbf{0}$; additionally, if $q_0 \in Q \smallsetminus Q(L_1, L_2)$, take that $\varphi'([q], a) = [q]$ and $\psi'([q], a) = \mathbf{0}$ for every $q \in Q \smallsetminus Q(L_1, L_2)$. Finally, by taking that $\varphi'([q], \emptyset) = [q]$ and $\psi'([q], \emptyset) = \mathbf{0}$ for every $q \in Q$, we have completely defined the functions φ' and ψ' (for every element of $Q' \times A$). For every $q \in Q$, by $\mathfrak{A}[L_1, L_2, q_0; q]$ denote the automaton $\mathfrak{A}[L_1, L_2, q_0]$ with the initial state $[q]$. Note that $|Q_{\mathfrak{A}[L_1, L_2, q_0]}| \leqslant |Q_{\mathfrak{A}}|$ for every $q_0 \in Q_{\mathfrak{A}}$.

Obviously the following assertion holds.

Proposition 3.5. *Let L_1 be an $\mathbf{e}$-labyrinth, L_2 be an $\mathbf{n}$-labyrinth, and let $\mathfrak{A}$ be an automaton such that $|Q_{\mathfrak{A}}| > 1$. If one of the labyrinths L_1, L_2, L_1^{-1}*

and L_2^{-1} is an absorbing labyrinth for $\mathfrak{A}$, then there exists $q_0 \in Q_{\mathfrak{A}}$ such that $|Q_{\mathfrak{A}[L_1,L_2,q_0]}| < |Q_{\mathfrak{A}}|$.

Further on, when we consider the automaton $\mathfrak{A}[L_1, L_2, q_0]$ and if the conditions of the previous proposition are satisfied, then q_0 is chosen in such a way that $|Q_{\mathfrak{A}[L_1,L_2,q_0]}| < |Q_{\mathfrak{A}}|$. If it is not important which q_0 satisfying the condition of the last proposition we have chosen, or if it is clear from the context which q_0 is meant, then instead of $\mathfrak{A}[L_1, L_2, q_0]$ we write simply $\mathfrak{A}[L_1, L_2]$.

The first of the following two assertions follows from Propositions 3.2 and 3.3, and the second one is obvious.

Proposition 3.6. *If $\mathfrak{A}$ is an automaton satisfying $|Q_{\mathfrak{A}}| = 1$, then there exist an $\mathbf{e}$-labyrinth L_1 and an $\mathbf{n}$-labyrinth L_2 such that the automaton $\mathfrak{A}[L_1, L_2]$ is trivial.*

Proposition 3.7. *If $\mathfrak{A}$ is a trivial automaton, then $\mathfrak{A}[L_1, L_2]$ is trivial for every $\mathbf{e}$-labyrinth L_1 and $\mathbf{n}$-labyrinth L_2.*

Suppose that $\mathfrak{A}$ is an initial automaton, $(L; y_0)$ is a labyrinth and $W \subseteq V(L)$. Let $x_0, x_1, \ldots$ be the W-trajectory of $\mathfrak{A}$ in $(L; y_0)$. The sequence (finite or infinite) which we obtain by replacing each maximal block of equal elements in $x_0, x_1, \ldots$ with one of these elements (i.e., every finite segment $x, y, \ldots, y, z$ and, if it exists, the infinite segment $x, y, y, \ldots$ in $x_0, x_1, \ldots$ is replaced with x, y, z and x, y respectively; here, $x \neq y$ and $y \neq z$) is called the *cleaned W-trajectory* of $\mathfrak{A}$ in $(L; y_0)$.

Given two automata $\mathfrak{A}_1$ and $\mathfrak{A}_2$, two labyrinths L_1 and L_2, and a mapping $f\colon Q_{\mathfrak{A}_2} \to Q_{\mathfrak{A}_1}$. We say that the pair $(\mathfrak{A}_1, L_1)$ *f-imitates* the pair $(\mathfrak{A}_2, L_2)$, and we write $(\mathfrak{A}_1, L_1) \leqslant_f (\mathfrak{A}_2, L_2)$, if:

(1) there exist a $V' \subseteq V(L_2)$ and a bijection $g\colon V(L_1) \to V'$ such that for every $x_0 \in V(L_1)$ and for every $q \in Q_{\mathfrak{A}_2}$ the cleaned V'-trajectory of $(\mathfrak{A}_2)_q$ in $(L_2; g(x_0))$ is either the infinite sequence $g(x_0), g(x_1), \ldots$ or the finite sequence $g(x_0), g(x_1), \ldots, g(x_m)$ for some $m \geqslant 0$, where
$$\tau((\mathfrak{A}_1)_{f(q)}; (L_1; x_0)) = x_0, x_1, \ldots;$$
(2) for an $x_0 \in V(L_1)$ and a $q \in Q_{\mathfrak{A}_2}$, $(L_1; x_0)$ is a regular trap for $(\mathfrak{A}_1)_{f(q)}$, then $(L_2; g(x_0))$ is a regular trap for $(\mathfrak{A}_2)_q$.

We say that $\mathfrak{A}_1$ *imitates* $\mathfrak{A}_2$, and we write $\mathfrak{A}_1 \leqslant \mathfrak{A}_2$, if there exists a mapping $f\colon Q_{\mathfrak{A}_2} \to Q_{\mathfrak{A}_1}$ such that for every labyrinth L_1 there exists a labyrinth L_2 satisfying that $(\mathfrak{A}_1, L_1) \leqslant_f (\mathfrak{A}_2, L_2)$. The following theorem holds.

Theorem 3.1. *If $\mathfrak{A}$ is an automaton, L_1 is an $\mathbf{e}$-labyrinth and L_2 is an $\mathbf{n}$-labyrinth, then $\mathfrak{A}[L_1, L_2] \leqslant \mathfrak{A}$.*

Proof. Use the above given designations. Let $f(q) = [q]$ for every $q \in Q_{\mathfrak{A}}$. Given an arbitrary labyrinth L. Consider the labyrinth $L' = \Sigma(L; L_1, L_2)$. Take $g\colon V(L) \to V(L')$ such that $g(x) = x$ for every $x \in V(L)$. Directly from the definition of the automaton $\mathfrak{A}[L_1, L_2]$ we get: 1) the first condition of the above definition holds for the pairs $(\mathfrak{A}[L_1, L_2], L)$ and $(\mathfrak{A}, L')$; and 2) if $(L; x_0, y_0)$ is a regular trap for $(\mathfrak{A}[L_1, L_2])_{f(q)}$, then for some $\omega \in \mathfrak{D} \setminus [y_0]_L$, the labyrinth $(L'; g(x_0), (y_0, w))$ is a regular trap for $\mathfrak{A}_q$ (see Proposition 2.2). Therefore, $(\mathfrak{A}[L_1, L_2], L) \leqslant_f (\mathfrak{A}, L')$, and as L is an arbitrary labyrinth, we have that $\mathfrak{A}[L_1, L_2] \leqslant \mathfrak{A}$. ☐

THEOREM 3.2. *The relation $\leqslant$ in the set of all automata is transitive.*

PROOF. Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_3$ be automata. Suppose that $\mathfrak{A}_1 \leqslant \mathfrak{A}_2$ and $\mathfrak{A}_2 \leqslant \mathfrak{A}_3$. From $\mathfrak{A}_1 \leqslant \mathfrak{A}_2$ [$\mathfrak{A}_2 \leqslant \mathfrak{A}_3$] it follows that there exists a mapping $f_1\colon Q_{\mathfrak{A}_2} \to Q_{\mathfrak{A}_1}$ [$f_2\colon Q_{\mathfrak{A}_3} \to Q_{\mathfrak{A}_2}$] such that for every labyrinth K_1 [M_1] there exists a labyrinth K_2 [M_2] satisfying $(\mathfrak{A}_1, K_1) \leqslant_{f_1} (\mathfrak{A}_2, K_2)$ [$(\mathfrak{A}_2, M_1) \leqslant_{f_2} (\mathfrak{A}_3, M_2)$]. Let $f_3 = f_1 \circ f_2$. Take an arbitrary labyrinth L_1. As we have seen, there exists a labyrinth L_2 such that $(\mathfrak{A}_1, L_1) \leqslant_{f_1} (\mathfrak{A}_2, L_2)$, and there exists a labyrinth L_3 such that $(\mathfrak{A}_2, L_2) \leqslant_{f_2} (\mathfrak{A}_3, L_3)$. Showing that $(\mathfrak{A}_1, L_1) \leqslant_{f_3} (\mathfrak{A}_3, L_3)$, we get that $\mathfrak{A}_1 \leqslant \mathfrak{A}_3$, which proves the theorem.

So, there exist a $W_1 \subseteq V(L_2)$ and a bijection $g_1\colon V(L_1) \to W_1$ such that for every $(q, x_0) \in Q_{\mathfrak{A}_2} \times V(L_1)$ the cleaned W_1-trajectory of $(\mathfrak{A}_2)_q$ in $(L_2; g_1(x_0))$ is either $g_1(x_0), g_1(x_1), \ldots$ or $g_1(x_0), \ldots, g_1(x_m)$ for an $m \geqslant 0$, where $\tau((\mathfrak{A}_1)_{f_1(q)}; (L_1; x_0)) = x_0, x_1, \ldots$ Also, there exist a $W_2 \subseteq V(L_3)$ and a bijection $g_2\colon V(L_2) \to W_2$ such that for every $(q', y_0) \in Q_{\mathfrak{A}_3} \times V(L_2)$ the cleaned W_2-trajectory of $(\mathfrak{A}_3)_{q'}$ in $(L_3; g_2(y_0))$ is either $g_2(y_0), g_2(y_1), \ldots$ or $g_2(y_0), \ldots, g_2(y_n)$ for an $n \geqslant 0$, where $\tau((\mathfrak{A}_2)_{f_2(q')}; (L_2; y_0)) = y_0, y_1, \ldots$

By g_3 denote the bijection $g_2 \circ g_1\colon V(L_1) \to W_3$, where $W_3 = g_2(W_1) \subseteq V(L_3)$. Take an $\hat{x}_0 \in V(L_1)$ and a $\hat{q} \in Q_{\mathfrak{A}_3}$. Let $\tau((\mathfrak{A}_2)_{f_2(\hat{q})}; (L_2; g_1(\hat{x}_0))) = \hat{y}_0, \hat{y}_1, \ldots$ and $\tau((\mathfrak{A}_1)_{f_3(\hat{q})}; (L_1; \hat{x}_0)) = \hat{x}_0, \hat{x}_1, \ldots$ But from our assumption it follows that there exists a finite or an infinite sequence of integers $0 = i_0 < i_1 < i_2 < \ldots$ such that $\hat{y}_{i_j} = g_1(\hat{x}_j)$ for every element i_j of this sequence, and $\hat{y}_{i_0}, \hat{y}_{i_1}, \ldots$ is the cleaned W_1-trajectory of $(\mathfrak{A}_2)_{f_2(\hat{q})}$ in $(L_2; g_1(\hat{x}_0))$. As $W_3 \subseteq W_2$, and as the cleaned W_2-trajectory of $(\mathfrak{A}_3)_{\hat{q}}$ in $(L_3; g_2(g_1(\hat{x}_0)))$ is either $g_2(\hat{y}_0), g_2(\hat{y}_1), \ldots$ or $g_2(\hat{y}_0), \ldots, g_2(\hat{y}_{\hat{n}})$ for some $\hat{n} \geqslant 0$, then the cleaned W_3-trajectory of $(\mathfrak{A}_3)_{\hat{q}}$ in $(L_3; g_3(\hat{x}_0))$ is either $g_2(\hat{y}_{i_0}), g_2(\hat{y}_{i_1}), \ldots$ or $g_2(\hat{y}_{i_0}), \ldots, g_2(\hat{y}_{i_{\hat{m}}})$, i.e., is either $g_3(\hat{x}_0), g_3(\hat{x}_1), \ldots$ or $g_3(\hat{x}_0), \ldots, g_3(\hat{x}_{\hat{m}})$, for some $\hat{m} \geqslant 0$.

Further, if for an $x_0 \in V(L_1)$ and a $q \in Q_{\mathfrak{A}_3}$, $(L_1; x_0)$ is a regular trap for $(\mathfrak{A}_1)_{f_1 \circ f_2(q)}$, then $(L_2; g_1(x_0))$ is a regular trap for $(\mathfrak{A}_2)_{f_2(q)}$, and, consequently, $(L_3; g_2 \circ g_1(x_0))$ is a regular trap for $(\mathfrak{A}_3)_q$. So we have that $(\mathfrak{A}_1, L_1) \leqslant_{f_3} (\mathfrak{A}_3, L_3)$, and our assertion is true. $\square$

THEOREM 3.3. *Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be arbitrary automata. Then there exist automata $\mathfrak{A}'_1, \ldots, \mathfrak{A}'_n$, satisfying that each of them is irreducible or trivial, such that $\mathfrak{A}'_i \leqslant \mathfrak{A}_i$ for every $1 \leqslant i \leqslant n$.*

PROOF. Take an arbitrary **e**-labyrinth L_1 and an arbitrary **n**-labyrinth L_2. Let $\mathfrak{A}_i^{(0)} = \mathfrak{A}_i[L_1, L_2]$ for every $1 \leqslant i \leqslant n$. If for some $1 \leqslant j \leqslant n$, either $|Q_{\mathfrak{A}_j^{(0)}}| > 1$ and $\mathfrak{A}_j^{(0)}$ is reducible or $|Q_{\mathfrak{A}_j^{(0)}}| = 1$ and $\mathfrak{A}_j^{(0)}$ is non-trivial, then Propositions 3.5 and 3.6 imply that there exist an **e**-labyrinth $L_1^{(0)}$ and **n**-labyrinth $L_2^{(0)}$ such that $|Q_{\mathfrak{A}_j^{(0)}[L_1^{(0)}, L_2^{(0)}]}| < |Q_{\mathfrak{A}_j^{(0)}}|$ or $\mathfrak{A}_j^{(0)}[L_1^{(0)}, L_2^{(0)}]$ is trivial respectively. Let $\mathfrak{A}_i^{(1)} = \mathfrak{A}_i^{(0)}[L_1^{(0)}, L_2^{(0)}]$ for every $1 \leqslant i \leqslant n$. Continue with this procedure and suppose that we obtain automata $\mathfrak{A}_i^{(k)}$, $1 \leqslant i \leqslant n$. If for a $1 \leqslant j \leqslant n$, either $|Q_{\mathfrak{A}_j^{(k)}}| > 1$ and $\mathfrak{A}_j^{(k)}$

is reducible or $|Q_{\mathfrak{A}_j^{(k)}}| = 1$ and $\mathfrak{A}_j^{(k)}$ is non-trivial, then there exist an **e**-labyrinth $L_1^{(k)}$ and **n**-labyrinth $L_2^{(k)}$ such that

$$|Q_{\mathfrak{A}_j^{(k)}[L_1^{(k)},\, L_2^{(k)}]}| < |Q_{\mathfrak{A}_j^{(k)}}| \tag{3.1}$$

or $\mathfrak{A}_j^{(k)}[L_1^{(0)},\, L_2^{(k)}]$ is trivial respectively. Let $\mathfrak{A}_i^{(k+1)} = \mathfrak{A}_i^{(k)}[L_1^{(k)},\, L_2^{(k)}]$ for every $1 \leqslant i \leqslant n$. Because of (3.1) and Proposition 3.7 this procedure must be finished, i.e., there exists a minimal integer $m \geqslant 0$ such that $\mathfrak{A}_i^{(m)}$ is trivial or irreducible for every $1 \leqslant i \leqslant n$. Let $\mathfrak{A}'_i = \mathfrak{A}_i^{(m)}$ for every $1 \leqslant i \leqslant n$. Fix an $1 \leqslant i \leqslant n$. From Theorem 3.1 it follows immediately that $\mathfrak{A}_i^{(k+1)} = \mathfrak{A}_i^{(k)}[L_1^{(k)},\, L_2^{(k)}] \leqslant \mathfrak{A}_i^{(k)}$. Hence, we have that $\mathfrak{A}'_i = \mathfrak{A}_i^{(m)} \leqslant \cdots \leqslant \mathfrak{A}_i^{(1)} \leqslant \mathfrak{A}_i^{(0)} \leqslant \mathfrak{A}_i$, and from Theorem 3.2, we get that $\mathfrak{A}'_i \leqslant \mathfrak{A}_i$. □

Let $\mathfrak{A}_{q_0} = (A, Q, B, \varphi, \psi, q_0)$ be an initial automaton, σ be one of the two permutations

$$\sigma_{\mathbf{r}} = \begin{pmatrix} \mathbf{e} & \mathbf{n} & \mathbf{w} & \mathbf{s} \\ \mathbf{n} & \mathbf{w} & \mathbf{s} & \mathbf{e} \end{pmatrix} \quad \text{and} \quad \sigma_{\mathrm{l}} = \begin{pmatrix} \mathbf{e} & \mathbf{s} & \mathbf{w} & \mathbf{n} \\ \mathbf{s} & \mathbf{w} & \mathbf{n} & \mathbf{e} \end{pmatrix}$$

of the set $\mathfrak{D}$ and $\omega_0 \in \mathfrak{D}$. Let us introduce two special classes of labyrinths which play an important role in the sequel.

Suppose that $L_0 = L_2 = \langle\omega_0\rangle$, L_1 is a $\sigma(\omega_0)\omega_0$-tree, and $y_0 \in V(L_1)$. Let $L' = \mathrm{Cross}(L_0L_1L_2, \{x_0, x_1, x_2, x_3\})$, where $x_i = (L_0L_1L_2; i)$ for every $0 \leqslant i \leqslant 3$, and $\pi = \pi(\mathfrak{A}_{q_0}; L')$. The 4-tuple $(L_0, L_1, L_2; y_0)$ is a *pre-absorbing* (or (ω_0, σ)-*pre-absorbing*) *labyrinth* for $\mathfrak{A}_{q_0}$ if:

(1) for every $\delta > 0$ and $\Delta > 0$ there exists a extraperfect embedding μ of $(L_0L_1L_2\ {}_{x_1}{+}_{x_3}\langle\omega_0\rangle\langle\sigma(\omega_0)\rangle; x_0, y_0)$ such that $|\overline{\mu(x_1)\mu(x_1\sigma(\omega_0))}| > \Delta$, $|\overline{\mu(x_2)\mu(x_2\overline{\omega_0})}| > \Delta$ and $\operatorname{diam} \mu(V(L_0L_1L_2) \smallsetminus \{x_0, x_1, x_2, x_3\}) < \delta$;

(2) $\psi(q_0, \mathfrak{D}) = \omega_0$, $\omega_1 = \mathrm{dr}_0(\pi, \{x_3\}) \in \{\sigma(\omega_0), \omega_0\}$, $\mathrm{dr}_0(\pi, \{x_1\}) = \sigma(\omega_0)$ and $\mathrm{pl}(\pi, \{x_3, y_0\}) = x_3$

(in Fig. 1a is given an $(\mathbf{e}, \sigma_{\mathrm{r}})$-pre-absorbing labyrinth).

Note that the figures given in the paper represent plane labyrinths or the embeddings of labyrinths by their realizations. A point v of such a realization W is a vertex of the given plane labyrinth or of the given embedding iff there does not exist an open disk with the center at v whose intersection with W is an open segment or v is marked with a small black closed disk.

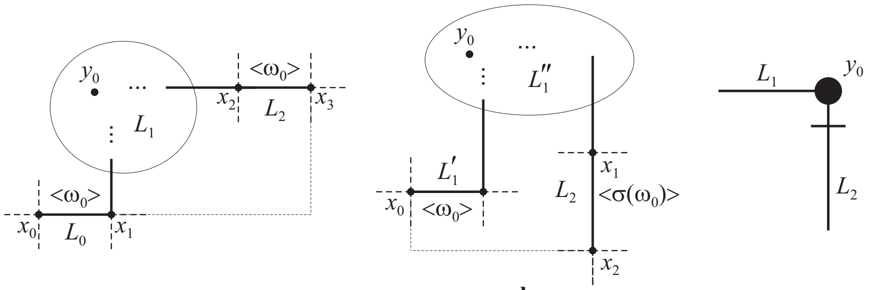
Assume that $L'_1 = \langle\omega_0\rangle$, L''_1 is a $\omega_1\sigma(\omega_0)$-tree, where $\omega_1 \neq \overline{\omega_0}$, and $L_2 = \langle\sigma(\omega_0)\rangle$. Let $L_1 = \mathrm{Cross}(L'_1L''_1, \{z_1\})$ and $y_0 \in V(L_1)$, where $z_i = (L'_1L''_1; i)$ for each $0 \leqslant i \leqslant 2$. Also, let $x_j = (L_1L_2; j)$ for each $0 \leqslant j \leqslant 2$. The ordered 3-tuple $(L_1, L_2; y_0)$ is an (ω_0, σ)-*incomplete pre-absorbing labyrinth* for $\mathfrak{A}_{q_0}$ if:

(1) for every $\delta, \Delta \in \mathbf{R}^+$ there exists a perfect embedding μ of the labyrinth $(L_1L_2\ {}_{x_0}{+}_{x_2}\langle\sigma(\omega_0)\rangle\langle\omega_0\rangle; x_0, y_0)$ satisfying

$$|\overline{\mu(x_0)\mu(x_0\omega_0)}| > \Delta,\ \ |\overline{\mu(x_1)\mu(x_2)}| > \Delta,\ \ \operatorname{diam} \mu(V(L_1L_2) \smallsetminus \{x_0, x_2\}) < \delta;$$

(2) $\psi(q_0, \mathfrak{D}) = \omega_0$ and $\mathrm{pl}(\pi(\mathfrak{A}_{q_0}; \mathrm{Cross}(L_1L_2, \{x_0, x_1, x_2\})), \{x_2, y_0\}) = x_2$.

Because of condition (1) from the above definition, an (ω_0, σ)-incomplete pre-absorbing labyrinth $(L_1, L_2; y_0)$, given in Fig. 1b, will be depicted as in Fig. 1c (here, $\omega_0 = \mathbf{e}$, $\omega_1 = \mathbf{n}$, and $\sigma = \sigma_1$).



a **b** **c**

FIGURE 1.

THEOREM 3.4. *Let $\mathfrak{A}$ be an automaton. If for a $q_0 \in Q_{\mathfrak{A}}$ there exists a pre-absorbing labyrinth $(L_0, L_1, L_2; z_0)$ for $\mathfrak{A}_{q_0}$, then $\mathfrak{A}$ is reducible.*

PROOF. We use the above given designations. Obviously $m = |Q_{\mathfrak{A}}| > 1$. Let $L_4 = (V_4, E_4; x_4', x_4'') = \langle\overline{\omega_0}\rangle^{m_1}$, $L_5 = (V_5, E_5; x_5', x_5'') = \langle\overline{\omega_0}\rangle^{m_2}$, $V_4' = V_4 \setminus \{x_4', x_4''\}$ and $V_5' = V_5 \setminus \{x_5', x_5''\}$; here m_1 and m_2 are natural numbers which we are going to determine. Obviously there exists an $\omega_1\sigma^{-1}(\omega_0)$-tree L_3 such that for every m_1 and m_2 the labyrinth

$$L = ((L_0 \dots L_5 \;{}_{x_5}{+}_{x_3}\; \langle\sigma(\omega_0)\rangle) \;{}_{x_6}{+}_{x_1}\; \langle\overline{\omega_0}\rangle; x_0, z_0),$$

where $x_i = (L_0 \dots L_5; i)$ for every $0 \leqslant i \leqslant 6$, is extraperfectly embeddable. Suppose that $\omega_0 = \omega_1 = \mathbf{w}$ and $\sigma = \sigma_1$ (see Fig. 2); the other cases are discussed similarly. Take some $m_1 \geqslant m + 1$, and put $L' = \mathrm{Cross}(L, \{x_i \mid 0 \leqslant i \leqslant 6\} \cup V_4' \cup V_5')$. Let $\pi = \pi(\mathfrak{A}_{q_0}; L')$ and $\pi_t = \pi_t(\mathfrak{A}_{q_0}; L')$ for every $t \geqslant 0$. Prove the theorem by contradiction, that is, suppose that $\mathfrak{A}$ is irreducible.

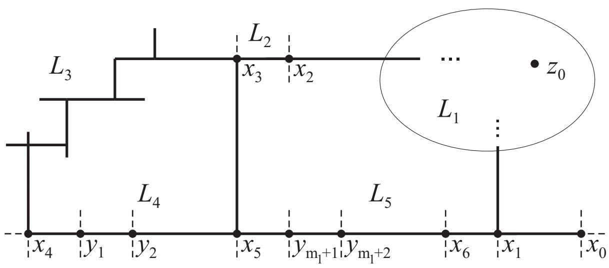


FIGURE 2.

Since $\mathrm{dr}_0(\pi, \{x_1\}) = \mathbf{n}$ and $\mathrm{dr}_0(\pi, \{x_3\}) = \mathbf{w}$, then from Proposition 3.1 it follows that $\mathfrak{A}_{q_0}$ cannot find itself again in x_1 and x_3 until it searches all the vertices of the set $V_4' \cup \{x_5\}$. Note that if $\mathfrak{A}_{q_0}$ finds itself in some vertex $z \in V_4' \cup V_5' \cup \{x_1, x_5, x_6\}$ at some moment $t > 1$ and $|(\mathrm{pr}_2(\pi_{t-1}), \mathrm{pr}_2(\pi_t))| = \mathbf{e}$, then $\overline{\mathrm{dr}}_0(\pi, t) = (\mathbf{sn})^{k_0}\mathbf{n}\ldots$ (case 1 for z) or $\overline{\mathrm{dr}}_0(\pi, t) = (\mathbf{sn})^{k_0}\mathbf{e}\ldots$ (case 2 for z) for some $k_0 \geqslant 0$ (since $\mathfrak{A}$ is irreducible, $\overline{\mathrm{dr}}_0(\pi, t) \neq (\mathbf{sn})^{k_0}\mathbf{w}\ldots$; otherwise $\dashv \mathbf{e}^2 \vdash$ reduces $\mathfrak{A}$).

Let $y_i = x_4(\overline{\omega_0})^i$ for every $1 \leqslant i \leqslant m_1 + m_2$; note that $y_{m_1} = x_5$ and $y_{m_1+m_2} = x_6$. If $\mathfrak{A}_{q_0}$ finds itself for the first time in x_5, and for x_5 case 2 holds, then, since $m_1 \geqslant m + 1$, there exists $1 \leqslant m_0 \leqslant m$ such that case 2 holds for all the vertices $y_{m_1+jm_0}$, where $jm_0 < m_2$ and $j \in \mathbf{N}$. Now if we choose $m_2 = 2m_0 - 1$, then for x_1 case 2 takes place, and, consequently, $\mathfrak{A}$ is $(L'; x_0, z_0)$-reducible. Hence case 1 holds for x_5, and $\mathfrak{A}_{q_0}$ finds itself at a moment $t' > t$ again in x_3 (assume that t' is the first of such moments). Now if $\overline{\mathrm{dr}}_0(\pi, t') = (\mathbf{ns})^{k_0}\mathbf{e}\ldots$ for a $k_0 \geqslant 0$, then the labyrinth

$$\tilde{L} = \mathrm{Cross}(L_2 \ldots L_5 \;{}_{x_3'}{+}_{x_1'}\; \langle\sigma(\omega_0)\rangle, \{x_1', x_2', x_3'\} \cup V_4')$$

reduces $\mathfrak{A}$, where $x_i' = (L_2 \ldots L_5; i)$, $0 \leqslant i \leqslant 4$, and $m_2 = 1$; if $\overline{\mathrm{dr}}_0(\pi, t') = (\mathbf{ns})^{k_1}\mathbf{s}\ldots$ for a $k_1 \geqslant 0$, then $\dashv \mathbf{nw} \vdash$ reduces $\mathfrak{A}$. Therefore, we may suppose that $\overline{\mathrm{dr}}_0(\pi, t') = (\mathbf{ns})^{k_2}\mathbf{w}\ldots$ for a $k_2 \geqslant 0$. Hence $\mathfrak{A}_{q_0}$ reaches x_5 again without visiting vertex x_3, and we may repeat our reasoning. But if $\mathfrak{A}_{q_0}$ visits x_5 more than m times and case 1 always takes place for x_5, then $\mathfrak{A}_{q_0}$ “moves in loops”, and $\tilde{L}$ again reduces $\mathfrak{A}$. Contradiction. □

Suppose that $\mathfrak{A}$ is an initial automaton, $(L; x_0)$ is a labyrinth and $\tau = y_0, y_1, \ldots$ is the cleaned trajectory of $\mathfrak{A}$ in $(L; x_0)$ (τ is finite or infinite). Let $z \in V(L) \smallsetminus \mathrm{Lf}(L)$, where $\mathrm{Lf}(L)$ is the set of all leaves of L. A finite segment $\tau' = y_m, y_{m+1}, \ldots, y_n$ ($0 \leqslant m \leqslant n$) or an infinite segment $\tau' = y_m, y_{m+1}, \ldots$ ($m \geqslant 0$) of τ is called a *z-block* of τ if τ' has at least one appearance of z and contains, besides z, only leaves of L. A *z-block* τ' of τ is *regular* if it is maximal, i.e., there is no other different *z-block* which contains τ'. A segment τ' of τ is a *regular block* if there exists a $z \in V(L)$ such that τ' is a regular z-block.

If τ contains at least one $z \notin \mathrm{Lf}(L)$, then it has at least one regular block, its regular blocks cover it, and any two different regular blocks of τ are disjoint.

Suppose that τ' is a regular block of τ. Let $z \in V(L) \smallsetminus \mathrm{Lf}(L)$ be such that τ' is a regular z-block. Perform the following procedure on the elements of τ' beginning with the first and taking them one by one: if a segment of the form z, w, z, where $w \in \mathrm{Lf}(L)$, appears in τ' for the first time, do not touch it, and if we have already had a segment exactly the same, then replace it by z (e.g., this procedure transforms z-block $z, w_1, z, w_2, z, w_2, z, w_1, z, w_3, z$, where w_1, w_2 and w_3 are different leaves of L, into $z, w_1, z, w_2, z, w_3, z$). Perform the above procedure on each regular block of τ. The obtained sequence is called the *doubly cleaned trajectory* of $\mathfrak{A}$ in $(L; x_0)$.

An initial automaton $\mathfrak{A}_{q_0}$ is a *snakelike σ_0-walker*, $\sigma_0 \in \{\sigma_\mathrm{r}, \sigma_\mathrm{l}\}$, if $\omega_1 = \psi_{\mathfrak{A}_{q_0}}(q_0, \mathfrak{D}) \neq \mathbf{0}$, and if for every $\alpha = \omega_1 \ldots \omega_n \in \mathrm{Sim}(\mathfrak{D})$, $n \geqslant 2$, the doubly cleaned trajectory $y_0, y_1, \ldots$ of $\mathfrak{A}_{q_0}$ in $\vdash \alpha \vdash$ is such that there exists $i_0 = \min\{j \mid y_j = x_\mathrm{f}(\vdash \alpha \vdash)\}$, and it holds that $|(y_i, y_{i+1})| = \sigma_0(|(y_i, y_{i-1})|)$ for every $1 \leqslant i \leqslant i_0 - 1$ satisfying that y_i is not a leaf. A snakelike σ_0-walker is a *snakelike rightwalker* [*snakelike leftwalker*] if $\sigma_0 = \sigma_\mathrm{r}$ [$\sigma_0 = \sigma_\mathrm{l}$].

Theorem 3.5. *If $\mathfrak{A}$ is an irreducible automaton, and if for a $q_0 \in Q_{\mathfrak{A}}$ and a $\sigma_0 \in \{\sigma_{\mathrm{r}}, \sigma_{\mathrm{l}}\}$ the initial automaton $\mathfrak{A}_{q_0}$ is not a snakelike σ_0-walker, then there exists a $(\psi_{\mathfrak{A}}(q_0, \mathfrak{D}), \sigma_0)$-incomplete pre-absorbing labyrinth for $\mathfrak{A}_{q_0}$.*

Proof. Note that Proposition 3.2 implies that $\omega_1 = \psi_{\mathfrak{A}}(q_0, \mathfrak{D}) \neq \mathbf{0}$. As Propositions 3.1 and 3.4 hold, and as $\mathfrak{A}_{q_0}$ is not a snakelike σ_0-walker, then there exist the shortest word $\alpha = \omega_1 \dots \omega_m \in \mathrm{Sim}(\mathfrak{D})$, $m \geqslant 2$, such that $y_{k+1} = x_{\mathrm{f}}(\vdash \alpha \vdash)$, $|(y_k, y_{k+1})| = \omega_m$, $\sigma_0(|(y_k, y_{k-1})|) \neq \omega_m$, and $y_j \neq x_{\mathrm{f}}(\vdash \alpha \vdash)$ for some $k \geqslant 1$ and for each $0 < j < k$, where $y_0, y_1, \dots$ is the doubly cleaned trajectory of $\mathfrak{A}_{q_0}$ in $\vdash \alpha \vdash$. Let $\alpha_1 = \alpha \sigma_0(\omega_m) \overline{\alpha} \sigma_0(\omega_1)$, $L_1 = \dashv \alpha_1 \vdash$, $L_2 = \langle \sigma_0(\omega_1) \rangle$, and

$$z_0 = (x_{\mathrm{s}}(L_1)\omega_1 \dots \omega_{m-1}\sigma_0(|(y_k, y_{k-1})|))_{L_1}.$$

Clearly, $(L_1, L_2; z_0)$ is an (ω_1, σ_0)-incomplete absorbing labyrinth for $\mathfrak{A}_{q_0}$. $\square$

In the sequel, wherever we use the result of the last assertion, by $*\alpha|$ denote the word $\alpha \sigma_0(\omega_m) \overline{\alpha} \sigma_0(\omega_1)$, by $|\alpha*$ denote $\sigma_0(\omega_1)$, and let $*\alpha* = *\alpha||\alpha*$. Denote the (ω_1, σ_0)-incomplete pre-absorbing labyrinth $(L_1, L_2; z_0)$ constructed in the proof of Theorem 3.5 by $(L_1, L_2; z_0; \alpha)$.

By $\mathcal{L}_{\mathrm{plwl}}$ denote the class of all plane labyrinths without leaves. An initial automaton $\mathfrak{A}_{q_0}$ is a *σ_0-walker*, $\sigma_0 \in \{\sigma_{\mathrm{r}}, \sigma_{\mathrm{l}}\}$, if for every $L \in \mathcal{L}_{\mathrm{plwl}}$ and every $x_0 \in V(L)$ the doubly cleaned $V(L)$-trajectory $y_0, y_1, \dots$ of $\mathfrak{A}_{q_0}$ in $(\mathrm{Cross}(L); x_0)$ is infinite and $|(y_i, y_{i+1})| = \sigma_0(|(y_i, y_{i-1})|)$ for every $i \geqslant 1$ satisfying that $y_i \in V(L)$; we say that a σ_0-walker $\mathfrak{A}_{q_0}$ is a *σ_0-walker with the guided vector* ω if $|(y_0, y_1)| = |(x_0, y_1)|$ is always the first element from $[y_0]_L$ which goes after $\overline{\omega}$ according to σ_0. A σ_0-walker $\mathfrak{A}_{q_0}$ is said to be a *rightwalker* [*leftwalker*] if $\sigma_0 = \sigma_{\mathrm{r}}$ [$\sigma_0 = \sigma_{\mathrm{l}}$].

Proposition 3.8. *Let $\mathfrak{A}$ be an irreducible automaton, $q_0 \in Q_{\mathfrak{A}}$, and $\alpha = \omega_1 \dots \omega_n \in \mathrm{Sim}(\mathfrak{D})$, where $\omega_1 = \psi_{\mathfrak{A}}(q_0, \mathfrak{D})$ and $n \geqslant 2$. If $\mathfrak{A}_{q_0}$ is a snakelike σ_0-walker, $\sigma_0 \in \{\sigma_{\mathrm{r}}, \sigma_{\mathrm{l}}\}$, then $\mathfrak{A}_{q_1}$, where $q_1 = \mathrm{st}(\pi(\mathfrak{A}_{q_0}; \vdash \alpha \vdash), \{x_{\mathrm{f}}(\vdash \alpha \vdash)\})$, is a σ_0-walker with the guided vector ω_n.*

Assume that $\mathfrak{A} = (A, Q, B, \varphi, \psi)$ is an automaton and $\omega \in \mathfrak{D}$. A $q \in Q$ is called an *ω-turning state* of $\mathfrak{A}$ if for some $\sigma_0 \in \{\sigma_{\mathrm{l}}, \sigma_{\mathrm{r}}\}$ it holds that $\psi(q, \mathfrak{D}) = \omega$ and $\psi(\varphi(q, \mathfrak{D}), \mathfrak{D}) = \sigma_0(\overline{\omega})$. A state q of $\mathfrak{A}$ is a *turning state* of $\mathfrak{A}$ if there exists $\omega \in \mathfrak{D}$ such that q is an ω-turning state of $\mathfrak{A}$.

Let q_0 be a state of an automaton $\mathfrak{A}$ such that $\omega_1 = \psi_{\mathfrak{A}}(q_0, \mathfrak{D}) \neq \mathbf{0}$, and let $\alpha = \omega_1 \dots \omega_n \in \mathrm{Sim}(\mathfrak{D})$, $n \geqslant 1$. Suppose that $\pi = \pi(\mathfrak{A}_{q_0}; \vdash \alpha \vdash) = (q_0, x_0), (q_1, x_1), \dots$ and $z_j = (\langle \alpha \rangle; j)$ for each $0 \leqslant j \leqslant n$. We say that q_0 *can be turned by* α if there exists $t_1 \geqslant 0$ such that $x_{t_1} = z_{n-1}$, $q_{t_1} = \mathrm{trn}(q_0, \alpha)$ is a ω_n-turning state of $\mathfrak{A}$, and $x_t \neq z_n$ for every $0 \leqslant t \leqslant t_1$. A state q of $\mathfrak{A}$ *can be turned* if there exists $\alpha \in \mathrm{Sim}(\mathfrak{D})$ such that q can be turned by α.

Theorem 3.6. *Each state of an irreducible automaton $\mathfrak{A} = (A, Q, B, \varphi, \psi)$ can be turned.*

Proof. Suppose our assertion does not hold for a $q \in Q$. Without loss of generality take that $\psi(q, \mathfrak{D}) = \mathbf{w}$. Consider the labyrinth $(L; y_0)$ given in Fig. 3. Let $L' = \mathrm{Cross}((L; y_0), \{y_0\})$, $\tau(\mathfrak{A}_q; L') = x_0, x_1, \dots$ and $\pi = \pi(\mathfrak{A}_q; L')$. Proposition

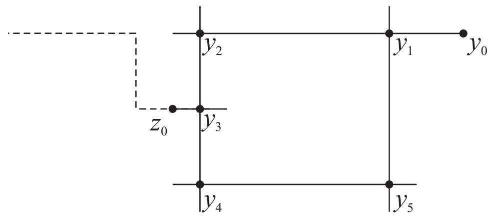


FIGURE 3.

3.2 implies that there is no $i \geqslant 0$ such that $x_i = x_{i+1} = y_j$ for an $1 \leqslant j \leqslant 5$. Now from our assumption we get that $x_0 = y_0$, $x_1 = y_1$ and $x_2 = y_2$. Notice that from the irreducibility of $\mathfrak{A}$ it follows that $\mathrm{pl}(\pi, \{y_0, z_0\}) \neq y_0$. According to Proposition 3.4, $\mathfrak{A}_q$ does not return on any of the words $\mathbf{w}^2\mathbf{s}(\mathbf{senws})^k$, $k \geqslant 0$. As Proposition 3.1 holds and q cannot be turned by any of the words $\mathbf{w}^2\mathbf{s}(\mathbf{senws})^k$, $0 \leqslant k \leqslant |Q|$, then the labyrinth $(L; y_0, z_0)$ reduces $\mathfrak{A}_q$. Contradiction. □

Let $\mathfrak{A}$ be an automaton. A state $q \in Q_{\mathfrak{A}}$ *orients* $\mathfrak{A}$ [*with the guided vector* ω] if $\mathfrak{A}_q$ is an leftwalker or an rightwalker [with the guided vector ω].

Assume that $\mathfrak{A}_{q_0}$ is an initial automaton, L_1 is a snakelike $\mathbf{e}$-labyrinth and L_2 is a snakelike $\mathbf{n}$-labyrinth. Let $L = \Delta(L(\mathfrak{D}); \dashv\alpha(L_1)\vdash, \dashv\alpha(L_2)\vdash)$. We say that the pair (L_1, L_2) *orients* $\mathfrak{A}_{q_0}$ if the state $\mathrm{st}(\pi(\mathfrak{A}_{q_0}; L), V'(\mathfrak{D}))$ exists and orients $\mathfrak{A}$ with the guided vector ω, where $x_\omega = \mathrm{pl}(\pi(\mathfrak{A}_{q_0}; L), V'(\mathfrak{D}))$; denote the state $\mathrm{st}(\pi(\mathfrak{A}_{q_0}; L), V'(\mathfrak{D}))$ by $q(\mathfrak{A}_{q_0}; L_1, L_2)$.

THEOREM 3.7. *If an automaton* $\mathfrak{A} = (A, Q, B, \varphi, \psi)$ *is irreducible, then for every* $q \in Q_{\mathfrak{A}}$ *there exist a snakelike* $\mathbf{e}$*-labyrinth* L_1 *and a snakelike* $\mathbf{n}$*-labyrinth* L_2 *such that the pair* (L_1, L_2) *orients* $\mathfrak{A}_q$.

PROOF. Fix a $q_0 \in Q_{\mathfrak{A}}$. Introduce four variables $\beta_{\mathbf{e}}$, $\beta_{\mathbf{w}}$, $\beta_{\mathbf{n}}$ and $\beta_{\mathbf{s}}$ whose values are words from the set $\mathrm{Sim}(\mathfrak{D}) \cup \{\Lambda\}$ and take Λ as their initial value, i.e., take that $\beta_{\mathbf{e}} := \Lambda$, $\beta_{\mathbf{w}} := \Lambda$, $\beta_{\mathbf{n}} := \Lambda$, and $\beta_{\mathbf{s}} := \Lambda$.

As $\mathfrak{A}$ is irreducible, then $\omega = \varphi(q_0, \mathfrak{D}) \neq \mathbf{0}$ for an $\omega \in \mathfrak{D}$, and from Theorem 3.6 it follows that there exists a word $\alpha_0 \in \mathrm{Sim}(\mathfrak{D})$ such that q_0 can be turned by α_0. Let $q_1 = \mathrm{trn}(q_0, \alpha_0)$ be an ω_0-turning state and $\psi(q_2, \mathfrak{D}) = \omega_1 = \sigma_0(\overline{\omega_0})$, where $q_2 = \varphi(q_1, \mathfrak{D})$ and $\sigma_0 \in \{\sigma_{\mathrm{l}}, \sigma_{\mathrm{r}}\}$ (in Fig. 4 we suppose that $\omega_0 = \mathbf{w}$, $\omega_1 = \mathbf{s}$ and $\sigma_0 = \sigma_{\mathrm{l}}$).

If $\mathfrak{A}_{q_2}$ is a snakelike σ_0-walker, then put $\beta_\omega := \alpha_0\omega_1$. If $\mathfrak{A}_{q_2}$ is not a snakelike σ_0-walker, then from Theorem 3.5 it follows that there exists an (ω_1, σ_0)-incomplete pre-absorbing labyrinth $(L_1, L_2; y_0; \alpha_1)$ for $\mathfrak{A}_{q_2}$, where $L_1 = \dashv * \alpha_1 | \vdash$ and $L_2 = \langle \omega_0 \rangle$. Let $L_0 = \langle \omega_0 \rangle$, $\tilde{L}_1 = L_0 L_1 L_2$, and let

$$L' = \mathrm{Cross}(\tilde{L}_1, \{x_0', x_1', x_2', x_3'\}),$$

where $x_i' = (\tilde{L}_1; i)$ for every $0 \leqslant i \leqslant 3$. As $\mathfrak{A}$ is irreducible and $L' \cong \vdash\omega_0\alpha_1\dashv$, then from Propositions 3.1, 3.2, and 3.4 it follows that $\omega_2 = \mathrm{dr}(\pi(\mathfrak{A}_{q_1}; L'), \{x_3'\}) = \mathrm{dr}_0(\pi(\mathfrak{A}_{q_1}; L'), \{x_3'\})$ exists and $\omega_2 \neq \overline{\omega_0}$. Now, if $\omega_2 \in \{\omega_0, \omega_1\}$, then the ordered tuple $(L_0, L_1, L_2; y_0)$ is a pre-absorbing labyrinth for $\mathfrak{A}_{q_1}$, and from Theorem 3.4 we get the contradiction. Hence $\omega_2 = \overline{\omega_1}$.

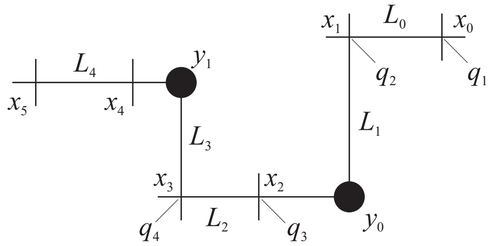


FIGURE 4.

Let $\pi(\mathfrak{A}_{q_1}; L') = (\hat{x}_0, \hat{q}_0), (\hat{x}_1, \hat{q}_1), \ldots$ and $t = \mathrm{tm}(\pi(\mathfrak{A}_{q_1}; L'), \{x'_3\})$ (obviously, $(\hat{x}_0, \hat{q}_0) = (x'_0, q_1)$ and $\hat{x}_t = x'_3$). It is clear that $\hat{x}_{t-1} = x'_2$ and $\hat{x}_{t+1} = x'_3\overline{\omega_1}$. Let $q_3 = \hat{q}_{t-1}$ and $q_4 = \hat{q}_t$. If the automaton $\mathfrak{A}_{q_4}$ is not a snakelike σ_0^{-1}-walker, then Theorem 3.4 implies the existence of an $(\overline{\omega_1}, \sigma_0^{-1})$-incomplete pre-absorbing labyrinth $(L_3, L_4; y_1; \alpha_2)$ for $\mathfrak{A}_{q_4}$, where $L_3 = \dashv *\alpha_2| \vdash$ and $L_4 = \langle \omega_0 \rangle$. Let $\tilde{L}_2 = L_0L_1L_2L_3L_4$ and $L'' = \mathrm{Cross}(\tilde{L}_2, \{x_i \mid 0 \leqslant i \leqslant 5\})$, where $x_i = (\tilde{L}_2; i)$ for every $0 \leqslant i \leqslant 5$. It is obvious again that $\omega_3 = \mathrm{dr}(\pi(\mathfrak{A}_{q_1}; L''), \{x_5\}) = \mathrm{dr}_0(\pi(\mathfrak{A}_{q_1}; L''), \{x_5\})$ exists and $\omega_3 \neq \overline{\omega}_0$. Now, if $\omega_3 \in \{\omega_0, \omega_1\}$, then the 4-tuple $(L_0, L_1L_2L_3, L_4; y_0)$ is a pre-absorbing labyrinth for $\mathfrak{A}_{q_1}$, and if $\omega_3 = \overline{\omega_1}$, then the 4-tuple $(L_2, L_3, L_4; y_1)$ is a pre-absorbing labyrinth for $\mathfrak{A}_{q_3}$. From Theorem 3.4 we get the contradiction and, consequently, we have to suppose that $\mathfrak{A}_{q_4}$ is a σ_0^{-1}-walker. Put that $\beta_\omega := \alpha_0(*\alpha_1*)\overline{\omega_1}$.

It is clear that there exists a simple word γ_1 [γ_2] over $\mathfrak{D}$ such that $\alpha_1 = \beta_{\mathbf{e}}\gamma_1(\beta_{\mathbf{w}})^{-1}$ [$\alpha_2 = \beta_{\mathbf{n}}\gamma_1(\beta_{\mathbf{s}})^{-1}$] is also a simple word over $\mathfrak{D}$, and $L_1 = \langle \alpha_1 \rangle$ [$L_2 = \langle \alpha_2 \rangle$] is an **e**-labyrinth [**n**-labyrinth]. Now from Proposition 3.8 we get that the pair (L_1, L_2) orients $\mathfrak{A}_{q_0}$. $\square$

By way of illustration, let us prove an assertion which is an analogy for the plane labyrinths of the main theorem from [**2**].

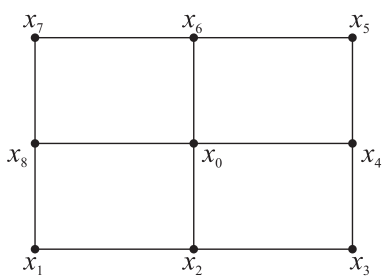


FIGURE 5.

THEOREM 3.8. *For every initial automaton there exists a regular trap.*

PROOF. Let $\mathfrak{A}_{q_0}$ be an initial automaton, and let $\mathfrak{A}$ be the corresponding non-initial automaton. Theorem 3.3 implies that there exists an irreducible or trivial automaton $\mathfrak{A}'$ and a mapping $f: Q_{\mathfrak{A}} \to Q_{\mathfrak{A}'}$ such that for every labyrinth L' there exists a labyrinth L such that $(\mathfrak{A}', L') \leqslant_f (\mathfrak{A}, L)$. Let $(K'; x_0)$ be the labyrinth given in Fig. 5, and let K be a labyrinth such that $(\mathfrak{A}', K') \leqslant_f (\mathfrak{A}, K)$. Also, let g be the corresponding mapping from the definition of $\leqslant_f$ for the pairs $(\mathfrak{A}, K)$ and $(\mathfrak{A}', K')$. Consider the automaton $\mathfrak{A}'_{f(q_0)}$. Now if $\mathfrak{A}'$ is trivial, then $(K'; x_0)$ is a regular trap for $\mathfrak{A}'_{f(q_0)}$ (take x_4 as an exit) and, consequently, $(K; g(x_0))$ is a regular trap for $\mathfrak{A}_{q_0}$. If $\mathfrak{A}'$ is irreducible, then by Theorem 3.7, there exist a snakelike $\mathbf{e}$-labyrinth L_1 and a snakelike $\mathbf{n}$-labyrinth L_2 such that the pair (L_1, L_2) orients $\mathfrak{A}'_{f(q_0)}$. Consider the labyrinth $M' = \mathrm{Cross}(\Delta(K'; L_1, L_2))$ and note that $\Delta(K'; L_1, L_2) \in \mathcal{L}_{\mathrm{plwl}}$. Without loss of generality take that $\psi_{\mathfrak{A}'}(f(q_0), \mathfrak{D}) = \mathbf{n}$. But then for a $\sigma_0 \in \{\sigma_{\mathrm{l}}, \sigma_{\mathrm{r}}\}$ the automaton $\mathfrak{A}'_{q_1}$, where $q_1 = q(\mathfrak{A}'_{f(q_0)}; L_1, L_2)$, is a σ_0-walker with the guided vector $\mathbf{n}$. So, $(M'; x_0, x_2\mathbf{s})$ is a trap for $\mathfrak{A}'_{f(q_0)}$, and, because of Proposition 2.2, it is a regular trap for $\mathfrak{A}'_{f(q_0)}$. Now, since $\mathfrak{A}' \leqslant \mathfrak{A}$, it follows that there exists a regular trap for $\mathfrak{A}_{q_0}$. $\square$

By carefully analyzing the proof of Theorem 3.8 and the proofs of all the theorems cited in the proof, we come to the conclusion that for every initial automaton it is possible to construct a regular trap in an effective way.

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Faculty of Project and Innovation Management
EDUCONS University
Belgrade
Serbia
gkilibar@gmail.com

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