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# RULED THREE-DIMENSIONAL CR SUBMANIFOLDS OF THE SPHERE $\mathbf{S}^6(1)$

**Miroslava Antić**

Abstract. We investigate proper, three-dimensional CR submanifolds of the nearly Kähler sphere $\mathbf{S}^6(1)$ ruled by totally geodesic spheres $\mathbf{S}^2(1)$, and classify them by using a sphere curve and a vector field along that curve.

## 1. Introduction

It is well known that by using multiplication of the octonions $\mathcal{O}$ and identifying the space of $\operatorname{Im}\mathcal{O}$ with $\mathbb{R}^7$, it is possible to introduce a vector cross product $\times$ in the space $\mathbb{R}^7$. This cross product induces an almost complex structure $J$ on the standard unit sphere $\mathbf{S}^6(1)$ in $\mathbb{R}^7$ which is Hermitian and almost complex, and moreover gives a nearly Kähler structure to $\mathbf{S}^6(1)$.

Recall that a submanifold $M$ of a manifold with an almost complex structure $J$ is also called almost complex, if its tangent bundle is invariant for $J$, i.e., $JT_pM \subset T_pM$, $p \in M$. If $JT_pM \subset T_p^{\perp}M$, $p \in M$, where $T^{\perp}M$ is a normal bundle of the submanifold, $M$ is called a totally real submanifold. One of the generalizations of almost complex and totally real submanifolds are CR submanifolds. By the definition of Bejancu [**2**], a submanifold $M$ is called a CR submanifold if there exists on $M$ a differentiable almost complex distribution $U$ such that its orthogonal complement $U^{\perp} \subset TM$ is a totally real distribution. If a CR submanifold is neither almost complex, nor totally real, it is a proper CR submanifold. Due to the dimension restrictions it is clear that a proper CR submanifold of the sphere $\mathbf{S}^6(1)$ can be of dimensions three, four and five. All hypersurfaces of the sphere are trivially CR so the focus of the investigation is mostly on those of dimension three and four. Here, we deal with the three dimensional case. Such CR submanifolds have been previously studied amongst others by K. Mashimo, H. Hashimoto, K. Sekigawa, M. Djorić and L. Vrancken. Particularly in [**7**] and [**6**] one of the first known families of the three dimensional minimal CR submanifolds was introduced, and in [**3**] was obtained the classification of the minimal CR submanifolds which

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satisfy Chen's basic equality. We also recall that the submanifold $M$ of a Riemannian manifold $(\widetilde{M}, g)$ is said to be ruled, if it admits a foliation with leaves that are totally geodesically immersed into $(\widetilde{M}, g)$. Then, trivially, the second fundamental form vanishes on the distribution corresponding to the foliation. Such distribution is said to be totally geodesic.

In [**1**] it was shown that for a three dimensional CR submanifold of $\mathbf{S}^6(1)$ it is equivalent

(1) the CR submanifold is minimal and contained in a totally geodesic hypersphere
(2) the CR submanifold is $U$ and $U^\perp$ totally geodesic

and that examples of [**6**] and [**3**] are of this type. Moreover, there it was shown that such a submanifold is locally congruent to the immersion

$$\begin{aligned} F(s, y_1, y_2, y_3) = y_1(\cos(\mu_1 s)e_1 + \sin(\mu_1 s)e_5) + y_2(\cos(\mu_2 s)e_2 \\ + \sin(\mu_2 s)e_6) + y_3(\cos(\mu_3 s)e_3 + \sin(\mu_3 s)e_7), \end{aligned}$$
$$\mu_1 + \mu_2 + \mu_3 = 0, \quad \mu_1^2 + \mu_2^2 + \mu_3^2 \neq 0, \tag{1.1}$$

where $e_1, \dots, e_7$ is a standard $G_2$ basis of the space $\mathbb{R}^7$ and $y_1^2 + y_2^2 + y_3^2 = 1$.

Note that these submanifolds are also ruled by totally geodesic spheres $\mathbf{S}^2(1)$. Here, we prove

Theorem 1.1. *Let $M$ be a proper three dimensional CR submanifold of $\mathbf{S}^6(1)$ ruled by $\mathbf{S}^2(1)$. If $|\cos\phi|$ is the length of the projection of the unit normal to the leaf of ruling at a point, on the almost complex distribution, then $\phi$ is constant. Moreover $M$ is locally congruent to the immersion:*

a) *for* $\cos\phi \neq 0$

$$\begin{aligned} p(x_1, x_2, x_3) &= \sin(x_2 + h)\gamma \times \gamma' \\ &\quad + \cos(x_2 + h)(\cos x_1 \sigma + \sin x_1(\cos\phi\gamma' - \sin\phi\gamma \times \gamma') \times \sigma), \end{aligned}$$

*where $\gamma$ is a curve in the sphere $\mathbf{S}^6(1)$, with arc length parameter $x_3$, such that $\langle \gamma \times \gamma', \gamma'' \rangle = 0$, $\sigma$ is a curve in $\mathbf{S}^6(1)$ parameterized by $x_3$ orthogonal to $\gamma$, $\gamma'$, $\gamma \times \gamma'$ such that*

$$\langle \sigma', \gamma \times \gamma' \rangle = \langle \sigma \times \sigma', \gamma \rangle = 0, \quad \langle \sigma \times \sigma', \sin\phi\gamma' + \cos\phi\gamma \times \gamma' \rangle = \tfrac{1}{2}\cos\phi,$$
$$\langle \sigma, \cos\phi(\gamma \times \gamma') \times \gamma'' + \sin\phi\gamma' \times \gamma'' \rangle = 0,$$

*and $h$ is a function of $x_3$ such that $\cos(x_2 + h) > 0$.*

b) *for* $\cos\phi = 0$

$$f(x_1, x_2, x_3) = \cos x_1 \cos x_2 \gamma + \sin x_1 \cos x_2 A_2 + \sin x_2 \gamma \times A_2,$$

*where $\gamma$ is a non constant curve in $\mathbf{S}^6(1)$ parameterized by $x_3$ and $A_2$ a vector field along $\gamma$ orthogonal to $\gamma, \gamma'$ and $\gamma \times \gamma'$.*

Remark 1.1. Note that immersion (1.1) is of the second type. At least one of the $\mu_i$ is different from zero, so we can assume that $\mu_1 \neq 0$. Then we can parameterise the sphere so that $y_1 = \cos x_1 \cos x_2$, $y_2 = \sin x_1 \cos x_2$, $y_3 = \sin x_2$ and take $\gamma$ to be the curve $s \mapsto \cos(\mu_1 s)e_1+\sin(\mu_1 s)e_5$ with $A_2 = \cos(\mu_2 s)e_2+\sin(\mu_2 s)e_6$ to obtain the immersion (1.1).

Remark 1.2. In Lemma 2.2 we prove that a three dimensional, totally real submanifold of $\mathbf{S}^6(1)$ ruled by $\mathbf{S}^2(1)$ is totally geodesic.

## 2. Preliminaries

Here, we give a short exposition of how the standard nearly Kähler structure on $\mathbf{S}^6(1)$ arises from the multiplication of the octonions $\mathcal{O}$.

A vector cross product $\times$ of the purely imaginary octonions $\operatorname{Im}\mathcal{O} = \mathbb{R}^7$ is given by $u \times v = \frac{1}{2}(uv - vu)$. This cross product has many properties similar to those of the cross product in the space $\mathbb{R}^3$. In particular, if $\langle , \rangle$ denotes the standard inner product of the space $\mathbb{R}^7$ we have that, see [**5**],

$$u \times (v \times w) + (u \times v) \times w = 2\langle u, w\rangle v - \langle u, v\rangle w - \langle w, v\rangle u, \tag{2.1}$$

$$\langle u \times v, u \times w\rangle = \langle u, u\rangle\langle v, w\rangle - \langle u, v\rangle\langle u, w\rangle. \tag{2.2}$$

An ordered orthonormal basis, respectively moving frame $e_1, \dots, e_7$ is said to be a $G_2$-basis, respectively frame, if

$$e_3 = e_1 \times e_2, \quad e_5 = e_1 \times e_4, \quad e_6 = e_2 \times e_4, \quad e_7 = e_3 \times e_4.$$

The unit orthogonal vector fields $e_1$ and $e_2$ and further $e_4$ orthogonal to $e_1, e_2$ and $e_1 \times e_2$ determine a $G_2$-frame uniquely. We also give the multiplication table for the cross product.

| $\times$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $e_3$ | $-e_2$ | $e_5$ | $-e_4$ | $-e_7$ | $e_6$ |
| $e_2$ | $-e_3$ | $0$ | $e_1$ | $e_6$ | $e_7$ | $-e_4$ | $-e_5$ |
| $e_3$ | $e_2$ | $-e_1$ | $0$ | $e_7$ | $-e_6$ | $e_5$ | $-e_4$ |
| $e_4$ | $-e_5$ | $-e_6$ | $-e_7$ | $0$ | $e_1$ | $e_2$ | $e_3$ |
| $e_5$ | $e_4$ | $-e_7$ | $e_6$ | $-e_1$ | $0$ | $-e_3$ | $e_2$ |
| $e_6$ | $e_7$ | $e_4$ | $-e_5$ | $-e_2$ | $e_3$ | $0$ | $-e_1$ |
| $e_7$ | $-e_6$ | $e_5$ | $e_4$ | $-e_3$ | $-e_2$ | $e_1$ | $0$ |

The standard nearly Kähler structure on $\mathbf{S}^6(1)$ is then obtained as follows:

$$JX = p \times X, \quad X \in T_p\mathbf{S}^6(1), \quad p \in \mathbf{S}^6(1).$$

It is clear that $J$ is an orthogonal almost complex structure on $\mathbf{S}^6(1)$. Also, straightforwardly we have that the group $G_2$ of automorphisms of $\mathcal{O}$ is precisely the group of isometries of $\mathbb{R}^7$ preserving the vector cross product and the almost complex structure of the sphere.

If we denote by $\langle\ ,\ \rangle$, $\bar{D}$ and $\widetilde{D}$ metric and Levi Civita connections of $M$ and $\widetilde{M}$, respectively, and by $D^\perp$ the corresponding normal connection of the immersion $M \to \widetilde{M}$ then the formulas of Gauss and Weingarten are given respectively by

$$\widetilde{D}_X Y = \bar{D}_X Y + h(X, Y), \quad \widetilde{D}_X \xi = -A_\xi X + D^\perp_X \xi,$$

where $X$ and $Y$ are vector fields on $M$ and $\xi$ is a normal vector field on $M$, and $h$ and $A$ are the second fundamental form and the shape operator, respectively. The second fundamental form and the shape operator are related by

$$\langle h(X,Y),\xi\rangle = \langle A_\xi X, Y\rangle$$

Let us denote by $\nabla, \widetilde{\nabla}$ and $D$ the Levi–Civita connections of $M, \mathbf{S}^6(1)$ and $\mathbb{R}^7$. Let $h$ and $\tilde{h}$ be the second fundamental forms corresponding to the immersions $M \to \mathbf{S}^6(1)$ and $\mathbf{S}^6(1) \to \mathbb{R}^7$, respectively. If we denote by $p$ the position vector field of the immersion of $M$ into $\mathbb{R}^7$, we have $\tilde{h}(X,Y) = -\langle X,Y\rangle p$, and $D_X p = X$, where $X, Y \in TM$. Further, the Gauss and Codazzi equations imply that for $X, Y \in TM$ and $\xi \in T^\perp M, \xi \in T\mathbf{S}^6(1)$ it holds

$$\begin{aligned} D_X Y &= \widetilde{\nabla}_X Y + \tilde{h}(X,Y) = \nabla_X Y + h(X,Y) - \langle X,Y\rangle p,\\ D_X \xi &= \widetilde{\nabla}_X \xi + \tilde{h}(X,\xi) = \widetilde{\nabla}_X \xi - \langle X,\xi\rangle p = -A_\xi X + \nabla^\perp_X \xi, \end{aligned}$$

where $\nabla^\perp$ denotes the normal connection corresponding to the immersion of $M$ into $\mathbf{S}^6(1)$.

Also, if we denote

$$(\nabla h)(X,Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

for $X, Y, Z \in T(M)$, then the Gauss, Codazzi and Ricci equations yield

$$\begin{aligned} R(X,Y,Z,W) &= \langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\\ &\qquad + \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle,\\ (\nabla h)(X,Y,Z) &= (\nabla h)(Y,X,Z),\\ \langle R^\perp(X,Y)\xi,\mu\rangle &= \langle [A_\xi, A_\mu]X, Y\rangle. \end{aligned}$$

Also, the following lemma holds straightforwardly.

Lemma 2.1. $D_X(Y \times Z) = D_X Y \times Z + Y \times D_X Z$.

Lemma 2.2. *Let $M$ be a totally real, three dimensional submanifold of* $\mathbf{S}^6(1)$*, ruled by totally geodesic* $\mathbf{S}^2(1)$*. Then $M$ is locally congruent to a totally geodesic sphere* $\mathbf{S}^3(1)$*.*

Proof. Assume that the two dimensional totally geodesic distribution $\mathcal{D}$ is spanned by unit and orthogonal vector fields $E_1, E_2$. Then we have that the second fundamental form vanishes on $\mathcal{D}$. Denote by $E_3$ the unit, tangent vector field orthogonal to $\mathcal{D}$. Then the vector fields $JE_1, JE_2, JE_3$ span the normal bundle. Since every three dimensional, totally real submanifold of $\mathbf{S}^6(1)$ is minimal, see [**4**], we have that $h(E_3, E_3) = 0$. Note that, for tangent vector fields $X, Y, Z$ we have

$$\langle h(X,Y), JZ\rangle = \langle A_{JZ}X, Y\rangle = -\langle Jh(Z,X), Y\rangle = \langle h(Z,X), Y\rangle$$

implying that the form $\langle h(X,Y), JZ\rangle$ is symmetric in all three components. Therefore, $h$ vanishes identically and $M$ is three dimensional, totally geodesic submanifold of $\mathbf{S}^6(1)$. $\square$

## 3. Proof of the main theorem

Here we are dealing with proper, three dimensional, CR submanifolds of $\mathbf{S}^6(1)$. Since the almost complex distribution is even dimensional, and nontrivial, it follows that $\dim U = 2$ and then $\dim U^\perp = 1$. We present one of the convenient moving frames to work with and the relations between the connection coefficients in it, for details see [**1, 3**].

We denote by $p$ the position vector field of the submanifold. Let then $E_1$ and $E_2 = p \times E_1 = JE_1$ be the unit vector fields which span the almost complex distribution, and $E_3$ the unit vector field which spans the totally real distribution. Straightforwardly, the normal vector fields are obtained by $E_4 = JE_3, E_5 = E_1 \times E_3$ and $E_6 = E_2 \times E_3$. Note, that the choice of $E_3$ is unique up to a sign while we have a freedom of rotation in the almost complex distibution, which further reflects to a rotation in the $\mathrm{Span}(E_5, E_6)$.

We denote

$$\omega_{ij}^k = \langle D_{E_i} E_j, E_k \rangle, \quad h_{ij}^k = \langle D_{E_i} E_j, E_{k+3} \rangle, \quad \beta_{ij}^k = \langle D_{E_i} E_{j+3}, E_{k+3} \rangle,$$

for $1 \leqslant i, j, k \leqslant 3$. Since the connection is metric and the second fundamental form symmetric, we have

$$\omega_{ij}^k = -\omega_{ik}^j, \quad h_{ij}^k = h_{ik}^j, \quad \beta_{ij}^k = -\beta_{ik}^j.$$

By taking in Lemma 2.1 $X \in \{E_1, E_2, E_3\}$ and $Y, Z \in \{p, E_1, \dots, E_6\}$, we get the following lemma, (see [**1**]).

LEMMA 3.1. *For the previously defined coefficients the following relations hold*

$$\begin{gathered}
\beta_{11}^3 = -h_{13}^2, \quad \beta_{11}^2 = 1 + h_{13}^3, \quad h_{11}^1 = -\omega_{12}^3, \quad h_{12}^1 = \omega_{11}^3, \quad \beta_{21}^3 = 1 - h_{23}^2, \\
\beta_{21}^2 = h_{23}^3, \quad h_{22}^1 = \omega_{21}^3, \quad \omega_{22}^3 = -\omega_{11}^3, \quad \beta_{31}^2 = h_{33}^3, \quad \beta_{31}^3 = -h_{33}^2, \\
h_{23}^1 = \omega_{31}^3, \quad h_{13}^1 = -\omega_{32}^3, \quad h_{11}^3 = h_{12}^2, \quad h_{11}^2 = -h_{12}^3, \quad h_{22}^2 = h_{12}^3, \\
h_{22}^3 = -h_{12}^2, \quad h_{23}^2 = h_{13}^3 - 1, \quad h_{23}^3 = -h_{13}^2, \quad \beta_{12}^3 = \omega_{11}^2 - \omega_{32}^3, \\
\beta_{22}^3 = \omega_{21}^2 + \omega_{31}^3, \quad \beta_{32}^3 = \omega_{31}^2 + h_{33}^1.
\end{gathered}$$

Now we take that $M$ is ruled by $\mathbf{S}^2(1)$. Then the complementary foliation is spanned by a unit vector field $W$ which can have non vanishing components both in the almost complex and the totally real distribution. Since we still have a freedom of rotating the vector fields $E_1$ and $E_2$ we can assume that the orthogonal projection of $W$ to $U$ is orthogonal to $E_2$, i.e., $W = \cos\phi E_1 + \sin\phi E_3$, for some differential function $\phi$. Then the vector fields $V = -\sin\phi E_1 + \cos\phi E_3$ and $E_2$ span the totally geodesic distribution. Further the inner products of $D_V V, D_V E_2, D_{E_2} E_2$ with $W, E_4, E_5$ and $E_6$ vanish. Straightforwardly this yields the following lemma.

Lemma 3.2. *Let $M$ be a proper, ruled, three-dimensional CR submanifold of $\mathbf{S}^6(1)$. Then the following relations hold*

$$h_{11}^3 = 0, \quad h_{12}^3 = 0, \quad \omega_{21}^3 = 0, \quad E_2(\phi) = 0, \quad -\omega_{21}^2 \cos\phi - h_{12}^1 \sin\phi = 0,$$
$$h_{23}^1 \cos\phi - h_{12}^1 \sin\phi = 0, \quad (-1 + h_{13}^3)\cos\phi = 0, \quad h_{13}^2 \cos\phi = 0,$$
$$-\cos\phi E_3(\phi) + \sin\phi E_1(\phi) = 0, \quad h_{33}^1 \cos^2\phi - \omega_{12}^3 \sin^2\phi + \omega_{32}^3 \sin(2\phi) = 0,$$
$$\cos\phi(h_{33}^2 \cos\phi - 2h_{13}^2 \sin\phi) = 0, \quad \cos\phi(h_{33}^3 \cos\phi - 2h_{13}^3 \sin\phi) = 0.$$

Note, also, that the Codazzi equation $0 = R(E_1, E_2, E_1, E_5) = 2(h_{12}^1 - \omega_{12}^3 h_{13}^2)$ yields $h_{12}^1 = \omega_{12}^3 h_{13}^2$. However, now we have to consider separately cases $\cos\phi \neq 0$ and $\cos\phi = 0$.

**3.1. Case $\cos\phi \neq 0$.** By taking $\cos\phi \neq 0$, the relations from Lemma 3.2 straightforwardly reduce to

$$\omega_{21}^2 = -h_{12}^1 \tan\phi, \quad h_{23}^1 = h_{12}^1 \tan\phi, \quad h_{13}^2 = 0, \quad h_{13}^3 = 1, \quad h_{33}^2 = 0,$$
$$h_{33}^3 = 2\tan\phi, \quad h_{33}^1 = \tan\phi(-2\omega_{32}^3 + \omega_{12}^3 \tan\phi), \quad E_3(\phi) = \tan\phi E_1(\phi).$$

Lemma 3.3. *We have*

$$\omega_{32}^3 = -\omega_{11}^2 + 2\omega_{12}^3 \tan\phi, \quad E_2(\omega_{11}^2) = 1 + (\omega_{11}^2)^2 - \omega_{12}^3 \omega_{31}^2,$$
$$E_2(\omega_{12}^3) = -2\omega_{12}^3(-\omega_{11}^2 + \omega_{12}^3 \tan\phi), \quad E_1(\phi) = 0,$$
$$\omega_{31}^2 = \omega_{12}^3 \tan^2\phi, \quad E_3(\omega_{11}^2) = E_1(\omega_{12}^3)\tan^2\phi,$$
$$E_1(\omega_{11}^2) = 2E_1(\omega_{12}^3)\tan^2\phi - E_3(\omega_{12}^3).$$

Proof. Direct computation gives the following equations of Codazzi and Gauss, from which we derive the proclaim.

$$R(E_1, E_2, E_3, E_6) = \omega_{11}^2 + \omega_{32}^3 - 2\omega_{12}^3 \tan\phi = 0,$$
$$R(E_1, E_2, E_1, E_2) = 1 + (\omega_{11}^2)^2 - \omega_{12}^3 \omega_{31}^2 - E_2(\omega_{11}^2) = 0,$$
$$R(E_1, E_2, E_1, E_4) = E_2(\omega_{12}^3) + 2\omega_{12}^3(-\omega_{11}^2 + \omega_{12}^3 \tan\phi) = 0,$$
$$R(E_1, E_3, E_3, E_5) = \omega_{31}^2 - \omega_{12}^3 \tan^2\phi = 0,$$
$$R(E_1, E_3, E_3, E_6) = 2E_1(\phi)\sec^2\phi = 0,$$
$$R(E_1, E_3, E_1, E_2) = -E_3(\omega_{11}^2) + E_1(\omega_{12}^3)\tan^2\phi = 0,$$
$$R(E_1, E_3, E_1, E_4) = E_1(\omega_{11}^2) + E_3(\omega_{12}^3) - 2E_1(\omega_{12}^3)\tan\phi = 0.$$

Direct computation shows that the other Gauss, Codazzi and Ricci equations do not yield any new relations. □

Note that we now have $\phi = \text{const}$. Also we have

$$h(E_1, E_1) = -\omega_{12}^3 E_4, \quad h(E_1, E_2) = 0, \quad h(E_2, E_2) = 0,$$
$$h(E_3, E_3) = \tan\phi(2\omega_{11}^2 - 3\omega_{12}^3 \tan\phi E_4 + 2E_6)$$

so in a general case distributions $U$ and $U^\perp$ are not totally geodesic. We also note that $[\rho V, E_2]=0$, for $\rho=(1+(\omega_{11}^2-\omega_{12}^3\tan\phi)^2)^{-1/2}$. Moreover, we have

$$\rho V(\omega_{11}^2)=\frac{\sin\phi(E_3(\omega_{12}^3)-E_1(\omega_{12}^3)\tan\phi)}{\sqrt{1+(\omega_{11}^2-\omega_{12}^3\tan\phi)^2}},$$

$$\rho V(\omega_{12}^3)=\frac{\cos\phi E_3(\omega_{12}^3)-\sin\phi E_1(\omega_{12}^3)}{\sqrt{1+(\omega_{11}^2-\omega_{12}^3\tan\phi)^2}},$$

$$\rho V(\omega_{11}^2-\omega_{12}^3\tan\phi)=0,\quad E_2(\omega_{11}^2-\omega_{12}^3\tan\phi)=1+(\omega_{11}^2-\omega_{12}^3\tan\phi)^2. \tag{3.1}$$

We give further a construction of the immersion. Recall that $[\rho V, E_2]=0$. Therefore, there exist local coordinates $x_1,x_2,x_3$ in the neighborhood of a point $p\in M$ such that $\rho V=\partial_{x_1}, E_2=\partial_{x_2}$. Here, the choice of the coordinate $x_3$ is essentially arbitrary. Moreover, (3.1) implies that

$$\partial_{x_1}(\omega_{11}^2-\omega_{12}^3\tan\phi)=0,\quad \partial_{x_2}(\omega_{11}^2-\omega_{12}^3\tan\phi)=1+(\omega_{11}^2-\omega_{12}^3\tan\phi)^2.$$

Hence, we have $\omega_{11}^2-\omega_{12}^3\tan\phi=\tan(x_2+h(x_3))$, where the differentiable function $h$ is independent of $x_1,x_2$. Also, we have a freedom of choosing $h$ up to an addend $\pi$, so we can assume that $\cos(x_2+h)>0$. Further we have $E_2(\omega_{12}^3)=\partial_{x_2}\omega_{12}^3=2\omega_{12}^3\tan(x_2+h)$. Therefore there exists a differentiable function $d$ depending on $x_1$ and $x_3$ such that

$$\omega_{12}^3=d\sec^2(x_2+h),\quad \text{and further}\quad \omega_{11}^2=d\sec^2(x_2+h)\tan\phi+\tan(x_2+h).$$

Now, straightforward computation shows that

$$D_{\partial_{x_1}}\partial_{x_1}=\partial^2_{x_1x_1}p=-\cos^2(x_2+h)p+\cos(x_2+h)\sin(x_2+h)\partial_{x_2}p,$$

$$D_{\partial_{x_1}}\partial_{x_2}=\partial^2_{x_1x_2}p=-\tan(x_2+h)\partial_{x_1}p,$$

$$D_{\partial_{x_2}}\partial_{x_2}=\partial^2_{x_2x_2}p=-p. \tag{3.2}$$

The last equation yields $p=\cos x_2 A+\sin x_2 B$ for some vector fields $A$ and $B$ depending on $x_1$ and $x_3$. Since $p$ is unit, straightforwardly we get that $A$ and $B$ are unit and mutually orthogonal. Then, the first equation of (3.2) reduces to

$$\cos x_2(\partial^2_{x_1x_1}A+\cos h(A\cos h-B\sin h))$$
$$+\sin x_2(\partial^2_{x_1x_1}B-\sin h(A\cos h-B\sin h))=0,$$

which further implies

$$\partial^2_{x_1x_1}A+\cos h(A\cos h-B\sin h)=0,$$

$$\partial^2_{x_1x_1}B-\sin h(A\cos h-B\sin h)=0.$$

Further, it follows $\partial^2_{x_1x_1}(A\cos h-B\sin h)=-(A\cos h-B\sin h)$, so we can put

$$A\cos h-B\sin h=\cos x_1 P+\sin x_1 Q, \tag{3.3}$$

for some vector fields $P$ and $Q$ depending only on $x_3$. Similarly as before $P$ and $Q$ are unit and orthogonal. Finally, the second equation of (3.2) simplifies to $\sec(h+x_2)(\cos h\,\partial_{x_1}B+\sin h\,\partial_{x_1}A)=0$. Integration over $x_1$ then gives $\sin hA+\cos hB=G$, where the unit vector field $G$ depends only on $x_3$, and is orthogonal to vector field (3.3), for arbitrary $x_1$. Therefore $G$ is orthogonal to both $P$ and $Q$. Using the last equation along with (3.3) we get

$$\begin{aligned}A &= \cos h(\cos x_1 P + \sin x_1 Q) + \sin h G,\\ B &= -\sin h(\cos x_1 P + \sin x_1 Q) + \cos h G.\end{aligned}$$

Now, the immersion is given by

$$p(x_1, x_2, x_3) = \cos(x_2 + h)(\cos x_1 P + \sin x_1 Q) + \sin(x_2 + h)G,$$

and the coordinate vector fields by

$$\begin{aligned}\partial_{x_1} p &= \cos(x_2 + h)(-\sin x_1 P + \cos x_1 Q) = \cos(x_2 + h)(-\sin\phi E_1 + \cos\phi E_3),\\ \partial_{x_2} p &= -\sin(x_2 + h)(\cos x_1 P + \sin x_1 Q) + \cos(x_2 + h)G = E_2,\\ \partial_{x_3} p &= \cos(x_2 + h)(\partial_{x_3} h G + \cos x_1 \partial_{x_3} P + \sin x_1 \partial_{x_3} Q)\\ &\quad + \sin(x_2 + h)(\partial_{x_3} G - \partial_{x_3} h(\cos x_1 P + \sin x_1 Q)).\end{aligned}$$

Then $E_1 = E_2 \times p = G \times (\cos x_1 P + \sin x_1 Q)$, and further

$$\begin{aligned}-\sin\phi &= \langle -\sin x_1 P + \cos x_1 Q, E_1\rangle\\ &= \cos^2 x_1 \langle Q, G \times P\rangle - \sin^2 x_1 \langle P, G \times Q\rangle = \langle P \times Q, G\rangle.\end{aligned}$$

It follows that

$$\begin{gathered}\langle G + \sin\phi P \times Q, P \times Q\rangle = 0, \quad G + \sin\phi P \times Q \perp P, Q,\\ \langle G + \sin\phi P \times Q, G + \sin\phi P \times Q\rangle = \cos^2\phi \neq 0.\end{gathered}$$

Since we can assume that $\cos\phi > 0$ (we can change the sign of $W$) the following lemma holds.

Lemma 3.4. *The vector fields $P$, $Q$ and $T = \sec\phi(G + \sin\phi P \times Q)$ determine a $G_2$-frame.*

Therefore, we can denote $e_1 = P$, $e_2 = Q$, $e_4 = T$, and other $e_i$ accordingly to the relations in a $G_2$ frame. Here we have $G = \cos\phi e_4 - \sin\phi e_3$. Moreover, it follows that

$$\begin{aligned}E_3 &= \sec\phi(\cos x_1 Q - \sin x_1 P + \sin\phi E_1)\\ &= -\sin x_1(\cos\phi e_1 + \sin\phi e_5) + \cos x_1(\cos\phi e_2 - \sin\phi e_2),\\ E_4 &= p \times E_3 = \cos(h + x_2)(\cos\phi e_3 + \sin\phi e_4) + \sin(h + x_2)(\sin x_1 e_5 - \cos x_1 e_6),\\ E_5 &= E_1 \times E_3 = e_7,\\ E_6 &= -\sin(h + x_2)(\cos\phi e_3 + \sin\phi e_4) + \cos(h + x_2)(\sin x_1 e_5 - \cos x_1 e_6).\end{aligned}$$

Denote by $z_{ij} = \langle \partial_{x_3} e_i, e_j\rangle$, the differentiable functions of $x_3$. Since the connection is metrical, we have $z_{ji} = -z_{ij}$.

Lemma 3.5. *It holds*

$$\begin{aligned}&z_{34} = z_{16} - z_{25}, \quad z_{35} = z_{17} + z_{24}, \quad z_{36} = -z_{14} + z_{27}, \quad z_{37} = -z_{15} - z_{26},\\ &z_{56} = z_{12} + z_{47}, \quad z_{57} = z_{13} - z_{46}, \quad z_{67} = z_{23} + z_{45}.\end{aligned}$$

PROOF. From Lemma 2.1 we have $\partial_{x_3}(e_i \times e_j) = \partial_{x_3} e_i \times e_j + e_i \times \partial_{x_3} e_j$. Taking $i, j \in \{1, \dots, 7\}$, we obtain the assertion. $\square$

The previous lemma obviously holds for an arbitrary $G_2$-frame. Let us now consider this particular one.

LEMMA 3.6. *Coefficients* $z_{ij}$ *satisfy*

$$z_{26} = z_{15} \neq 0, \quad z_{16} = 0, \quad z_{17} = 0, \quad z_{25} = 0, \quad z_{27} = 0,$$
$$z_{46} = -z_{13} - 2z_{14} \tan\phi, \quad z_{45} = z_{23} + 2z_{24} \tan\phi, \quad z_{13} = -z_{14} \tan\phi,$$
$$z_{23} = -z_{24} \tan\phi, \quad z_{47} = -2z_{15} \tan\phi.$$

PROOF. For $p$ to be the CR immersion of the needed type we have to impose the following conditions. The vector field $\partial_{x_3} p$ has to be independent of $\partial_{x_1} p$ and $\partial_{x_2} p$, and orthogonal to the vector fields $E_4$, $E_5$ and $E_6$. We have

$$\begin{aligned} 0 = \langle \partial_{x_3} p, E_5 \rangle &= \cos(h + x_2)(z_{17} \cos x_1 + z_{27} \sin x_1) \\ &\quad + \sin(h + x_2)(z_{47} \cos\phi + (z_{15} + z_{26}) \sin\phi)). \end{aligned}$$

Since $\cos x_1$ and $\sin x_1$ are the only functions of $x_1$ in this relation, we get that $z_{17} = z_{27} = 0$ and $z_{47} \cos\phi + (z_{15} + z_{26}) \sin\phi = 0$. The relation $z_{15} = z_{26}$ and other ones are obtained similarly, taking in the expressions for $\langle \partial_{x_3} p, E_4 \rangle$ and $\langle \partial_{x_3} p, E_6 \rangle$ the coefficients multiplying independent functions of $x_1$. Finally, if we denote by $\partial_{x_3} p^{pr}$ the projection of $\partial_{x_3} p$ to $\mathrm{Span}(\partial_{x_1} p, \partial_{x_2} p)$ we get

$$\partial_{x_3} p - \partial_{x_3} p^{pr} = z_{15} \cos(h + x_2)(\cos x_1 e_5 + \sin x_1 e_6) \neq 0$$

which finishes the proof. $\square$

Note that now it follows that $e_7' = 2z_{15}(e_3 + \tan\phi e_4) \neq 0$. By possible rescaling of the coordinate $x_3$, we can assume that $\|e_7'\| = 1$, i.e., that the sphere curve $\gamma(x_3) = e_7(x_3)$ is parameterized by arc length and that $2z_{15} = \cos\phi$. Then

$$\gamma \times \gamma' = e_7 \times (\cos\phi e_3 + \sin\phi e_4) = -\sin\phi e_3 + \cos\phi e_4 = G,$$

and further

$$e_3 = \cos\phi\gamma' - \sin\phi\gamma \times \gamma', \quad e_4 = \sin\phi\gamma' + \cos\phi\gamma \times \gamma'.$$

Since

$$e_1' = z_{12} e_2 + z_{14}(-\tan\phi e_3 + e_4) + z_{15} e_5 \neq 0, \tag{3.4}$$
$$e_2' = -z_{12} e_1 + z_{24}(-\tan\phi e_3 + e_4) + z_{15} e_6 \tag{3.5}$$

we also have that $\sigma = e_1$ is a nonconstant sphere curve, not necessarily parameterized by its arc length. Since $e_1, e_3$ and $e_4$ determine a $G_2$-frame, we have that $\sigma$ is orthogonal to $\gamma$, $\gamma'$ and $\gamma \times \gamma'$. From (3.4) we deduce that $\sigma'$ is orthogonal to $G = \gamma \times \gamma'$, $e_7 = \gamma$ and $e_6 = e_1 \times e_7 = \sigma \times \gamma$ which is, by (2.2), equivalent to $\langle \sigma' \times \sigma, \gamma \rangle = 0$. Also, (3.4) implies

$$\langle \sigma', \sigma \times e_4 \rangle = \langle \sigma' \times \sigma, \sin\phi\gamma' + \cos\phi\gamma \times \gamma' \rangle = \langle e_1', e_5 \rangle = \tfrac{1}{2} \cos\phi = \mathrm{const}\,.$$

Similarly, (3.5) implies that $(e_3\times\sigma)'$ is orthogonal to $G$ and $e_5=e_1\times e_4$. We note that $\langle (e_3\times e_1)', e_1\times e_4\rangle = \langle e_3'\times e_1, e_1\times e_4\rangle + \langle e_3\times e_1', e_1\times e_4\rangle$. By using (2.2) and (2.1) we get

$$\langle e_3\times e_1', e_1\times e_4\rangle = -\langle e_1', e_3\times (e_1\times e_4)\rangle = \langle e_1', e_6\rangle = \langle e_1'\times e_1,\gamma\rangle = 0.$$

Therefore, we have $0=\langle e_3'\times e_1, e_1\times e_4\rangle = \langle -e_3', e_4\rangle$ and further

$$\begin{aligned}0=\langle e_3', e_4\rangle &= \langle \cos\phi\gamma'' - \sin\phi(\gamma\times\gamma')', \sin\phi\gamma' + \cos\phi(\gamma\times\gamma')\rangle\\ &= \cos^2\phi\langle\gamma'',\gamma\times\gamma'\rangle - \sin^2\phi\langle\gamma\times\gamma'',\gamma'\rangle = -\langle\gamma'',\gamma\times\gamma'\rangle.\end{aligned}$$

Here we use the fact that the sphere curves $\gamma'$ and $\gamma\times\gamma'$ are orthogonal to their tangent vector fields $\gamma''$ and $(\gamma\times\gamma')'$, respectively.

Let us investigate the orthogonality condition for $(e_3\times\sigma)'$ and $G$. We have

$$\begin{aligned}\langle e_3\times e_1', \gamma\times\gamma'\rangle &= -\langle e_1', e_3\times(\gamma\times\gamma')\rangle\\ &= -\langle e_1', \cos\phi\gamma'\times(\gamma\times\gamma')\rangle = -\cos\phi\langle e_1',\gamma\rangle = 0,\end{aligned}$$

so we are left with

$$\begin{aligned}0 &= \langle e_3'\times e_1, \gamma\times\gamma'\rangle = \langle e_3', e_1\times(\gamma\times\gamma')\rangle = \langle e_1, (\gamma\times\gamma')\times e_3'\rangle\\ &= \langle e_1, \cos\phi(\gamma\times\gamma')\times\gamma'' - \sin\phi(\gamma\times\gamma')\times(\gamma\times\gamma'')\rangle.\end{aligned}\tag{3.6}$$

Since (2.1) implies

$$\begin{aligned}(\gamma\times\gamma')\times(\gamma\times\gamma'') &= \gamma'\times(\gamma\times(\gamma\times\gamma''))\\ &= \gamma'\times(\langle\gamma,\gamma''\rangle\gamma - \langle\gamma,\gamma\rangle\gamma'') = -\gamma'\times(\gamma+\gamma'')\end{aligned}$$

and since $\sigma$ and $\gamma\times\gamma'$ are orthogonal, relation (3.6) becomes

$$\langle\sigma, \cos\phi(\gamma\times\gamma')\times\gamma'' + \sin\phi\gamma'\times\gamma''\rangle = 0.$$

Straightforward computation shows that taking curves $\gamma$ and $\sigma$ that satisfy the listed conditions satisfy the relations of Lemma 3.6 and, moreover that we obtain a CR submanifold of the required form.

**3.2. Case $\cos\phi = 0$.** In this case we have $W=E_3$ and the submanifold is foliated by almost complex spheres. We will now present a construction of the submanifold following the method given in [**1**]. If $p\in M$ is a point of the submanifold, then there exists a $G_2$ basis $e_1,\dots,e_7$ of the space $\mathbb{R}^7$ such that $e_1=p$, $e_4=E_3(p)$ and the tangent space of the totally geodesic leaf at the point $p$ is spanned by $e_2$ and $e_3=p\times e_2$. We can parameterize that leaf by

$$(\cos x_1\cos x_2, \sin x_1\cos x_2, \sin x_2, 0,0,0,0)$$

for $x_1,x_2$ in some neighborhood of $(0,0)$. We denote by $\gamma$ the integral curve for the vector field $E_3$, parameterized by $x_3$ such that $\gamma(0)=p$. If $\gamma(x_3)$ is a point of the curve, then there also exists a $G_2$ transformation $A(x_3)$ mapping, respectively, $p$ into $\gamma(x_3)$, and the vectors $E_1(p)$ and $E_3(p)$ into $E_1(\gamma(x_3))$ and $E_3(\gamma(x_3))$. Note that we have a possibility of choosing the vector field $E_1$ belonging to the almost complex distribution. Now, we have that locally the immersion is given by

$f(x_1,x_2,x_3) = A(x_3)(\cos x_1 \cos x_2, \sin x_1 \cos x_2, \sin x_2)^t$, for a differentiable $G_2$ matrix function $A(x_3)$. Denoting by $A_i$ the columns of $A$, we obtain

$$f(x_1,x_2,x_3) = \cos x_1 \cos x_2 A_1(x_3) + \sin x_1 \cos x_2 A_2(x_3) + \sin x_2 A_3(x_3).$$

Since the point $p$, obtained for $x_1 = x_2 = 0$, is mapped into $\gamma(x_3)$, we have that $A_1 = \gamma$. Similarly $e_4$ is mapped into $\gamma'$, so $e_2$ is mapped into $A_2$ orthogonal to $\gamma, \gamma'$ and $\gamma \times \gamma'$. Moreover we have $A_3 = A_1 \times A_2 = \gamma \times A_2$. Therefore, $\gamma, A_2$ and $\gamma'$ determine a $G_2$ frame.

We have

$$\begin{aligned}\partial_{x_1} f &= -\sin x_1 \cos x_2 A_1 + \cos x_1 \cos x_2 A_2,\\ \partial_{x_2} f &= -\sin x_2 \cos x_1 A_1 - \sin x_2 \sin x_1 A_2 + \cos x_2 A_3,\end{aligned}$$

and further

$$\begin{aligned}\partial^2_{x_1x_1} f &= -\cos x_2(\cos x_1 A_1 + \sin x_1 A_2) = -\cos x_2^2 f + \sin x_2 \cos x_2 \partial_{x_2} f,\\ \partial^2_{x_1x_2} f &= \sin x_2(\sin x_1 A_1 - \cos x_1 A_2) = -\tan x_2 \partial_{x_1} f,\\ \partial^2_{x_2x_2} f &= -p.\end{aligned}$$

Also, straightforwardly we have $f \times \frac{\partial_{x_1} f}{\cos x_2} = \partial_{x_2} f$, so $\partial_{x_1} f$ and $\partial_{x_2} f$ span an almost complex distribution, with integral manifolds being totally geodesic spheres, parameterized by $x_1, x_2$. Moreover this makes $f$ a CR immersion of a required form. Also, note that by reparameterization of the curve $\gamma$ we obtain the same CR submanifold, so the condition that $x_3$ is the arc-length parameter is not necessary. This completes the proof.

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Faculty of Mathematics
University of Belgrade
Belgrade
Serbia
mira@math.rs

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