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L'HÔPITAL'S MONOTONE RULE, GROMOV'S THEOREM, AND OPERATIONS THAT PRESERVE THE MONOTONICITY OF QUOTIENTS

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Abstract. We study several operators T that when applied to both the numerator and denominator of an increasing, or decreasing, function u/v produce another increasing, or decreasing, function $T(u)/T(v)$. We also give new proofs of the monotone form of L'Hôpital's rule and of Gromov's theorem.

1. Introduction

In his book, *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes*, published in 1696 and considered to be the first textbook on calculus, Guillaume de l'Hôpital included the well known rule to compute limits presently called L'Hôpital's rule[1]. In its basic form it says the following.

Theorem 1.1. *Suppose f, g are continuous functions defined in*[2] (a, b), *differentiable in* (a, b) *and such that* $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$, *or such that* $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = \pm\infty$. *Assume* $g'(x) \neq 0$ *for all* $x \in (a, b)$ *Then, whenever* $\lim_{x\to a^+} f'(x)/g'(x) = L$, *we also have that* $\lim_{x\to a^+} f(x)/g(x) = L$.

We would like to point out the interesting structure of this result. Indeed, one applies the *same* operation to both the numerator and the denominator of a quotient, and, when some extra conditions are satisfied, then a certain characteristic of the fraction (having a limit equal to L, in this case) is preserved. This is perhaps better appreciated if we rewrite L'Hôpital's rule in an integral form[3], which reads as follows.

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[1]It must be said that the rule was probably discovered by Johann Bernoulli.

[2]We shall assume that $-\infty < a < b < \infty$ in the whole article, but it is clear that many results will also hold if $a = -\infty$ or $b = \infty$.

[3]The integral form is actually a little stronger than Theorem 1.1. Similarly, Gromov's theorem (Theorem 1.3) is a little stronger than the monotone form of L'Hôpital's rule (Theorem 1.4). These matters are discussed in Section 4.

THEOREM 1.2. *Suppose u and v are locally Lebesgue integrable in $(a, b]$, with v strictly positive. Suppose that*

$$\lim_{x\to a^+} \frac{u(x)}{v(x)} = L.$$

Then if they are both integrable at a then

$$\lim_{x\to a^+} \frac{\int_a^x u(t)\,dt}{\int_a^x v(t)\,dt} = L,$$

while if both integrals diverge to $\pm\infty$ at a then

$$\lim_{x\to a^+} \frac{\int_x^b u(t)\,dt}{\int_x^b v(t)\,dt} = L.$$

In recent years there has been an interest in studying whether the procedure employed in L'Hôpital's rule, namely to apply the same operation to both the numerator and the denominator of a fraction, preserves the *monotonicity* of the fraction. This interest has been motivated by the need of such results in diverse areas of mathematics such as differential geometry [**7**] or conformal mappings [**1**], but, of course, it has become clear that the results have an interest of their own [**2, 14, 15**], while applications in other areas have been found [**2, 3**]. The first such result is Gromov's theorem [**7**, p. 42], which is the monotonic version of Theorem 1.2.

THEOREM 1.3 (Gromov's Theorem). *Let u and v be Lebesgue integrable functions on $[a, b]$, with v strictly positive. Suppose that u/v is increasing (decreasing) on $[a, b]$. Then the function of x*

$$\frac{\int_a^x u(t)\,dt}{\int_a^x v(t)\,dt},$$

is also increasing (decreasing) on $[a, b]$.

There is also a monotonic version of Theorem 1.1, called the monotone form of L'Hôpital's rule [**2, 14, 15**].

THEOREM 1.4 (Monotone form of L'Hôpital's rule). *Let f, g be continuous functions defined in $[a, b]$, differentiable in (a, b). Suppose $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$, and assume that $g'(x) \neq 0$ for all $x \in (a, b)$. If f'/g' is increasing (decreasing) on (a, b) then so is f/g.*

The main aim of this article is to give new, alternative proofs of these two results. Indeed, Section 2 contains a new, rather direct proof of the monotone form of L'Hôpital's rule, obtained after a change of variables, while Section 3 offers a simple proof of Gromov's theorem obtained also by a suitable change of variables. We actually give another proof of Theorem 1.4, as well as several extensions, in Section 6 by approximating the integrals with Riemann sums and employing the results of Section 5, where we give results for the preservation of the monotonicity of quotients of *sequences*. Interestingly, there is a discrete version of L'Hôpital's rule, known as the Stolz–Cesàro theorem [**6, 16**]: If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of real numbers, with $\{b_n\}_{n=1}^{\infty}$ strictly monotone and divergent and if

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = L,$$

then we also have that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

Actually in Section 5 we show that if $a_n \in \mathbb{R}$, $b_n > 0$, and $\{a_n/b_n\}_{n=1}^{\infty}$ is increasing then the sequences

$$\left\{ \frac{\sum_{j=1}^{n} a_j}{\sum_{j=1}^{n} b_j} \right\}_{n=1}^{\infty}$$

and, for a fixed m,

$$\left\{ \frac{\sum_{j=n}^{n+m} a_j}{\sum_{j=n}^{n+m} b_j} \right\}_{n=1}^{\infty}$$

are also increasing.

In Section 7 we give a useful extension of Gromov's theorem that allows us to give another proof of the fact [**5, 11**] that if $b_n > 0$, $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing, and the power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge for $|x| < R$, then the function of x

$$\frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n},$$

is likewise increasing in $[0, R)$.

Finally in Section 8, we give an application of Gromov's theorem in the theory of analytic functions, namely we prove that if $0 < p < \infty$ and $M_p(r; f)$ is the integral mean of a function $f(z)$ analytic in $|z| < R$, while $A_p(r; f)$ is the corresponding area mean, then $M_p(r; f)/A_p(r; f)$ is an increasing function of r.

This article is an example of what may become a new form of cooperative mathematical work in the future. Indeed, the second author learned of these ideas through contacts with fellow mathematicians in *Research Gate* and, as a result, wrote another proof of the monotone form of L'Hôpital's rule in his technical report [**12**], and, furthermore, found an application of Gromov's theorem in the theory of analytic functions, which he wrote as another technical report [**13**]. The first author received a communication from *Research Gate* of these technical reports, became interested in these ideas, also found new proofs of several results and a collaborative effort began. The present paper summarizes our ideas, born of this cooperation, in this fascinating area.

REMARK 1.1. Almost all of the results of this article have an increasing version and a decreasing version. *We shall only give the statements and proofs for the increasing case*, but the reader should know that in all results the corresponding decreasing case is also true and the proof is basically the same.

2. Proof of L'Hôpital's monotone rule

We shall now give a proof of the monotone form of L'Hôpital's rule by employing a suitable change of variables.

THEOREM 2.1 (Monotone form of L'Hôpital's rule). *Let f, g be continuous functions defined in $[a, b]$, differentiable in (a, b). Suppose $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$, and assume that $g'(x) \neq 0$ for all $x \in (a, b)$. If f'/g' is increasing on (a, b) then so is f/g.*

PROOF. Since g' never vanishes in (a, b), then Darboux's theorem[4] implies that we should have $g'(x) > 0$ for all x or $g'(x) < 0$ for all x; let us suppose that the first possibility holds, so that g is strictly increasing on $[a, b]$, and denote by h the inverse function. Consider the function $F(y) = f(h(y))$, $y \in [g(a), g(b)]$. It is differentiable in $(g(a), g(b))$ with derivative

$$F'(y) = f'(h(y))h'(y) = \frac{f'(h(y))}{g'(h(y))},$$

and thus F' is an increasing function. Hence F is convex, which implies that for each fixed d, the function $\frac{F(y)-F(d)}{y-d}$, is an increasing function of y. If we now take $y = g(x)$ and $d = g(c)$, we conclude that $\frac{f(x)-f(c)}{g(x)-g(c)}$, is an increasing function of $x \in [a, b]$ for all $c \in [a, b]$, and the result follows by taking $c = a$ or $c = b$, as needed. □

It is interesting to observe that the theorem remains valid if we replace increasing by *strictly* increasing.

3. Proof of Gromov's theorem

We shall now give a proof of Gromov's theorem by also employing a change of variables. We start with the following very simple special case.

LEMMA 3.1. *Let U be defined, positive, and increasing on $[a, b]$. Then the average function,*

$$A_U(x) = \frac{1}{x-a}\int_a^x U(t)\,dt,$$

is likewise positive and increasing on $[a, b]$.

PROOF. Suppose $a < x < y$. Then the change $t = a + \left(\frac{y-a}{x-a}\right)(s-a)$ yields

$$\begin{aligned} A_U(y) = \frac{1}{y-a}\int_a^y U(t)\,dt &= \frac{1}{x-a}\int_a^x \Big(a + \Big(\frac{y-a}{x-a}\Big)(s-a)\Big)ds \\ &\geqslant \frac{1}{x-a}\int_a^x U(s)\,ds = A_U(x), \end{aligned}$$

as required. Observe, since A_U is increasing, then $A_U(a^+)$ is well defined, so that the above inequality holds in $[a, b]$. □

[4] Darboux's theorem states that if the derivative g' exists *everywhere*, then it satisfies the intermediate value property, that is, g' is a Darboux function.

Hence we obtain the ensuing proof of Gromov's theorem.

Theorem 3.1 (Gromov's Theorem). *Let u and v be Lebesgue integrable on $[a, b]$, with v strictly positive, and such that u/v is increasing on $[a, b]$. Then*

$$I(x) = \frac{\int_a^x u(t)\,dt}{\int_a^x v(t)\,dt},$$

is also increasing on $[a, b]$.

Proof. Let us take $h(x) = \int_a^x v(t)\,dt$, so that h is strictly increasing in $[a, b]$, and apply Lemma 3.1 with $U(x) = u(h^{-1}(x))/v(h^{-1}(x))$. Then $A_U(h(x))$ is increasing, but

$$A_U(h(x)) = \frac{1}{h(x)} \int_0^{h(x)} \frac{u(h^{-1}(t))}{v(h^{-1}(t))}\,dt = \frac{\int_a^x u(t)\,dt}{\int_a^x v(t)\,dt} = I(x),$$

since

$$\int_0^{h(x)} \frac{u(h^{-1}(t))}{v(h^{-1}(t))}\,dt = \int_a^x u(t)\,dt. \qquad \square$$

4. Equivalence of the results

It is an interesting question whether Gromov's theorem, Theorem 1.3 is stronger than the monotone form of L'Hôpital's rule, Theorem 1.4. The same question can be asked about Theorems 1.1 and 1.2. We can answer these questions by recalling some facts from integration theory [**9**].

Lemma 4.1. *Let f be continuous in $[a, b]$ such that $f'(t)$ exists for all $t \in (a, b)$. Then f' is integrable in the sense of Denjoy on $[a, b]$ and*

$$\int_a^b f'(t)\,dt = f(b) - f(a). \tag{4.1}$$

We need to employ the integral of Denjoy, not the integral of Lebesgue in Lemma 4.1 because, in general, the derivative f' of a continuous function f, even if it exists at all points, is not Lebesgue integrable. On the other hand, equation (4.1) may not hold if f' exists just almost everywhere, whether one uses the Denjoy or the Lebesgue integral[5]. However, we immediately obtain from Lemma 4.1 the following.

Lemma 4.2. *Let f be continuous in $[a, b]$ such that $f'(t)$ exists for all $t \in (a, b)$. If f' is integrable in the sense of Lebesgue on $[a, b]$, then (4.1) holds in the sense of Lebesgue.*

We also recall the following fact.

Lemma 4.3. *Let f be a Denjoy integrable function in some interval I. If f is positive in I then it is Lebesgue integrable in I.*

[5]This is shown by employing a Cantor type function.

One may employ these lemmas, for instance, to conclude that if a function is increasing in $[a, b]$ and differentiable in (a, b), then f' is Lebesgue integrable and f is absolutely continuous, that is, it satisfies (4.1). We may also use these lemmas to obtain that Theorem 1.3 is stronger than Theorem 1.4.

PROPOSITION 4.1. *Suppose f and g satisfy the following conditions of Theorem* 1.4*, namely, f, g are continuous functions defined in $[a, b]$, differentiable in (a, b), with $f(a) = g(a) = 0$, such that $g'(x) \neq 0$ for all $x \in (a, b)$, and such that f'/g' is increasing on (a, b). Let $u = f'$ and $v = g'$; then these functions satisfy the conditions of Theorem* 1.3*, that is, both are Lebesgue integrable on $[a, b]$, with v strictly positive, u/v increasing on $[a, b]$, and furthermore*

$$f(x) = \int_a^x u(t)\,dt, \quad g(x) = \int_a^x v(t)\,dt. \tag{4.2}$$

PROOF. We just need to prove that u and v are Lebesgue integrable on $[a, b]$, since (4.2) would then follow from Lemma 4.2. Now, that $v = g'$ is Lebesgue integrable in $[a, b]$ follows from Lemma 4.3 because it is Denjoy integrable and always positive or always negative there. The function u is likewise Denjoy integrable, and since u/v is increasing, we have that u has a constant sign in $[a, b]$ or there exists $c \in (a, b)$ such that u has a constant sign in $[a, c]$ and in $[c, b]$; in either case, Lemma 4.3 yields the Lebesgue integrability of u. □

It should also be clear that Gromov's theorem, Theorem 1.3, is not a *direct* consequence of the monotone form of L'Hôpital's rule, Theorem 1.4 since if u and v are Lebesgue integrable, and f and g are given by (4.2), then f' and g' would not exist at all points, but just almost everywhere.

We can also settle another matter at this point. Indeed, in Theorem 1.3, Gromov's theorem, the functions u and v are supposed Lebesgue integrable. What we would get if we used another, stronger integral, such as the Denjoy integral or the distributional integral [**8**]? Well, the proof of Proposition 4.1 also shows that no new result is obtained, because the other conditions of the theorem imply that these functions must be Lebesgue integrable.

A similar argument shows that we also have the following result on the relationship between Theorems 1.1 and 1.2. The Lebesgue integrability of v follows as before, while the Lebesgue integrability of u in a neighborhood of $x = a$ is obtained by the comparison criterion.

PROPOSITION 4.2. *Suppose f and g satisfy the following conditions of Theorem* 1.1*, namely, f, g are continuous functions defined in $[a, b]$, differentiable in (a, b), with $f(a) = g(a) = 0$, such that $g'(x) \neq 0$ for all $x \in (a, b)$, and such that* $\lim_{x\to a^+} f'(x)/g'(x) = L$*. Let $u = f'$ and $v = g'$; then these functions satisfy the conditions of Theorem* 1.2 *in a neighborhood of $x = a$; actually both are Denjoy integrable on $[a, b]$ and Lebesgue integrable on $[a, c]$, for some c, with v strictly positive, and* $\lim_{x\to a^+} u(x)/v(x) = L$; *furthermore* (4.2) *holds in $[a, b]$.*

Hence Theorem 1.1 can be obtained immediately from Theorem 1.2.

5. Discrete versions

We shall now discuss some discrete versions of Theorems 2.1 and 3.1. As we shall see, not only are those discrete analogs correct, but they will allow us to obtain new proofs of those theorems and some generalizations in the next section.

Our basic tool is the following inequality.

PROPOSITION 5.1. *Let $a_1, a_2 \in \mathbb{R}$, $b_1, b_2 > 0$ and suppose that*

$$\frac{a_1}{b_1} \leqslant \frac{a_2}{b_2}. \tag{5.1}$$

Then

$$\frac{a_1}{b_1} \leqslant \frac{a_1 + a_2}{b_1 + b_2} \leqslant \frac{a_2}{b_2}, \tag{5.2}$$

with strict inequalities in (5.2) *if the inequality in* (5.1) *is strict.*

PROOF. An old trick for solving differential equations is based on the observation that if $a_1/b_1 = a_2/b_2 = \lambda$, then one also has that $(a_1 + a_2)/(b_1 + b_2) = \lambda$. In the case $a_1/b_1 < a_2/b_2$ we proceed by observing that the function

$$f(x) = \frac{a_1 + a_2 x}{b_1 + b_2 x} = \frac{a_2}{b_2} - b_1\Big(\frac{a_2}{b_2} - \frac{a_1}{b_1}\Big)\frac{1}{(b_1 + b_2 x)}$$

is strictly increasing in $[0, \infty)$, and we have that $f(0) = a_1/b_1$, $f(1) = (a_1+a_2)/(b_1+b_2)$, while $\lim_{x\to\infty} f(x) = a_2/b_2$. □

Using an inductive argument we immediately obtain the following result.

PROPOSITION 5.2. *Let $a_n \in \mathbb{R}$ and b_n be strictly positive numbers for $1 \leqslant n \leqslant N$. Suppose that $\{a_n/b_n\}_{n=1}^N$ is increasing:*

$$\frac{a_j}{b_j} \leqslant \frac{a_k}{b_k} \quad \text{if} \quad 1 \leqslant j < k \leqslant N.$$

Then

$$\frac{a_1}{b_1} \leqslant \frac{a_1 + \cdots + a_N}{b_1 + \cdots + b_N} \leqslant \frac{a_N}{b_N},$$

with strict inequalities unless $\{a_n/b_n\}_{n=1}^N$ is a constant sequence.

It is worth to point out that this proposition can be rephrased as follows: If $c_n \in \mathbb{R}$ and d_n are strictly positive numbers for $1 \leqslant n \leqslant N$, then

$$\min_{1\leqslant n\leqslant N} \frac{c_n}{d_n} \leqslant \frac{c_1 + \cdots + c_n}{d_1 + \cdots + d_n} \leqslant \max_{1\leqslant n\leqslant N} \frac{c_n}{d_n},$$

with strict inequalities unless all the quotients c_n/d_n coincide.

We can improve these inequalities as follows.

THEOREM 5.1. *Let $a_n \in \mathbb{R}$ and b_n be strictly positive numbers for $1 \leqslant n \leqslant N$. Suppose that $\{a_n/b_n\}_{n=1}^N$ is increasing. Let $M_1, M_2, N_1, N_2 \in \{1, \ldots, N\}$ with $M_1 \leqslant N_1$, $M_2 \leqslant N_2$, $M_1 \leqslant M_2$, $N_1 \leqslant N_2$. Then*

$$\frac{\sum_{n=M_1}^{N_1} a_n}{\sum_{n=M_1}^{N_1} b_n} \leqslant \frac{\sum_{n=M_2}^{N_2} a_n}{\sum_{n=M_2}^{N_2} b_n},$$

the inequality being strict if the sequence $\{a_n/b_n\}_{n=1}^N$ *is strictly increasing and* $(M_1, N_1) \neq (M_2, N_2)$.

Proof. If $N_1 \leqslant M_2$ then we have that

$$\frac{\sum_{n=M_1}^{N_1} a_n}{\sum_{n=M_1}^{N_1} b_n} \leqslant \frac{a_{N_1}}{b_{N_1}} \leqslant \frac{a_{M_2}}{b_{M_2}} \leqslant \frac{\sum_{n=M_2}^{N_2} a_n}{\sum_{n=M_2}^{N_2} b_n},$$

while if $N_1 \geqslant M_2$ we may use the case already established and Proposition 5.1 to obtain

$$\begin{aligned}\frac{\sum_{n=M_1}^{N_1} a_n}{\sum_{n=M_1}^{N_1} b_n} &= \frac{\sum_{n=M_1}^{M_2-1} a_n + \sum_{n=M_2}^{N_1} a_n}{\sum_{n=M_1}^{M_2-1} b_n + \sum_{n=M_2}^{N_1} b_n} \leqslant \frac{\sum_{n=M_2}^{N_1} a_n}{\sum_{n=M_2}^{N_1} b_n} \\ &\leqslant \frac{\sum_{n=M_2}^{N_1} a_n + \sum_{n=N_1+1}^{N_2} a_n}{\sum_{n=M_2}^{N_1} b_n + \sum_{n=N_1+1}^{N_2} b_n} = \frac{\sum_{n=M_2}^{N_2} a_n}{\sum_{n=M_2}^{N_2} b_n}.\end{aligned}$$

It is not hard to see that we obtain a strict inequality if $\{a_n/b_n\}_{n=1}^N$ is strictly increasing and $(M_1, N_1) \neq (M_2, N_2)$. $\square$

Therefore we immediately obtain that certain operations when applied at the same time to the numerator and to the denominator of an increasing sequence produce another increasing sequence.

Theorem 5.2. *Let* $a_n \in \mathbb{R}$ *and* b_n *be strictly positive numbers and suppose that* $\{a_n/b_n\}_{n=1}^\infty$ *is increasing. If* $\{M_k\}_{k=1}^\infty$ *and* $\{N_k\}_{k=1}^\infty$ *are sequences of strictly positive integers with* $M_k \leqslant N_k$, $M_k \leqslant M_{k+1}$, $N_k \leqslant N_{k+1}$, *then the sequence*

$$\left\{\frac{\sum_{n=M_k}^{N_k} a_n}{\sum_{n=M_k}^{N_k} b_n}\right\}_{k=1}^\infty \tag{5.3}$$

is increasing. In particular, the sequences

$$\left\{\frac{\sum_{j=1}^{k} a_j}{\sum_{j=1}^{k} b_j}\right\}_{k=1}^\infty \tag{5.4}$$

and, for a fixed m,

$$\left\{\frac{\sum_{j=k}^{k+m} a_j}{\sum_{j=k}^{k+m} b_j}\right\}_{k=1}^\infty \tag{5.5}$$

are also increasing.

If $\{a_n/b_n\}_{n=1}^\infty$ *is strictly increasing and* $(M_k, N_k) \neq (M_{k+1}, N_{k+1})$ *for all* k, *then the sequence* (5.3) *is likewise strictly increasing and, in particular, so are* (5.4) *and* (5.5).

Several remarks are in order. Notice first that the monotonicity of (5.4) is the direct analog of Gromov's theorem, and thus one may wonder if the results corresponding to (5.3) and (5.5) hold for integrals; we shall see that indeed such results are correct in the next section.

In the case $k = 1$ in (5.5) we obtain that if $\{a_n/b_n\}_{n=1}^N$ is increasing, then the sequence

$$\left\{\frac{a_n + a_{n+1}}{b_n + b_{n+1}}\right\}_{n=1}^{\infty}$$

is increasing, a fact that was already clear from Proposition 5.1. Actually, inequality (5.2) shows that for any constants $\alpha \geqslant 0$ and $\beta \geqslant 0$, $(\alpha, \beta) \neq (0, 0)$, the sequence

$$\left\{\frac{\alpha a_n + \beta a_{n+1}}{\alpha b_n + \beta b_{n+1}}\right\}_{n=1}^{\infty}$$

is also increasing. However, even though

$$\left\{\frac{a_n + a_{n+1} + a_{n+2}}{b_n + b_{n+1} + b_{n+2}}\right\}_{n=1}^{\infty}$$

is increasing, it is not true that

$$\left\{\frac{\alpha a_n + \beta a_{n+1} + \gamma a_{n+2}}{\alpha b_n + \beta b_{n+1} + \gamma b_{n+2}}\right\}_{n=1}^{\infty}$$

should be increasing for all constants $\alpha \geqslant 0$, $\beta \geqslant 0$, and $\gamma \geqslant 0$; in fact one can see this if $\alpha = \gamma = 1$ and $\beta = 0$ since there are examples of increasing sequences $\{a_n/b_n\}_{n=1}^{\infty}$ and $\{c_n/d_n\}_{n=1}^{\infty}$ such that $\{(a_n + c_n)/(b_n + d_n)\}_{n=1}^{\infty}$ is *not* increasing.

6. Integrals, again

We shall now show that an analog of Theorem 5.2 holds in the continuous case.

Let us recall that if $[a, b]$ is a closed interval, then a tagged subpartition of $[a, b]$ is a finite collection $\mathcal{P} = \{(I_j, \tau_j)\}_{j=1}^n$, where $I_1, \dots, I_n$ are closed, nonoverlaping intervals[6] and where the tags τ_j are elements of them, $\tau_j \in I_j$. The Riemann sum of a function $f\colon [a, b] \to \mathbb{R}$ at $\mathcal{P}$ is

$$S(f; \mathcal{P}) = \sum_{j=1}^{n} f(\tau_j)|I_j|.$$

If f is integrable in the Henstock–Kurzweil sense – which is equivalent to be Denjoy integrable [**9**] – over $[a, b]$, in particular if f is Lebesgue integrable, then the Riemann sums of f converge to its integral [**4, 9**] in the sense that for each $\varepsilon > 0$ there exists a gauge δ such that for any tagged δ-fine subpartition[7] of $[a, b]$, we have that

$$\left| \int_a^b f(x)\, dx - S(f; \mathcal{P}) \right| < \varepsilon. \tag{6.1}$$

Actually since the minimum of two gauges is also a gauge, if u and v are two integrable functions in the Henstock–Kurzweil sense over $[a, b]$, and $\varepsilon > 0$, we can find a gauge δ such that for any tagged δ-fine subpartition (6.1) holds for $f = u$ and for $f = v$. Hence we obtain the ensuing auxiliary result.

[6] We shall always assume that I_j is to the left of I_k if $j < k$.

[7] A gauge δ over the interval $[a, b]$ is any function $\delta\colon [a, b] \longrightarrow (0, \infty)$. We say that $\mathcal{P} = \{(I_j, \tau_j)\}_{j=1}^n$ is δ−fine if $I_j \subset [\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)]$.

LEMMA 6.1. *Let u and v be two integrable functions in the Henstock–Kurzweil sense over $[a,b]$. Then we can find a sequence of tagged subpartitions of $[a,b]$, $\{\mathcal{P}_n\}_{n=1}^{\infty}$, such that*

$$\lim_{n\to\infty} S(u;\mathcal{P}_n) = \int_a^b u(x)\,dx \quad \textit{and} \quad \lim_{n\to\infty} S(v;\mathcal{P}_n) = \int_a^b v(x)\,dx. \tag{6.2}$$

We are now ready to give yet another proof of Gromov's theorem, as well as certain related results.

PROPOSITION 6.1. *Let u and v be two Lebesgue*[8] *integrable functions over $[a,b]$, with v strictly positive at all points of this interval. Suppose that u/v is increasing. Suppose $[\alpha_k,\beta_k]\subset[a,b]$ for $k=1,2$, while $\beta_1\leqslant\alpha_2$. If $\mathcal{P}^k$ are tagged subpartitions of $[\alpha_k,\beta_k]$, $k=1,2$, then*

$$\frac{S(u;\mathcal{P}^1)}{S(v;\mathcal{P}^1)} \leqslant \frac{S(u;\mathcal{P}^2)}{S(v;\mathcal{P}^2)}, \tag{6.3}$$

with strict inequality unless u/v is constant on $[\alpha_1,\beta_2]$.

PROOF. The inequality follows at once from Theorem 5.1. □

If we now employ Lemma 6.1 we obtain the following inequality.

THEOREM 6.1. *Let u and v be two Lebesgue integrable functions over $[a,b]$, with v strictly positive at all points of this interval. Suppose that u/v is increasing. Suppose $[\alpha_k,\beta_k]\subset[a,b]$ for $k=1,2$, are two non degenerate closed intervals such that $\alpha_1\leqslant\alpha_2$ and $\beta_1\leqslant\beta_2$. Then*

$$\frac{\int_{\alpha_1}^{\beta_1} u(x)\,dx}{\int_{\alpha_1}^{\beta_1} v(x)\,dx} \leqslant \frac{\int_{\alpha_2}^{\beta_2} u(x)\,dx}{\int_{\alpha_2}^{\beta_2} v(x)\,dx}. \tag{6.4}$$

If $[\alpha_1,\beta_1]\neq[\alpha_2,\beta_2]$ and u/v is not constant, then the inequality is strict.

PROOF. If $\beta_1\leqslant\alpha_2$ the inequality is obtained from (6.3) by choosing sequences of subpartitions $\{\mathcal{P}_n^k\}_{n=1}^{\infty}$ of $[\alpha_k,\beta_k]$ that satisfy (6.2). When $\beta_1>\alpha_2$ we observe that what we already have proved yields

$$\frac{\int_{\alpha_1}^{\alpha_2} u(x)\,dx}{\int_{\alpha_1}^{\alpha_2} v(x)\,dx} \leqslant \frac{\int_{\alpha_2}^{\beta_1} u(x)\,dx}{\int_{\alpha_2}^{\beta_1} v(x)\,dx} \leqslant \frac{\int_{\beta_1}^{\beta_2} u(x)\,dx}{\int_{\beta_1}^{\beta_2} v(x)\,dx},$$

so that (6.4) follows from Theorem 5.1. □

From this theorem we immediately obtain the following results on the preservation of monoticity.

[8]It was seen in Section 4 that if u and v are two integrable functions in the Denjoy–Henstock–Kurzweil sense, or even in the distributional sense [**8**], over $[a,b]$, with v strictly positive at all points of this interval and u/v increasing, then they actually must be Lebesgue integrable.

Theorem 6.2. *Let u and v be two locally Lebesgue integrable functions over $[a,\infty)$, with v strictly positive at all points of this interval. Suppose that u/v is increasing. Let $\{[\alpha_k,\beta_k]\}_{k=1}^{\infty}$ be a family of closed non degenerate subintervals of $[a,b)$, with $\alpha_k \leqslant \alpha_{k+1}$ an $\beta_k \leqslant \beta_{k+1}$. Then the sequence*

$$\left\{\frac{\int_{\alpha_k}^{\beta_k} u(t)\,dt}{\int_{\alpha_k}^{\beta_k} v(t)\,dt}\right\}_{k=1}^{\infty}$$

is increasing.

Theorem 6.3. *Let u and v be two locally Lebesgue integrable functions over $[a,b)$, with v strictly positive at all points of this interval. Suppose that u/v is increasing. Then the following functions are also increasing,*

$$x \mapsto \frac{\int_a^x u(t)\,dt}{\int_a^x v(t)\,dt},$$

in (a,b), and

$$x \mapsto \frac{\int_x^{x+c} u(t)\,dt}{\int_x^{x+c} v(t)\,dt},$$

in $(a,b-c)$.

7. Extensions and applications

We shall now consider some interesting extensions of Gromov's theorem.

Theorem 7.1. *Let u and v be two Lebesgue integrable functions over $[a,b]$, with v strictly positive at all points of this interval. Suppose that u/v is increasing. Let A and B be constants, with $B>0$, such that*

$$\frac{A}{B} \leqslant \frac{u(x)}{v(x)}, \quad x \in [a,b].$$

Then the function of x

$$\frac{A+\int_a^x u(t)\,dt}{B+\int_a^x v(t)\,dt}, \tag{7.1}$$

is increasing for $x \in [a,b]$.

Proof. Consider the functions

$$u_1(x) = \begin{cases} u(x), & x \in [a,b],\\ A, & x \in [a-1,a),\end{cases} \qquad v_1(x) = \begin{cases} v(x), & x \in [a,b],\\ B, & x \in [a-1,a),\end{cases}$$

and apply Gromov's theorem in $[a-1,b]$. Since u_1/v_1 is increasing we obtain that (7.1) is increasing if $x \in [a,b]$. $\square$

In the differential form we have the following result.

THEOREM 7.2. *Let f, g be continuous functions defined in $[a,b]$, differentiable in (a,b) with $g'(x) \neq 0$ for all $x \in (a,b)$. Suppose $f(a) = A$, $g(a) = B \neq 0$, and*

$$\frac{A}{B} \leqslant \frac{f'(x)}{g'(x)}, \quad x \in [a,b].$$

If f'/g' is increasing on (a,b) then so is f/g.

PROOF. It follows from the previous theorem if $u = f'$ and $v = g'$, ☐

Let $\{a_n/b_n\}_{n=0}^{\infty}$ be an increasing sequence with $b_n > 0$. We can apply the above results inductively starting with $u \equiv a_0$, $v \equiv b_0$, to obtain that the following functions

$$\frac{a_0 + a_1 x}{b_0 + b_1 x}, \quad \frac{a_0 + a_1 x + a_2 x^2}{b_0 + b_1 x + b_2 x^2}, \quad \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{b_0 + b_1 x + b_2 x^2 + b_3 x^3},$$

and more generally $\sum_{j=0}^{n} a_j x^j / \sum_{j=0}^{n} b_j x^j$ are increasing functions of $x \in [0, \infty)$. Thus we obtain another proof of the following result [**5, 11**].

THEOREM 7.3. *Let $\{a_n/b_n\}_{n=0}^{\infty}$ be an increasing sequence with $b_n > 0$. Then the functions*

$$\frac{\sum_{n=0}^{N} a_n x^n}{\sum_{n=0}^{N} b_n x^n}, \tag{7.2}$$

are increasing for $x \in [0, \infty)$, while if both series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge in $[0, R)$ then

$$\frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n}, \tag{7.3}$$

is increasing in $[0, R)$.

PROOF. The only thing left to show is that (7.3) is increasing, but this is obtainable since this function is the limit of the increasing sequence of functions (7.2) in $[0, R)$. ☐

8. Integral means of analytic functions

We shall now consider an application of the ideas of this article. A strictly positive function $f(r)$ defined on some subinterval of $(0, \infty)$ is said to be a log-convex function of $\log r$ if the function $\log f(e^x)$ is convex on the corresponding interval. This means that f satisfies the inequality[9]

$$f(a^{\lambda} b^{1-\lambda}) \leqslant f(a)^{\lambda} f(b)^{1-\lambda}, \quad a < b, \quad 0 \leqslant \lambda \leqslant 1.$$

It is easy to check that if f is differentiable, then it is log-convex of $\log r$ if and only if $\frac{x f'(x)}{f(x)}$ increases in x.

Therefore we obtain the ensuing result.

LEMMA 8.1. *If $f(r)$ is differentiable and log-convex of* $\log r$ *on an interval $(0; R)$ and $f(0+)$ is finite, then the function $r f(r) \; / \int_0^r f(x)\, dx$, increases in $r \in [0, R)$.*

[9]Very general extensions of the notion of convexity are considered in [**3**].

Proof. Indeed, by the Gromov theorem the following function is increasing in r:

$$\frac{\int_0^r xf'(x)\,dx}{\int_0^r f(x)\,dx} = \frac{rf(r) - \int_0^r f(x)\,dx}{\int_0^r f(x)\,dx}.$$

This proves the lemma. □

If f is log-convex of $\log r$, then so is $rf(r)$ and thus we obtain the following.

Corollary 8.1. *Under the hypothesis of Lemma* 8.1*, we have that the function*

$$\frac{r^2 f(r)}{\int_0^r xf(x)\,dx},$$

increases in $r \in [0, R)$.

If $f(z)$ is analytic in a disc $|z| < R$, then its integral mean $M_p(r) = M_p(r; f)$ is defined as

$$M_p^p(r; f) = \frac{1}{2\pi} \int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta, \quad 0 < p < \infty.$$

In what can be called the birth of the theory of H^p spaces, 100 years ago, exactly, Hardy [**10**] proved that the integral mean $M_p(r)$ of a function $f(z)$ analytic in a disc $|z| < R$ is a log-convex function of $\log r$.

Denote the corresponding area mean by $A_p(r) = A_p(r; f)$,

$$A_p^p(r; f) = \frac{1}{\pi r^2} \iint_{|x+iy|<r} |f(x + iy)|^p dx\,dy,$$

that is

$$A_p^p(r; f) = \frac{2}{r^2} \int_0^r M_p^p(\rho; f)\rho\,d\rho.$$

In view of the increasing property of $M_p(r; f)$, one has $A_p(r; f) \leqslant M_p(r; f)$. However, as a direct consequence of the corollary and Hardy's theorem, we actually have the following result.

Theorem 8.1. *If* $f(z)$ *is analytic in a disc* $|z| < R$*, then the function*

$$\frac{M_p(r; f)}{A_p(r; f)},$$

is increasing in r.

Observe, one can actually let $p \to \infty$ in this theorem, since $M_\infty(r; f)/A_\infty(r; f) \equiv 1$ is in fact increasing, although not strictly, of course, for $0 \leqslant r < R$.

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