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COMPLETENESS THEOREM FOR CONTINUOUS FUNCTIONS AND PRODUCT CLASS-TOPOLOGIES

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Abstract. We introduce an infinitary logic $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$ which is an extension of $L_{\mathbb{A}}$ obtained by adding new quantifiers O^n and C^n, for every $n \in \omega$. The corresponding models are topological class-spaces. An axiomatization is given and the completeness theorem is proved.

1. Introduction

In [**7**] a topological class logic $L_{\mathbb{A}}(O, C)$ appropriate for the study of topologies on proper classes was developed. The logic $L_{\mathbb{A}}(O, C)$ is an infinitary logic with new quantifiers O and C. The corresponding semantics consists of a classical first-order structure $\mathcal{K}$ whose domain $\mathbb{K}$ is a class, with addition of two classes $\mathbb{T}$ and $\mathbb{C}$ of subsets of $\mathbb{K}$ such that $(\mathbb{K}, \mathbb{T}, \mathbb{C})$ is a topological class-space introduced in [**2**] (see Definition 2.1). The intended meaning of $Ox\varphi(x)$ $(Cx\varphi(x))$ is that the set defined by $\varphi(x)$ belongs to the class $\mathbb{T}$ $(\mathbb{C})$. The logic $L_{\mathbb{A}}(O, C)$ is analogous to the logics $L_{\omega\omega}(O)$ and $L_{\omega_1\omega}(O)$ developed in [**14**] for ordinary topological spaces. In [**13**], the author investigated the logics $L_{\omega\omega}(O^n)_{n\in\omega}$ and $L_{\omega_1\omega}(O^n)_{n\in\omega}$ with quantifiers O^n for each $n \in \omega$ in order to give a formal treatment of continuous functions on product topologies.

In this paper, following the ideas from [**7**] and [**13**], we continue the study of topological class logics and introduce the logic $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$ appropriate for continuous functions and product class-topologies.

All above mentioned logics are of the form $L_{\mathbb{A}}(Q_1, Q_2, \ldots)$, i.e., they are extensions of an admissible fragment $L_{\mathbb{A}}$ of $L_{\omega_1\omega}$ with new quantifiers $Q_1, Q_2, \ldots$ Here, we give some general remarks and a short (mainly informal) overview of the notions relevant to our consideration.

The original purpose of admissible sets was to generalize classical computability theory from natural numbers to ordinals. The Kripke–Platek set theory, KP for

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short, is considered as a minimal subsystem of ZF necessary for a good notion of computation. KP arises from ZF by omitting the Power Set Axiom and restricting Separation and Collection to Δ_0-formulas. An admissible set is a transitive set $\mathbb{A}$ such that $(\mathbb{A}, \in)$ is a model of KP. The smallest example of an admissible set is the set of hereditarily finite sets $\mathbb{HF}$ which corresponds to classical computability theory. Another example of an admissible set, important in this paper, is the set $\mathbb{HC}$ of hereditarily countable sets. To emphasize the analogy with computability theory, for an arbitrary admissible set $\mathbb{A}$, the elements of $\mathbb{A}$ are called $\mathbb{A}$-finite, a subset of $\mathbb{A}$ that is definable in $(\mathbb{A}, \in)$ by Σ-formula, with parameters in $\mathbb{A}$, is called $\mathbb{A}$-computable enumerable ($\mathbb{A}$-c.e.), and a set $X \subseteq \mathbb{A}$ is called $\mathbb{A}$-computable if both X and $\mathbb{A} \smallsetminus X$ are $\mathbb{A}$-c.e.

If L is a countable first-order language and $\mathbb{A}$ is an admissible set such that $\mathbb{A} \subseteq \mathbb{HC}$ and $\omega \in \mathbb{A}$, the admissible fragment $L_{\mathbb{A}}$ is the set of all $L_{\omega_1\omega}$ formulas that belongs to $\mathbb{A}$. $L_{\mathbb{A}}$ is an $\mathbb{A}$-computable set, and is closed under basic syntactical operations. The Barwise compactness theorem is one of the most significant results related to admissible fragments: If $\mathbb{A}$ is a countable admissible set, and Γ is an $\mathbb{A}$-c.e. subset of $L_{\mathbb{A}}$ such that every $\mathbb{A}$-finite subset of Γ has a model, then Γ has a model.

Barwise compactness is in the heart of a techniques developed by Rašković in [**11**]. After that paper, the technique has been applied for proving completeness theorems for many infinitary logics with generalized quantifiers [**4**–**7**, **12**]. Roughly speaking, adding new quantifiers to infinitary logics is one of the most frequent and high acceptable ways to incorporate into the realm of logic those structures whose related concepts are left out of the first-order logic (such as: probability spaces, topological spaces, etc.). Here, our attention is focussed only on n-ary quantifiers ($n \geqslant 1$) which can be applied to a single formula and bind an n-tuple of variables. More precisely, if L is a first-order language (a vocabulary) and Q is an n-ary quantifier, then the set of formulas $L_{\omega_1\omega}(Q)$ is built in the standard way with the additional formation rule: if φ is a formula and $\vec{x}$ is an n-tuple of variables, then $Q\vec{x}\varphi$ is a formula. A weak-model for $L_{\omega_1\omega}(Q)$ is a structure of the form $(\mathcal{A}, q)$, where $\mathcal{A}$ is an L structure and q is a set of subsets of the universe A. The truthfulness of an $L_{\omega_1\omega}(Q)$-formula in such models is defined inductively in the usual way with the new clause:

$$(\mathcal{A}, q) \models Q\vec{x}\varphi \quad \text{iff} \quad \{\vec{a} \in A^n \mid (\mathcal{A}, q) \models \varphi[\vec{a}]\} \in q.$$

Of course, there are many concrete interpretations of Q (Q_∞ "there exist infinitely many"", $Q_{\aleph_1}$ "there exists uncountable many" [**10**], $Q_{\geqslant r}$ "to have probability at least r" [**9**, **12**], Q_{open} "to be open" [**14**], etc.).

Combining a consistency property argument and the Henkin construction yields an $L_{\omega_1\omega}(Q)$-version of the weak completeness theorem (see [6,11]). In order to obtain stronger completeness theorems, we have to carry out more interesting model constructions according to the intended meaning of a new quantifier. The key step of Rašković's technique is the construction of a middle model whose all subsets satisfy some desirable conditions being true in a weak model only for $L_{\omega_1\omega}(Q)$-definable subsets. If $K = L \cup C$ is introduced in the construction of a weak model

(C is a set of new constant symbols and $C \in \mathbb{A}$), then a desirable middle model is obtained by Barwise Compactness applied on a theory of many sorted logic whose language contains (at least) two kinds of variables: $X, Y, Z, \ldots$, variables for sets, and $x, y, z, \ldots$ variables for urelements. Predicates are $E_n(x_1, \ldots, x_n, X)$, $n \geqslant 1$, with canonical meaning $(x_1, \ldots, x_n) \in X$, and $Q(X)$. Constant symbols are A_φ for each $K_{\mathbb{A}}(Q)$-formula φ. Finally, when a middle model is properly constructed, it have to be transformed to a strong model. This strategy will be applied in the proof of Completeness theorem for $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$.

The rest of the paper is organized as follows. Section 2 contains a short overview of topological class-spaces. In Section 3, the syntax and the corresponding semantics of the logic $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$ are described. A sound and complete axiomatic system is given in Section 4. This section contains the main result of the paper, i.e., the completeness theorem for $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$. We conclude in Section 5.

2. Preliminaries

The notion of topological class-space was introduced in [**2**], and further developed in [**3**]. The metatheory is NBG class theory.

Definition 2.1. [**2**] Let $\mathbb{K}$ be a class and $\mathbb{T}$ and $\mathbb{C}$ classes of subsets of $\mathbb{K}$. We call the triple $(\mathbb{K}, \mathbb{T}, \mathbb{C})$ a topological class-space if the following axioms are satisfied:

(KT_1) if $u, v \in \mathbb{T}$, then $u \cap v \in \mathbb{T}$;
(KT_2) for any i, if $u_j \in \mathbb{T}$, $j \in i$, then $\bigcup_{j\in i} u_j \in \mathbb{T}$;
(KT_3) for any $x \in \mathbb{K}$ there exists $u \in \mathbb{T}$ such that $x \in u$;
(KT_4) if $u \in \mathbb{T}$ and $a \in \mathbb{C}$, then $u \smallsetminus a \in \mathbb{T}$;

(KC_1) if $a, b \in \mathbb{C}$, then $a \cup b \in \mathbb{C}$;
(KC_2) for any i, if $a_j \in \mathbb{C}$, $j \in i$, then $\bigcap_{j\in i} a_j \in \mathbb{C}$;
(KC_3) for any subset x of $\mathbb{K}$ there exists $a \in \mathbb{C}$ such that $x \subseteq a$;
(KC_4) if $u \in \mathbb{T}$ and $a \in \mathbb{C}$, then $a \smallsetminus u \in \mathbb{C}$.

Elements of $\mathbb{T}$ are open subsets, while elements of $\mathbb{C}$ are closed subsets of $\mathbb{K}$.

Many examples of topological class spaces can be found in [**2**,**3**]. The following theorem is an answer to the question when an ordered pair $(\mathbb{B}_T, \mathbb{B}_C)$ of classes of subsets of a class $\mathbb{K}$ defines a class-space over $\mathbb{K}$.

Theorem 2.1. [**3**] *Let $\mathbb{K}$ be a proper class and $\mathbb{B}_T$ and $\mathbb{B}_C$ be classes of subsets of $\mathbb{K}$ such that*

(1) $\forall x \in \mathbb{K}\ \exists s \in \mathbb{B}_T\ x \in s$.
(2) *for every subset $m \subseteq \mathbb{K}$ there is $f \in \mathbb{B}_C$ such that $m \subseteq f$.*
(3) $\forall s \in \mathbb{B}_T\ \forall f \in \mathbb{B}_C\ (s \smallsetminus f \in \mathbb{B}_T \wedge f \smallsetminus s \in \mathbb{B}_C)$.

Then there is the least class-topology $(\mathbb{K}, \mathbb{T}, \mathbb{C})$ such that $\mathbb{B}_T \subseteq \mathbb{T}$ and $\mathbb{B}_C \subseteq \mathbb{C}$.

If $\mathbb{K}$ and $\mathbb{L}$ are classes, recall that a map from $\mathbb{K}$ into $\mathbb{L}$ is every class $\mathbb{F} \subseteq \mathbb{K} \times \mathbb{L}$ such that $\forall x \in \mathbb{K}\ \exists_1 y \in \mathbb{L}\ (x, y) \in \mathbb{F}$.

DEFINITION 2.2. [**2**] Let $(\mathbb{K}, \mathbb{T}_\mathbb{K}, \mathbb{C}_\mathbb{K})$ and $(\mathbb{L}, \mathbb{T}_\mathbb{L}, \mathbb{C}_\mathbb{L})$ be class-spaces. A map $\mathbb{F}\colon \mathbb{K} \to \mathbb{L}$ is continuous iff the restriction $\mathbb{F} \restriction u$ to every $u \in \mathbb{T}_\mathbb{K}$ is a continuous function.

THEOREM 2.2. [**3**] *Let* $(\mathbb{K}, \mathbb{T}_\mathbb{K}, \mathbb{C}_\mathbb{K})$ *and* $(\mathbb{L}, \mathbb{T}_\mathbb{L}, \mathbb{C}_\mathbb{L})$ *be class-spaces and let* $\mathbb{F}\colon \mathbb{K} \to \mathbb{L}$ *be a map. Then the following are equivalent:*

(1) $\mathbb{F}$ *is continuous.*
(2) *For all* $u \in \mathbb{T}_\mathbb{K}$*, and all* $w \in \mathbb{T}_\mathbb{L}$*,* $u \cap \mathbb{F}^{-1}(w) \in \mathbb{T}_\mathbb{K}$*.*
(3) *For all* $v \in \mathbb{C}_\mathbb{K}$*, and all* $w \in \mathbb{C}_\mathbb{L}$*,* $v \cap \mathbb{F}^{-1}(w) \in \mathbb{C}_\mathbb{K}$*.*

The product of two class-spaces $(\mathbb{K}_1, \mathbb{T}_1, \mathbb{C}_1)$ and $(\mathbb{K}_2, \mathbb{T}_2, \mathbb{C}_2)$ is the class-space $(\mathbb{K}, \mathbb{T}, \mathbb{C})$, where $\mathbb{K} = \mathbb{K}_1 \times \mathbb{K}_2$ and the basis for $\mathbb{T}$ and $\mathbb{C}$ are the classes $\mathbb{T}_0 = \{\pi_1^{-1}(u) \cap \pi_2^{-1}(v) \mid u \in \mathbb{T}_1, v \in \mathbb{T}_2\}$ and $\mathbb{C}_0 = \{\pi_1^{-1}(u) \cap \pi_2^{-1}(v) \mid u \in \mathbb{C}_1, v \in \mathbb{C}_2\}$, where $\pi_1 : \mathbb{K} \to \mathbb{K}_1$ and $\pi_2 : \mathbb{K} \to \mathbb{K}_2$ are projection maps. The classes $\mathbb{T}_0$ and $\mathbb{C}_0$ do not satisfy the conditions of Theorem 2.1, so let $\mathbb{T}'$ be the class of all finite unions of the elements of the class $\mathbb{T}_0$, and $\mathbb{C}'$ be the class of all finite unions of the elements of the class $\mathbb{C}_0$. It is obvious that the classes $\mathbb{T}'$ and $\mathbb{C}'$ are closed under finite intersections. In fact, the classes $\mathbb{T}'$ and $\mathbb{C}'$ form the base of the topological class-space $(\mathbb{K}, \mathbb{T}, \mathbb{C})$. Note that $\mathbb{T}_0 \subseteq \mathbb{T}'$ and $\mathbb{C}_0 \subseteq \mathbb{C}'$ and the classes $\mathbb{T}'$ and $\mathbb{C}'$ satisfy the conditions of Theorem 2.1, and there is the least class-space $(\mathbb{K}, \overline{\mathbb{T}}, \overline{\mathbb{C}})$ such that $\mathbb{T}' \subseteq \overline{\mathbb{T}}$ and $\mathbb{C}' \subseteq \overline{\mathbb{C}}$. It is not difficult to see that the class $\overline{\mathbb{T}} = \{\cup x \mid x \text{ is a finite intersection of members of } \mathbb{T}'\}$ is exactly the class $\mathbb{T}$ of the open sets. Similarly, the class $\overline{\mathbb{C}} = \{\cap x \mid x \text{ is a finite union of members of } \mathbb{C}'\}$ is the class $\mathbb{C}$ of closed sets. Furthermore, each open set $O \in \mathbb{T}$ is a union of sets of the form $O_1 \times O_2$, where $O_1 \in \mathbb{T}_1$ and $O_2 \in \mathbb{T}_2$. This fact is of the great importance for our axiomatization given here.

3. Syntax and semantics of the logic $L_\mathbb{A}(O^n, C^n)_{n\in\omega}$

We assume that $\mathbb{A}$ is a countable admissible set such that $\mathbb{A} \subseteq \mathbb{HC}$ and $\omega \in \mathbb{A}$. We refer the reader to [**1**, **8**] for a detailed treatment of admissible sets and the infinitary logic $L_\mathbb{A}$. Briefly, we note that the set of formulas of $L_\mathbb{A}$ is the set of all expressions in $\mathbb{A}$ that are built from atomic formulas using negation, finite or infinite conjunction, and the quantifier $\forall$.

Let L be a Σ-definable set which contains a set of finitary relation symbols. The infinitary logic $L_\mathbb{A}(O^n, C^n)_{n\in\omega}$ is an extension of $L_\mathbb{A}$ obtained by adding new quantifiers $O^n x_1, \dots, x_n$ and $C^n x_1, \dots, x_n$, for every $n \in \omega$, where $x_1, \dots, x_n$ is a tuple of pairwise distinct variables.

DEFINITION 3.1. The set of formulas of $L_\mathbb{A}(O^n, C^n)_{n\in\omega}$ is the least set such that:

(1) Each atomic formula of first-order logic is a formula of $L_\mathbb{A}(O^n, C^n)_{n\in\omega}$;
(2) If φ is a formula of $L_\mathbb{A}(O^n, C^n)_{n\in\omega}$ then $\neg\varphi$ is a formula of $L_\mathbb{A}(O^n, C^n)_{n\in\omega}$;
(3) If φ is a formula of $L_\mathbb{A}(O^n, C^n)_{n\in\omega}$ then $\forall x\varphi$, $\exists x\varphi$, $O^n x_1, \dots, x_n\varphi$ and $C^n x_1, \dots, x_n\varphi$ are formulas of $L_\mathbb{A}(O^n, C^n)_{n\in\omega}$; occasionally, we use the abbreviation $Q^n\vec{x}$ for $Qx_1 \dots Qx_n$, $Q \in \{\forall, \exists\}$;
(4) If $\Phi \in \mathbb{A}$ is a set of formulas of $L_\mathbb{A}(O^n, C^n)_{n\in\omega}$ with only finitely many free variables, then $\bigwedge \Phi$ is a formula of $L_\mathbb{A}(O^n, C^n)_{n\in\omega}$.

Three types of models are relevant for our logic.

Definition 3.2. A *weak model* for $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$ is a structure $(\mathcal{K},\mathbb{T}_n,\mathbb{C}_n)_{n\in\omega}$, such that $\mathcal{K}$ is a first-order structure of a language L whose universe $\mathbb{K}$ is a class, and $\mathbb{T}_n$, $\mathbb{C}_n$ are classes of subsets of $\mathbb{K}^n$.

A *middle model* for $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$ is a weak model $(\mathcal{K},\mathbb{T}_n,\mathbb{C}_n)_{n\in\omega}$, such that:

(a) For each $(x_1,\dots,x_n)\in\mathbb{K}^n$ there is $U\in\mathbb{T}_n$ such that $(x_1,\dots,x_n)\in U$;
(b) For every subset $X\subseteq\mathbb{K}^n$ there is $F\in\mathbb{C}_n$ such that $X\subseteq F$;
(c) For all $U\in\mathbb{T}_n$ and $F\in\mathbb{C}_n$, $U\smallsetminus F\in\mathbb{T}_n$ and $F\smallsetminus U\in\mathbb{C}_n$.

A *complete topological class model* for $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$ is a weak model $(\mathcal{K},\mathbb{T}_n,\mathbb{C}_n)_{n\in\omega}$ such that for each $n\in\omega$, $(\mathbb{K}^n,\mathbb{T}_n,\mathbb{C}_n)$ is a topological class-space, $\mathbb{T}_n$ is the n-th topological product of $\mathbb{T}_1$ on $\mathbb{K}$ and $\mathbb{C}_n$ is the n-th topological product of $\mathbb{C}_1$ on $\mathbb{K}$.

The satisfaction relation $\models$ is defined inductively in the usual way with the new clauses:

$$(\mathcal{K},\mathbb{T}_n,\mathbb{C}_n)_{n\in\omega}\models O^k x_1,\dots,x_k\varphi[x_1,\dots,x_k,b_1,\dots,b_l]\quad\text{iff}$$
$$\{(a_1,\dots,a_k)\mid(\mathcal{K},\mathbb{T}_n,\mathbb{C}_n)_{n\in\omega}\models\varphi[a_1,\dots,a_k,b_1,\dots,b_l]\}\in\mathbb{T}_k$$

for $(b_1,\dots,b_l)\in\mathbb{K}^l$;

$$(\mathcal{K},\mathbb{T}_n,\mathbb{C}_n)_{n\in\omega}\models C^k x_1,\dots,x_k\varphi[x_1,\dots,x_k,b_1,\dots,b_l]\quad\text{iff}$$
$$\{(a_1,\dots,a_k)\mid(\mathcal{K},\mathbb{T}_n,\mathbb{C}_n)_{n\in\omega}\models\varphi[a_1,\dots,a_k,b_1,\dots,b_l]\}\in\mathbb{C}_k$$

for $(b_1,\dots,b_l)\in\mathbb{K}^l$.

4. A complete axiomatization

$L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$ has the following set of axioms, where $\Phi=\bigcup_{n\in\omega}\Phi_n$, $\Phi_n=\{\varphi\in\Phi\mid\varphi$ is a formula of $L_{\mathbb{A}}(O,C)$ with $n{+}1$ free variables$\}$, and $\Phi,\Phi_n\in\mathbb{A}$, $n\in\omega$:

(1) All axioms schemas for $L_{\mathbb{A}}$;
(2) $\forall x_1,\dots,x_n(\varphi\leftrightarrow\psi)\to(O^n x_1,\dots,x_n\varphi\leftrightarrow O^n x_1,\dots,x_n\psi)$;
$\forall x_1,\dots,x_n(\varphi\leftrightarrow\psi)\to(C^n x_1,\dots,x_n\varphi\leftrightarrow C^n x_1,\dots,x_n\psi)$;
(3) $O^n x_1,\dots,x_n\varphi(x_1,\dots,x_n)\to O^n y_1,\dots,y_n\varphi(y_1,\dots,y_n)$;
$C^n x_1,\dots,x_n\varphi(x_1,\dots,x_n)\to C^n y_1,\dots,y_n\varphi(y_1,\dots,y_n)$;
(4) $O^n x_1,\dots,x_n(x_1\neq x_1)$;
$C^n x_1,\dots,x_n(x_1\neq x_1)$;
(5) $O^n x_1,\dots,x_n\varphi\wedge O^n x_1,\dots,x_n\psi\to O^n x_1,\dots,x_n(\varphi\wedge\psi)$;
$C^n x_1,\dots,x_n\varphi\wedge C^n x_1,\dots,x_n\psi\to C^n x_1,\dots,x_n(\varphi\vee\psi)$;
(6) $O^n x_1,\dots,x_n\varphi\wedge C^n x_1,\dots,x_n\psi\to O^n x_1,\dots,x_n(\varphi\wedge\neg\psi)$;
$O^n x_1,\dots,x_n\varphi\wedge C^n x_1,\dots,x_n\psi\to C^n x_1,\dots,x_n(\neg\varphi\wedge\psi)$;
(7) $\forall y O^n x_1,\dots,x_n\varphi(x_1,\dots,x_n,y)\to O^n x_1,\dots,x_n\exists y\varphi(x_1,\dots,x_n,y)$;
$\forall y C^n x_1,\dots,x_n\varphi(x_1,\dots,x_n,y)\to C^n x_1,\dots,x_n\forall y\varphi(x_1,\dots,x_n,y)$;
(8) $\bigwedge\limits_{\varphi\in\Phi} O^n x_1,\dots,x_n\varphi\to O^n x_1,\dots,x_n\bigvee\Phi$;
$\bigwedge\limits_{\varphi\in\Phi} C^n x_1,\dots,x_n\varphi\to C^n x_1,\dots,x_n\bigwedge\Phi$;

(9) $O^n x_1,\dots,x_n\varphi \wedge O^m x_{n+1},\dots,x_{n+m}\psi \to O^{n+m}x_1,\dots,x_{m+n}(\varphi\wedge\psi)$;
$C^n x_1,\dots,x_n\varphi \wedge C^m x_{n+1},\dots,x_{n+m}\psi \to C^{n+m}x_1,\dots,x_{n+m}(\varphi\wedge\psi)$;

(10) $O^n x_1,\dots,x_n\varphi(x_1,\dots,x_n) \to O^k x_{i_1},\dots,x_{i_k}\varphi(x_{\sigma(1)},\dots,x_{\sigma(n)})$;
$C^n x_1,\dots,x_n\varphi(x_1,\dots,x_n) \to C^k x_{i_1},\dots,x_{i_k}\varphi(x_{\sigma(1)},\dots,x_{\sigma(n)})$,
where $\sigma\colon n\to n$, $|\sigma[n]| = k$ and $\sigma = \{i_1 < \cdots < i_k\}$;
[Informal meaning: The permutation, projection, or consolidation of an open set is open (see [**13**])]

(11) $O^n x_1,\dots,x_n\varphi(x_1,\dots,x_n) \to \forall x_1,\dots,x_k O^{n-k}x_{k+1},\dots,x_n\varphi(x_1,\dots,x_n)$;

[Informal meaning: Any projection of an open set is open.]

(12) $\forall^n\vec{x}\bigvee_m \bigvee_{\varphi\in\Phi_{m+n-1}} \exists^m\vec{y}\,\big(O^n\vec{z}\varphi(\vec{z},\vec{y})\wedge\varphi(\vec{x},\vec{y})\big)$;
[Axiom corresponds to the condition (KT_3).]

(13) $\bigwedge_n \bigwedge_{\varphi\in\Phi_{n+k-1}} \forall^n\vec{x}\bigvee_m \bigvee_{\psi\in\Phi_{m+k-1}} \exists^m\vec{y}\big(C^k\vec{z}\psi(\vec{z},\vec{y})\wedge\forall^k\vec{u}\big(\varphi(\vec{u},\vec{x})\to\psi(\vec{u},\vec{y})\big)\big)$;
[Axiom corresponds to the condition (KC_3).]

(14) $O^m\vec{y}\psi(\vec{y}) \to \bigwedge_l \bigwedge_{\theta\in\Phi_{n+l-1}} \forall^l\vec{v}\big(O^n\vec{x}\theta(\vec{x},\vec{v}) \to$
$$\to O^{m+n-k}\vec{z},y_{k+1},\dots,y_m\big(\exists y_1,\dots,y_k\big(\psi(\vec{y})\wedge\varphi(\vec{z},\vec{y})\big)\wedge\theta(\vec{z},\vec{v})\big)\big);$$
[Informal meaning: $\varphi(z_1,\dots,z_n,y_1,\dots,y_k)$ defines an (n,k)-ary continuous relation, i.e., under the restriction to any open set, the inverse image of a slice of an open set is open.]

(15) $\bigwedge_n \bigwedge_{\varphi\in\Phi_{n-1}} \forall^n\vec{u} \bigvee_{k\in\omega^n} \bigvee_{(\varphi_1,\dots,\varphi_n)\in\Phi_{k_1}\times\ldots\times\Phi_{k_n}} \exists x_{11}\dots x_{1k_1}\dots\exists x_{n1}\dots x_{nk_n}$
$$\Big(O^n\vec{x}\varphi(\vec{x})\wedge\varphi(\vec{u}) \to \Big(\bigwedge_{i=1}^n \big(Ox\varphi_i(x,x_{i1},\dots,x_{ik_i})\wedge\varphi_i(u_i,x_{i1},\dots,x_{ik_i})\big)\wedge$$
$$\wedge\forall y_1,\dots,y_n\Big(\bigwedge_{i=1}^n\varphi_i(y_i,x_{i1},\dots,x_{ik_i})\to\varphi(y_1,\dots,y_n)\Big)\Big)\Big),$$
(for simplicity the formula φ is written without parameters);
[Informal meaning: Every open set is the union of open boxes, i.e., if $(x_1,\dots,x_n)\in U\in\mathbb{T}_n$, then there exist $V_1,\dots,V_n\in\mathbb{T}_1$ such that $(x_1,\dots,x_n)\in V_1\times\cdots\times V_n\subseteq U$.]

(16) $\forall^n\vec{y}\bigwedge_k \bigwedge_{\varphi\in\Phi_{k+n-1}} \bigvee_{l\in\omega^n} \bigvee_{(\varphi_1,\dots,\varphi_n)\in\Phi_{l_1}\times\ldots\times\Phi_{l_n}} \exists x_{11}\dots x_{1l_1}\dots\exists x_{n1}\dots x_{nl_n}$
$$\Big(C^n\vec{x}\varphi(\vec{x})\wedge\neg\varphi(\vec{y}) \to \Big(\bigwedge_{i=1}^n Ox\varphi_i(x,x_{i1},\dots,x_{il_i})\wedge\varphi_i(y_i,x_{i1},\dots,x_{il_i})\Big)\wedge$$
$$\wedge\forall x_1,\dots,x_n\Big(\bigwedge_{i=1}^n\varphi_i(x_i,x_{i1},\dots,x_{il_i})\to\neg\varphi(x_1,\dots,x_n)\Big)\Big);$$
(for simplicity the formula φ is written without parameters);
[Informal meaning: If $(x_1,\dots,x_n)\notin F\in\mathbb{C}_n$, then there exist $V_1,\dots,V_n\in\mathbb{T}_1$ such that $(x_1,\dots,x_n)\in V_1\times\cdots\times V_n$ and $(V_1\times\cdots\times V_n)\cap F=\emptyset$ (see [**2**, **3**]).]

and the rules of inferences:

(R_1) From φ and $\varphi\to\psi$, infer ψ.
(R_2) From $\varphi\to\psi$, for all $\psi\in\Psi$, infer $\varphi\to\bigwedge_{\psi\in\Psi}\psi$.

(R_3) From φ, infer $\forall x\,\varphi$.

Axioms (2)–(8), (12) and (13) formalize our notion of topologies on classes and they are analogue to the axioms of the $L_{\mathbb{A}}(O,C)$. Axioms (9), (10) and (11) describe products of topological class-spaces. Axiom (14) enforces the constraint that an (n,k)-ary relation defined by a formula $\varphi(z_1,\dots,z_n,y_1,\dots,y_k)$ is continuous (see Theorem 2.2, and [**13**]). Axioms (15) and (16) capture the fact that for each $n \in \omega$, $\mathbb{T}_n$ is the n-th topological product of $\mathbb{T}_1$ on $\mathbb{K}$, and $\mathbb{C}_n$ is n-th topological product of $\mathbb{C}_1$ on $\mathbb{K}$.

The soundness theorem for $L_{\mathbb{A}}(O^n,C^n)_{n\in\omega}$ holds since all the axioms represent properties of the topological class-spaces. We prove that the axiomatization is complete with respect to the class of complete topological class models. In the proof we combine Keisler's construction for the weak completeness theorem in [**10**], with Sgro's construction from [**14**] and Rašković's middle-strong construction from [**11**]. Two simple statements will be used in the middle-strong construction:

(1) If $Y \subseteq \mathbb{K}$ and Y is not open, then there is $c \in Y$ such that if U is open and $U \subseteq Y$ then $c \notin U$.
(2) If $Y \subseteq \mathbb{K}$ and Y is not closed, then there is $c \notin Y$ such that if F is closed and $Y \subseteq F$ then $c \in F$.

Let T be a set of sentences of $L_{\mathbb{A}}(O^n,C^n)_{n\in\omega}$ such that T is Σ_1-definable over $\mathbb{A}$ and consistent with the axioms of $L_{\mathbb{A}}(O^n,C^n)_{n\in\omega}$. Let θ^{O^k} and θ^{C^k}, $k \in \omega$, be the sentences of $L_{\mathbb{A}}(O^n,C^n)_{n\in\omega}$ introduced for each formula $\theta(x_1,\dots,x_k,y_1,\dots,y_n)$ as follows:

(O) θ^{O^k} is $\forall^n\vec{y}\exists^k\vec{x}(\neg O^k\vec{z}\theta(\vec{z},\vec{y}) \to \bigwedge_m \bigwedge_{\chi\in\Phi_{m+k-1}} \theta^{O^k}_{\chi}(\vec{x},\vec{y}))$,
where $\theta^{O^k}_{\chi}(\vec{x},\vec{y})$ is
$\forall^m\vec{z}\big((O^k\vec{u}\chi(\vec{u},\vec{z}) \wedge \forall^k\vec{v}(\chi(\vec{v},\vec{z}) \to \theta(\vec{v},\vec{y}))) \to (\theta(\vec{x},\vec{y}) \wedge \neg\chi(\vec{x},\vec{z}))\big)$.
(C) θ^{C^k} is $\forall^n\vec{y}\exists^k\vec{x}(\neg C^k\vec{z}\theta(\vec{z},\vec{y}) \to \bigwedge_m \bigwedge_{\chi\in\Phi_{m+k-1}} \theta^{C^k}_{\chi}(\vec{x},\vec{y}))$,
where $\theta^{C^k}_{\chi}(\vec{x},\vec{y})$ is
$\forall^m\vec{z}\big((C^k\vec{u}\chi(\vec{u},\vec{z}) \wedge \forall^k\vec{v}(\theta(\vec{v},\vec{y}) \to \chi(\vec{v},\vec{z}))) \to (\neg\theta(\vec{x},\vec{y}) \wedge \chi(\vec{x},\vec{z}))\big)$.

Note that $\theta^{O^k}_{\chi}(x_1,\dots,x_k,y_1,\dots,y_n)$ is the set of k-typles $(x_1,\dots,x_k)$ wich are in θ, but not in any open subset of θ defined by χ using parameters, and θ^{O^k} means that for any parameters, if θ is not open then it is not equal to any union of open sets definable in $L_{\mathbb{A}}(O^n,C^n)_{n\in\omega}$. Similarly, $\theta^{C^k}_{\chi}(x_1,\dots,x_k,y_1,\dots,y_n)$ is the set of k-typles $(x_1,\dots,x_k)$ which are in any closed superset of θ defined by χ using parameters, and θ^{C^k} means that for any parameters, if θ is not closed then it is not equal to any intersection of closed sets definable in $L_{\mathbb{A}}(O^n,C^n)_{n\in\omega}$. It can be shown (as in [**7**]) that $\Gamma = T \cup \{\theta^{O^k}, \theta^{C^k} \mid \theta \text{ is a formula in }, k \in \omega\}$ is consistent in $L_{\mathbb{A}}(O^n,C^n)_{n\in\omega}$. By Keisler's construction of weak models, we get a set model with the properties of topological class-spaces (see [**7**] for more details). In the weak model $(\mathcal{K},\mathbb{T}_n,\mathbb{C}_n)_{n\in\omega}$ of the theory Γ, the condition (KC_3) holds, by Axiom

(13), only for $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$-definable sets. The construction of the middle model extends (KC_3) to all subsets of $\mathbb{K}^n$, $n \in \omega$.

Theorem 4.1 (Middle Completeness Theorem for $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$). *Let T be a set of sentences of $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$ such that T is Σ_1-definable over $\mathbb{A}$ and consistent with the axioms of $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$. Then there is a middle model of T.*

Proof. Let $K = L \cup C$ be the language introduced in the Keisler construction of a weak model of Γ where C is a set of new constant symbols and $C \in \mathbb{A}$. Let M be a language with two kinds of variables: $X, Y, Z, \dots$ variables for sets and $x, y, z, \dots$ variables for urelements. Predicates of our language are $E_n(x_1, \dots, x_n, X)$, $O^n(X)$ and $C^n(X)$, $n \in \omega$, with canonical meaning $(x_1, \dots, x_n) \in X$, X is an open set of n-typles and X closed set of n-typles. Constant symbols are C_φ for each formula φ of $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$. Let S be the following theory of $M_{\mathbb{A}}$:

(1) Axiom of well-definedness
$\forall X \bigwedge\limits_{n<m} \neg\exists x_1, \dots, x_n, y_{n+1}, \dots, y_m (E_m(x_1, \dots, x_n, y_{n+1}, \dots, y_m, X) \wedge$
$\wedge E_n(x_1, \dots, x_n, X))$,
where $\{x_1, \dots, x_n\} \cap \{y_{n+1}, \dots, y_m\} = \emptyset$;
(2) Axiom of extensionality
$\forall x_1, \dots, x_n \; (E_n(x_1, \dots, x_n, X) \leftrightarrow E_n(x_1, \dots, x_n, Y)) \leftrightarrow X = Y$;
(3) Axiom of specification
  (a) $\forall^n \vec{x} \; (\forall^m \vec{y} y_1, \dots, y_m E_{n+m}(\vec{x}, \vec{y}, C_R) \leftrightarrow E_{n+m}(\vec{x}, \vec{c}, C_R))$
  for each atomic formula $R(x_1, \dots, x_n, c_1, \dots, c_m)$ of $K_{\mathbb{A}}$;
  (b) $\forall^n \vec{x} \; (E_n(\vec{x}, C_{\neg\varphi}) \leftrightarrow \neg E_n(\vec{x}, C_\varphi))$;
  (c) $\forall^n \vec{x} \; (E_n(\vec{x}, C_{\wedge\Phi}) \leftrightarrow \bigwedge\limits_{\varphi\in\Phi} E_n(\vec{x}, C_\varphi))$;
  (d) $\forall^n \vec{x} \; (E_n(\vec{x}, C_{\forall x\varphi}) \leftrightarrow \forall x E_{1+n}(x, \vec{x}, C_\varphi))$;
  (e) $\forall^n \vec{x} \; (E_n(\vec{x}, C_{\exists x\varphi}) \leftrightarrow \exists x E_{1+n}(x, \vec{x}, C_\varphi))$;
  (f) $\forall^m \vec{x} \; \big(E_m(\vec{x}, C_{O^n \vec{y}\varphi}) \leftrightarrow \exists X (O^n(X) \wedge$
  $\wedge \forall^n \vec{y} (E_n(\vec{y}, X) \leftrightarrow E_{n+m}(\vec{y}, \vec{x}, C_\varphi)))\big)$;
  (g) $\forall^m \vec{x} \; \big(E_m(\vec{x}, C_{C^n \vec{y}\varphi}) \leftrightarrow \exists X (C^n(X) \wedge$
  $\wedge \forall \vec{y} (E_n(\vec{y}, X) \leftrightarrow E_{n+m}(\vec{y}, \vec{x}, C_\varphi)))\big)$;
(4) Axiom of subbases
  (a) $\forall^n \vec{x} \; \exists X \; \big(O^n(X) \wedge E_n(\vec{x}, X)\big)$;
  [see $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$-axiom (12)]
  (b) $\forall X \; \exists Y \; \big(\exists^n \vec{x} E_n(\vec{x}, X) \to C^n(Y) \wedge \forall^n \vec{x} (E_n(\vec{x}, X) \to E_n(\vec{x}, Y))\big)$;
  [see $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$-axiom (13)]
  (c) $\forall X \; \forall Y \; \forall Z \; \big(O^n(X) \wedge C^n(Y) \to \big((\forall^n \vec{x} \; (E_n(\vec{x}, Z) \leftrightarrow$
  $\leftrightarrow \neg E_n(\vec{x}, Y) \wedge E_n(\vec{x}, X)) \to O^n(Z))\big)\big)$;
  [see $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$-axiom (6)]
  (d) $\forall X \; \forall Y \; \forall Z \; \big(O^n(X) \wedge C^n(Y) \to \big((\forall^n \vec{x} \; \big(E_n(\vec{x}, Z) \leftrightarrow$
  $\leftrightarrow \neg E_n(\vec{x}, X) \wedge E_n(\vec{x}, Y)\big) \to C^n(Z))\big)\big)$;
  [see $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$-axiom (6)]
(5) Axioms of complete topological product and continuity

(a) $\forall X\, \forall Y\, \forall Z\, \big(O^n(X) \wedge O^m(Y) \wedge \forall^{n+1}\vec{x}\, \big(E_{n+m}(\vec{x}, Z) \leftrightarrow$
$\leftrightarrow E_n(x_1, \dots, x_n, X) \wedge E_m(x_{n+1}, \dots, x_{n+m}, Y)\big) \to O^{n+m}(Z)\big);$
[see $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$-axiom (9)]

(b) $\forall X\, \forall Y\, \big(O^n(X) \wedge \forall x_{i_1}, \dots, x_{i_k}\, \big(E_k(x_{i_1}, \dots, x_{i_k}, Y) \leftrightarrow$
$\leftrightarrow E_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}, X)\big) \to O^k(Y)\big),$
$\sigma\colon \{1, \dots, n\} \to \{1, \dots, n\}$ and the range of σ is $\{i_1 < i_2 < \dots < i_k\}$;
Similarly for the predicates C^n;
[see $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$-axiom (10)]

(c) $\forall X\, \forall Y\, \big(O^n(X) \wedge \forall^k\vec{x}\, \big(\exists y_{k+1}, \dots, y_n\, E_n(\vec{x}, y_{k+1}, \dots, y_n, X) \to$
$\to \forall x_{k+1}, \dots, x_n\, \big(E_{n-k}(x_{k+1}, \dots, x_n, Y) \leftrightarrow$
$\leftrightarrow E_n(x_1, \dots, x_n, X))\big) \to O^{n-k}(Y)\big);$
[see $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$-axiom (11)]

(d) $\forall X \forall^n\vec{x}\Big(O^n(X) \wedge E_n(\vec{x}, X) \to \exists Y_1 \dots \exists Y_n\, \Big(\bigwedge_{i=1}^{n} \Big(O(Y_i) \wedge E_1(x_i, Y_i)\Big) \wedge$
$\wedge \forall^n\vec{y}\, \Big(\bigwedge_{i=1}^{n} E_1(y_i, Y_i) \to E_n(\vec{y}, X)\Big)\Big)\Big);$
[see $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$-axiom (15)]

(e) $\forall X \forall^n\vec{x}\Big(C^n(X) \wedge \neg E_n(\vec{x}, X) \to \exists Y_1 \dots \exists Y_n \Big(\bigwedge_{i=1}^{n} \Big(O(Y_i) \wedge E_1(x_i, Y_i)\Big) \wedge$
$\wedge \forall^n\vec{y}\, \Big(\bigwedge_{i=1}^{n} E_1(y_i, Y_i) \to \neg E_n(\vec{y}, X)\Big)\Big)\Big);$
[see $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$-axiom (16)]

(f) $\forall X\, \forall Y\, \forall Z\, \big(O^m(X) \wedge O^n(Z) \to$
$\to \quad \big(\forall z_1, \dots, z_n, y_{k+1}, \dots, y_m\, \big(\big(\exists y_1, \dots, y_k(E_m(y_1, \dots, y_m, X) \wedge$
$\wedge E_{n+k}(z_1, \dots, z_n, y_1, \dots, y_k, C_\varphi)) \wedge E_n(z_1, \dots, z_n, Z) \leftrightarrow$
$\leftrightarrow E_{m+n-k}(z_1, \dots, z_n, y_{k+1}, \dots, y_m, Y)\big) \to O^{m+n-k}(Y)\big)\big)\big),$
for each (n, k)-ary relation φ;
[see $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$-axiom (14)]

(6) Axioms which are transformations of all axioms φ of $K_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$
$\forall x_1, \dots, x_n\, E_n(x_1, \dots, x_n, C_\varphi)$;

(7) Axioms of realizability of all sentences φ in Γ
$\forall x\, E_1(x, C_\varphi)$.

The theory S is Σ-definable over $\mathbb{A}$ and each $S_0 \subseteq S$, $S_0 \in \mathbb{A}$ has a standard model since the axiom

$$\bigwedge_n \bigwedge_{\varphi \in (S_0')_{n+k-1}} \forall^n\vec{x} \bigvee_m \bigvee_{\psi \in (S_0')_{m+k-1}} \exists^m\vec{y}\big(C^k\vec{z}\psi(\vec{z}, \vec{y}) \wedge \forall^k\vec{u}\big(\varphi(\vec{u}, \vec{x}) \to \psi(\vec{u}, \vec{y})\big)\big)$$

holds in the weak model $(\mathcal{K}, \mathbb{T}_n, \mathbb{C}_n)_{n\in\omega}$, which can be transformed to a standard model for S_0 (see [**7**]), where $S_0 \subseteq S_0'$, $S_0' \in \mathbb{A}$ is closed under substitution of the constant symbol from $\mathbb{K}$ and disjunction, and

$$(S_0')_n = \{\varphi \in S_0' \mid \varphi \text{ has } n+1 \text{ free variables}\}.$$

It follows by means of the Barwise compactness theorem (see [**1**]) that S has a standard model $\mathcal{B}$, which can be transformed to a middle model $(\mathcal{K}, \mathbb{T}_n, \mathbb{C}_n)_{n\in\omega}$ of Γ, similarly as in the logic $L_{\mathbb{A}}(O, C)$. $\square$

Let $(\mathcal{K}, \mathbb{T}'_n, \mathbb{C}'_n)_{n\in\omega}$ be the middle model of T where T is Σ_1-definable over $\mathbb{A}$ consistent with the axioms of $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$. Classes $\mathbb{T}'_n$ and $\mathbb{C}'_n$ for each $n \in \omega$ satisfy conditions 2.1, so we can form the classes

$$\overline{\mathbb{T}}_n = \{x \mid x \text{ is a finite intersection of members of } \mathbb{T}'_n\}$$

and

$$\overline{\mathbb{C}}_n = \{x \mid x \text{ is a finite unionof members of } \mathbb{C}'_n\}.$$

As in the logic $L_{\mathbb{A}}(O, C)$ we can prove that

$$(\mathcal{K}, \mathbb{T}'_n, \mathbb{C}'_n)_{n\in\omega} \equiv_{L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}} (\mathcal{K}, \mathbb{T}_n, \mathbb{C}_n)_{n\in\omega},$$

where $\mathbb{T}_n = \{\bigcup x \mid x \in \overline{\mathbb{T}}_n\}$ and $\mathbb{C}_n = \{\bigcap x \mid x \in \overline{\mathbb{C}}_n\}$ for each $n \in \omega$ using the sentences θ^{O^k} and θ^{C^k} for each formula $\theta(z_1, \dots, z_n, y_1, \dots, y_n)$ of $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$. The given structure will be a complete topological class model for T. We have thus proved the following theorem.

THEOREM 4.2 (Completeness Theorem for $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$). *If T is Σ_1-definable over $\mathbb{A}$ and consistent with the axioms of $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$, then there is a complete topological class model of T.*

5. Conclusion

We introduced the logic $L_{\mathbb{A}}(O^n, C^n)_{n\in\omega}$ related to topological class-spaces and suitable for studying continuous functions and product class-topologies. A sound and complete axiomatization is given, and the completeness theorem is proved. The proof illustrates the power of Rašković's middle-model method in producing new results. In general, it is worth noticing that Barwise compactness and the great expressive power of infinitary logics can provide more subtle models that can be obtained from ordinary compactness of the first-order logic.

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