PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 100(114)) (2016), 77–86 DOI: 10.2298/PIM1614077B

INFERENCE RULES FOR PROBABILITY LOGIC

Marija Boričić

Abstract. Gentzen's and Prawitz's approach to deductive systems, and Carnap's and Popper's treatment of probability in logic were two fruitful ideas of logic in the mid-twentieth century. By combining these two concepts, the notion of sentence probability, and the deduction relation formalized by means of inference rules, we introduce a system of inference rules based on the traditional proof-theoretic principles enabling to work with each form of probabilized propositional formulae. Namely, for each propositional connective, we define at least one introduction and one elimination rule, over the formulae of the form $A[a, b]$ with the intended meaning that 'the probability c of truthfulness of a sentence A belongs to the interval $[a, b] \subseteq [0, 1]$'. It is shown that our system is sound and complete with respect to the Carnap–Poper-type probability models.

1. Introduction

The need for formal consideration of 'probability of a proposition' is as old as the need for formalism covering the truthfulness of propositions. The interactions between logic and probability theory inspired many fruitful ideas [**1**,**5**–**7**,**15**,**22**,**25**]. The recent authors deal usually with Hilbert-style extensions of propositional calculi by probability operators making it possible to manipulate formally with proposition probabilities [**12**–**14**, **16**, **19**, **24**, **26**]. On the other side, the investigations of probability logic calculi in natural deduction style are very rare [**8**], although there are interesting papers treating other kinds of approximate reasoning through the inference rules and deduction relation [**10**, **11**, **21**].

In this paper we present a probability logic based on classical propositional calculus enabling one to work with formulae of the form $A[a, b]$ meaning that 'the probability c of truthfulness of a sentence A belongs to the interval $[a, b]$'. We present a syntactic system of inference rules which is sound and complete with respect to the Carnap–Popper-type probabilistic semantics [**6**, **16**, **17**, **22**]. Let us emphasize that, although Poper and Carnap do not belong to the same school of

2010 *Mathematics Subject Classification:* 03B48, 03B50, 03B05.

Key words and phrases: consistency; inference rules; probability; soundness; completeness.

This research was supported in part by the Ministry of Science and Technology of Serbia, grant number 174026.

Communicated by Žarko Mijajlović.

thought, in this field they have absolutely common ideas and equivalent approaches [**16**]. Our paper is organized as follows. The formalism of a probabilized natural deduction system **NKprob** is followed by the notions of consistency and **NKprob**-theory. The system **NKprob** consists of inference rules covering each propositional connective by at least one introduction rule, and one elimination rule, with the best possible probability bounds. Probabilistic models, inspired with Carnap–Poper's traditional approach, are accompanied by soundness and completeness results, and presented in the concluding part of the paper.

2. A system of probabilized natural deduction NKprob

In this paragraph we define and introduce the basic syntactic elements of deduction system **NKprob**. We suppose that the set of *propositional formulae* is defined inductively over a denumerable set of propositional letters and basic propositional connectives: $\neg$, $\wedge$, $\vee$ and $\rightarrow$. We also suppose that I is a finite subset of reals $[0,1]$ containing 0 and 1, closed under addition, meaning that, for instance, $a+b$ denotes the $\min(1, a+b)$, and $a+b-1$ denotes the $\max(0, a+b-1)$. An example of such a set is $I = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\}$, for each $n \in \mathbf{N}$. Also, let us note that the intersection of any two intervals of such form is an interval, too. Latin capital letters $A, B, C, \dots$, with or without subscripts, are metavariables for formulae, and $A \leftrightarrow B$, $\top$ and $\bot$, are used as abbreviations for $(A \rightarrow B) \wedge (B \rightarrow A)$, $A \vee \neg A$ and $A \wedge \neg A$, respectively.

Definition 2.1. For each propositional formula A and each $a, b \in I$, the object $A[a,b]$ is called a *probabilized formula*.

Particular cases of a probabilized formula $A[a,b]$ will be $A[a,a]$, for $a=b$, and $A\emptyset$, for $b<a$. Let us point out that $A[a,b]$ denotes $A[c,c]$, for some $c \in [a,b]$. As I is finite, each propositional formula A generates a finite list of probabilized formulae $A[a,b]$, $a, b \in I$. On the other side, we emphasize that combining probabilistic formulae by means of propositional connectives is not allowed.

Our system is defined over the set of all probabilized formulae.

The system **NKprob** has the following two types of axioms:

For each propositional formula A provable in classical logic (i.e. in Gentzen's original **NK**), the probabilized formula $A[1,1]$ is an axiom of the system **NKprob**.

The system **NKprob** consists of the following inference rules:

For each propositional connective we define the corresponding introduction (I-) and elimination (E-)rules, for
conjunction:

$$\frac{A[a,b] \quad B[c,d]}{(A \wedge B)[a+c-1, \min(b,d)]}(I\wedge), \qquad \frac{A[a,b] \quad (A \wedge B)[c,d]}{B[c, 1+d-a]}(E\wedge)$$

disjunction:

$$\frac{A[a,b] \quad B[c,d]}{(A \vee B)[\max(a,c), b+d]}(I\vee), \qquad \frac{A[a,b] \quad (A \vee B)[c,d]}{B[c-b, d]}(E\vee)$$

implication:

$$\frac{A[a,b] \quad B[c,d]}{(A\to B)[\max(1-b,c),1-a+d]}(I\to),\qquad \frac{A[a,b] \quad (A\to B)[c,d]}{B[a+c-1,d]}(E_1\to),$$
$$\frac{B[a,b] \quad (A\to B)[c,d]}{A[1-d,1-c+b]}(E_2\to)$$

negation:

$$\frac{A[a,b]}{(\neg A)[1-b,1-a]}(I\neg),\qquad \frac{(\neg A)[a,b]}{A[1-b,1-a]}(E\neg)$$

additivity rule:

$$\frac{A[a,b] \qquad B[c,d] \qquad (A\wedge B)[e,f]}{(A\vee B)[a+c-f,b+d-e]}(ADD)$$

which can be considered an additional case treating $(E\wedge)$ and $(I\vee)$ rules, two *monotonicity rules*:

$$\frac{A[a,b] \quad A[c,d]}{A[\max(a,c),\min(b,d)]}(M\downarrow),\qquad \frac{A[a,b]}{A[c,d]}(M\uparrow)$$

where, for $(M\uparrow)$ it is supposed that $[a,b]\subseteq[c,d]$ and, finally, two rules regarding *inconsistency*:

$$\frac{\dfrac{[A[c_1,c_1]]}{A\emptyset}\ \dfrac{[A[c_2,c_2]]}{A\emptyset}\ \cdots\ \dfrac{[A[c_m,c_m]]}{A\emptyset}}{A\emptyset}(I\emptyset),\qquad \frac{A\emptyset}{B[a,b]}(E\emptyset)$$

for any propositional formulae A and B, and any $a,b\in I=\{c_1,c_2,\dots,c_m\}$.

Note that the rules $(E_1\to)$ and $(E_2\to)$ are the two known probabilistic versions of *modus ponens* and *modus tollens*, respectively [**13**, **26**]. These two rules are interderivable, but for the reasons of system comfortability we keep both of them as basic rules. Also, from $A[a,b]$ and $A[c,d]$, by $(M\downarrow)$, in case when $[a,b]\cap[c,d]=\emptyset$, we infer $A\emptyset$, a conclusion connected with the inconsistency, which will be discussed in the next paragraph. The rule $(I\emptyset)$ provides that, for each propositional formula A, the probabilized formula $A[0,1]$ can be treated as an axiom of **NKprob**, with the meaning that each probabilized formula $A[c,c]$ holds, for some $c\in[0,1]$. The rule $(E\emptyset)$ enables to treat inconsistency in a traditional way.

We use the usual inductive definition of derivation from hypotheses:

Definition 2.2. We say that a formula $A[a,b]$ can be *derived from the set of hypotheses* Γ *in* **NKprob** if there is a finite sequence of probabilized formulae ending with $A[a,b]$, such that each formula is an axiom, it belongs to Γ or it is obtained by an **NKprob**-rule applied on some previous formulae of this sequence.

Particularly, the double lines appearing in the rule $(I\emptyset)$ denote the form of derivations just defined; each appearance of $[A[c,c]]$ means that it is possible to cancel $A[c,c]$ in the set of hypotheses Γ. More accurately, in this case the set of hypotheses is obtained by striking out some (or none, or all) occurrences, if any, of the formula $A[c,c]$ on the top of a derivation tree.

An alternative way to present the inference rules of **NKprob** could be in terms of the deduction relation. For example, the rules $(I\vee)$, $(E\vee)$ and $(I\emptyset)$ would have the following forms:

$$\frac{\Gamma\vdash A[a,b]\quad \Gamma\vdash B[c,d]}{\Gamma\vdash (A\vee B)[\max(a,c),b+d]}(I\vee),\qquad \frac{\Gamma\vdash A[a,b]\quad \Gamma\vdash (A\vee B)[c,d]}{\Gamma\vdash B[c-b,d]}(E\vee),$$

$$\frac{\Gamma\cup\{A[c_1,c_1]\}\vdash A\emptyset\quad \Gamma\cup\{A[c_2,c_2]\}\vdash A\emptyset\quad \dots\quad \Gamma\cup\{A[c_m,c_m]\}\vdash A\emptyset}{\Gamma\vdash A\emptyset}(I\emptyset),$$

where $\Gamma\vdash A[a,b]$ denotes the fact that formula $A[a,b]$ is derivable from the set of hypotheses Γ in **NKprob**.

Now, will show that each two classically equivalent formulae are equiprobable as well:

Lemma 2.1. *For any propositional formulae A and B, if $A\leftrightarrow B$ is provable in classical logic, and $A[a,b]$ is provable in* **NKprob***, then $B[a,b]$ is provable in* **NKprob**.

Proof. Note that if $A\leftrightarrow B$ is provable in classical logic, then, by definition, both $(A\to B)[1,1]$ and $(B\to A)[1,1]$ are the axioms of **NKprob**. From $A[a,b]$ and $(A\to B)[1,1]$, by *modus ponens* $(E_1\to)$, we infer $B[a,1]$. On the other side, from $A[a,b]$ and $(B\to A)[1,1]$, by *modus tollens* $(E_2\to)$, we infer $B[0,b]$. Finally, from $B[a,1]$ and $B[0,b]$, by $(M\downarrow)$, we have $B[a,b]$. □

Corollary 2.1. *For any propositional formulae A and B, if $A\leftrightarrow B$ is provable in classical logic, and $A[a,a]$ is provable in* **NKprob***, then $B[a,a]$ is provable in* **NKprob**.

3. NKprob-Theories

In order to obtain a good basis for completeness proof, we define the notion of a consistent theory.

Definition 3.1. By an **NKprob**-*theory* (or *theory*) we mean a set of formulas which are derivable from the set of hypothesis $\{A_1[a_1,b_1],\dots,A_n[a_n,b_n]\}$ in **NKprob**, denoted by $\mathbf{NKprob}(A_1[a_1,b_1],\dots,A_n[a_n,b_n])$. We say that a theory $\mathbf{NKprob}(A_1[a_1,b_1],\dots,A_n[a_n,b_n])$ is *inconsistent* if there is a proposition A such that both $A[a,b]$ and $A[c,d]$ are contained in $\mathbf{NKprob}(A_1[a_1,b_1],\dots,A_n[a_n,b_n])$, and $[a,b]\cap[c,d]=\emptyset$; otherwise, we say that it is *consistent*. A consistent theory is called a *maximal consistent theory* if each of its proper extensions is inconsistent.

Lemma 3.1. *Each consistent theory can be extended to a maximal consistent theory.*

Proof. Let $\mathcal{T}$ be a consistent theory, and let $A_0,A_1,\dots,A_n,\dots$ be the sequence of all propositional formulae and let $A_0[c,c],A_1[c,c],\dots,A_n[c,c],\dots,$ for each $c\in I$, be the sequence of the corresponding probabilized formulae. Let the sequence $(\mathcal{T}_n)$ of theories be defined inductively as follows: $\mathcal{T}_0=\mathcal{T}$, and $\mathcal{T}_{n+1}=\mathcal{T}_n\cup\{A_n[c_1,c_1]\}$, if it is consistent, but if it is not consistent, then: $\mathcal{T}_{n+1}=\mathcal{T}_n\cup\{A_n[c_2,c_2]\}$, if it is consistent, but if it is not, then... $\mathcal{T}_{n+1}=$

$\mathcal{T}_n \cup \{A_n[c_{m-1}, c_{m-1}]\}$, if it is consistent, and finally, $\mathcal{T}_{n+1} = \mathcal{T}_n \cup \{A_n[c_m, c_m]\}$, otherwise; where $\{c_1, c_2, \dots, c_m\} = I$. Let us note that the final result of this construction depends on the order of the points $c_1, c_2, \dots, c_m$ of the set I. Let $\mathcal{T}' = \bigcup_{n\in\omega} \mathcal{T}_n$. Then, by induction on n we will prove that $\mathcal{T}'$ is a maximal consistent extension of $\mathcal{T}$. First, we prove that if $\mathcal{T}_n$ is consistent, then $\mathcal{T}_{n+1}$ is consistent. The only interesting case is when $\mathcal{T}_{n+1} = \mathcal{T}_n \cup \{A_n[c_m, c_m]\}$, which is justified directly by the rule $(I\emptyset)$. In order to prove that $\mathcal{T}'$ is a *maximal* consistent extension of $\mathcal{T}$ we extend $\mathcal{T}'$ by the probabilized formula $A_k[a, b]$. In case that this is a proper extension, we already have that the theory $\mathcal{T}_{k+1} \subset \mathcal{T}'$ contains $A_k[c, c]$ for some $c \notin [a, b]$, and, consequently, this extension will be inconsistent. $\square$

4. Semantics, soundness and completeness

In this part we describe models for **NKprob**.

Definition 4.1. Let For be the set of all propositional formulae and I a finite subset of reals $[0, 1]$ closed under addition, containing 0 and 1. Then a mapping p : For $\to I$ will be an **NKprob**-*model* (or, simply, *model*), if it satisfies the following conditions:

(i) $p(\top) = 1$ and $p(\bot) = 0$;
(ii) if $p(A \wedge B) = 0$, then $p(A \vee B) = p(A) + p(B)$;
(iii) if $A \leftrightarrow B$ in classical logic, then $p(A) = p(B)$.

Let us emphasize that the above conditions roughly correspond to Carnap's and Popper's sentence probability axioms [**6**, **22**], variations of which were considered by Leblanc and van Fraassen [**16**–**18**] (see also [**3**, **4**]).

Justification of additivity and monotonicity rules is given by the following statement:

Lemma 4.1. (a) $p(A) + p(B) = p(A \vee B) + p(A \wedge B)$
(b) *If $A \to B$ in classical logic, then $p(A) \leqslant p(B)$.*

Proof. (a) From $p(A) = p((A \wedge \neg B) \vee (A \wedge B)) = p(A \wedge \neg B) + p(A \wedge B)$ and $p(A \vee B) = p((A \wedge \neg B) \vee B) = p(A \wedge \neg B) + p(B)$ we infer $p(A) + p(B) = p(A \wedge B) + p(A \vee B)$.

(b) $p(B) = p(\neg A \vee B) + p(\neg A \wedge B) - p(\neg A) = 1 + p(\neg A \wedge B) - 1 + p(A) \geqslant p(A)$, because $p(A \to B) = p(\neg A \vee B) = 1$. $\square$

Definition 4.2. Satisfiability in a model for the probabilized formulae is defined by clause

$$\models_p A[a, b] \text{ iff } a \leqslant p(A) \leqslant b$$

and we say that the probabilized formula $A[a, b]$ is *satisfied in the model p*. We say that a formula $A[a, b]$ is *valid* iff it is satisfied in each model, and this is denoted by $\models A[a, b]$.

I-rules will be justified by the following statement:

LEMMA 4.2. (a) $p(A) + p(B) - 1 \leqslant p(A \wedge B) \leqslant \min(p(A), p(B))$
(b) $\max(p(A), p(B)) \leqslant p(A \vee B) \leqslant p(A) + p(B)$
(c) $\max(1 - p(A), p(B)) \leqslant p(A \to B) \leqslant 1 - p(A) + p(B)$
(d) $p(\neg A) = 1 - p(A)$
The bounds in (a), (b), *and* (c) *are the best possible.*

PROOF. (a) Immediately, by Lemma 4.1, we have $p(A \wedge B) = p(A) + p(B) - p(A \vee B) \geqslant p(A) + p(B) - 1$. Also, we have $p(A) = p((A \wedge B) \vee (A \wedge \neg B)) = p(A \wedge B) + p(A \wedge \neg B)$, i.e., $p(A \wedge B) = p(A) - p(A \wedge \neg B) \leqslant p(A)$. Similarly we obtain $p(A \wedge B) \leqslant p(B)$.

(b) $p(A \vee B) = p(A) + p(\neg A \wedge B) \geqslant p(A)$, and similarly we have $p(A \vee B) \geqslant p(B)$. The upper bound is obtained by $p(A \vee B) = p(A) + p(B) - p(A \wedge B) \leqslant p(A) + p(B)$.

(c) This inequality follows from $p(A \to B) = p(\neg A \vee B)$ and part (b).

(d) $1 = p(\top) = p(A \vee \neg A) = p(A) + p(\neg A)$. □

A general approach to the best possible bounds computation of the probability of a logical function of events, based on linear programming methods, was developed by Hailperin (1965); according to him [**26**], Fréchet (1935) was first shown that the well known Boole's and Bonferroni's inequalities are the best possible.

E-rules correspond to the following statement:

LEMMA 4.3. (a) *If $a \leqslant p(A) \leqslant b$ and $c \leqslant p(A \wedge B) \leqslant d$, then $c \leqslant p(B) \leqslant d + 1 - a$.*
(b) *If $a \leqslant p(A) \leqslant b$ and $c \leqslant p(A \vee B) \leqslant d$, then $c - b \leqslant p(B) \leqslant d$.*
(c) *If $a \leqslant p(A) \leqslant b$ and $c \leqslant p(A \to B) \leqslant d$, then $a + c - 1 \leqslant p(B) \leqslant d$.*
(d) *If $a \leqslant p(\neg B) \leqslant b$ and $c \leqslant p(A \to B) \leqslant d$, then $a + c - 1 \leqslant p(\neg A) \leqslant d$.*
The bounds in (a), (b), (c) *and* (d) *are the best possible.*

PROOF. These inequalities follows from the observations below.

(a) We have that $p(B) = p(A \wedge B) + p(\neg A \wedge B) \geqslant c$ and $p(B) = p(A \vee B) + p(A \wedge B) - p(A) \leqslant d - a + 1$. For $p(\neg A \wedge B) = 0$ and $p(A \vee B) = 1$, respectively, the lower and upper bounds are reached, meaning that these bounds are the best possible.

(b) We have that $p(B) = p(A \vee B) - p(\neg B \wedge A) \leqslant d$ and $p(B) = p(A \vee B) + p(A \wedge B) - p(A) \geqslant c - b$. For $p(\neg B \wedge A) = 0$ and $p(A \wedge B) = 0$ the lower and upper bounds are reached.

(c) Using the part (b), Lemma 4.2 (d) and the fact that $p(A \to B) = p(\neg A \vee B)$ we obtain $a + c - 1 \leqslant p(B) \leqslant d$. For $p(\neg A \wedge B) = 0$ and $p(\neg A \wedge \neg B) = 0$ the lower and upper bounds are reached.

(d) Similarly as the part (c). □

Finally, we give a lemma, as formulated in [**2**], dealing with the *hypothetical syllogism* rule.

LEMMA 4.4. (a) *From $A[a, a]$, $B[b, b]$ and $C[c, c]$ in* **NKprob***, we can infer*

$$(A \to C)[\max(1 - a, b) + \max(1 - a, 1 - b, c) - 1, \min(1, 1 - b + c) + 1 - a].$$

(b) *From* $A[a,a]$, $C[c,c]$, $(A \to B)[r,r]$ *and* $(B \to C)[s,s]$ *in* **NKprob** *we can infer*

$$(A \to C)[\max(r-a, r+s-1), \min(s+1-a, r+c)].$$

(c) *From* $(A \to B)[a,b]$ *and* $(B \to C)[c,d]$ *in* **NKprob** *we can infer*

$$(A \to C)[\max(0, a+c-1), \min(b+d, 1)].$$

PROOF. (a) Let us consider the following Gentzen-type derivation:

$$\dfrac{\vdash A \to B \qquad \dfrac{A \vdash A \quad B \vdash C}{A, A \to B \vdash C}(*)}{A \vdash C}(**)$$

where the first step $(*)$ has done by the Gentzen's rule for 'introducing the implication in antecedent' $(\to\vdash)$ having the following general form:

$$\dfrac{A \vdash B \quad C \vdash D}{A, B \to C \vdash D}(\to\vdash)$$

while the step denoted by $(**)$ can be considered a particular case of the *cut rule*, i.e., *modus ponens* rule:

$$\dfrac{\vdash B \quad A, B \vdash C}{A \vdash C}$$

This derivation justifies that the hypothetical syllogism rule can be inferred as a logical consequence of the *modus ponens* rule. Let us us denote the probability of $(A \to ((A \to B) \to C))$ as $p(A \to ((A \to B) \to C)) = t$. Then, we have

$$\max(1-a, b) \leqslant r \leqslant \min(1, 1-a+b), \tag{4.1}$$

$$\max(1-b, c) \leqslant s \leqslant \min(1, 1-b+c), \tag{4.2}$$

$$(A \to C)[\max(1-a, c), \min(1, 1-a+c)] \tag{4.3}$$

From

$$\begin{aligned} t &= p(A \wedge (A \to B) \to C) = p(\neg A \vee \neg B \vee C) \\ &= p(B \to C) + p(\neg A) - p(\neg A \wedge (B \to C)) \end{aligned}$$

we conclude

$$\max(1-a, s) \leqslant t \leqslant s+1-a \tag{4.4}$$

On the other hand, bearing in mind Hailperin's result [**13**] related to the probability analogue of *modus ponens* rule, i.e., its particular case denoted by $(**)$, we have

$$(A \to C)[r+t-1, t] \tag{4.5}$$

From (4.2) and (4.4) we infer

$$\max(1-a, 1-b, c) \leqslant t \leqslant \min(1, 1-b+c) + 1 - a \tag{4.6}$$

and finally, from (4.1), (4.3), (4.5) and (4.6), we have

$$\max(1-a, b) + \max(1-a, 1-b, c) - 1 \leqslant p(A \to C) \leqslant \min(1, 1-a+c)$$

(b) From the previous relation, immediately, we have

$$(A \to C)[\max(r-a, r+s-1), \min(s+1-a, r+c, 1)]$$

bearing in mind $1-a \leqslant r$, and $1-a+c \leqslant r+c$.

(c) As an immediate consequence of (a) and (b), bearing in mind that $\max(r-a, r+s-1) \geqslant r+s-1$ and $\min(s+1-a, r+c, 1) \leqslant r+c \leqslant r+s$, we obtain $(A \to C)[\max(0, a+c-1), \min(b+d, 1)]$. □

Note that this Lemma generalizes and contains as its particular subcases both results Hailperin's *modus ponens* probabilized [**13**] and Wagner's *modus tollens* probabilized [**26**].

As an immediate consequence of the lemmata above, we obtain the soundness:

THEOREM 4.1. *If an* **NKprob**-*theory has a model, then it is consistent.*

PROOF. By induction on the length of the proof for any formula $A[a, b]$ provable in **NKprob** we can prove that the system **NKprob** is sound; this fact follows immediately from our justification of inference rules through Lemmata 4.2 and 4.3. Let us point out that the rule $(I\emptyset)$ is justified simply by the assumption that our model consists of a mapping $p : \mathrm{For} \to I$. Consequently, any satisfiable set of formulae $\{\sigma_1, \dots, \sigma_n\}$ is consistent with respect to **NKprob**. □

In order to prove the completeness part, we define the notion of canonical model. Let $\mathrm{Cn}\,\mathbf{NKprob}(\sigma_1, \dots, \sigma_n))$ be the set of all $\mathbf{NKprob}(\sigma_1, \dots, \sigma_n)$-provable formulae and $\mathrm{ConExt}(\mathrm{Cn}(\mathbf{NKprob}(\sigma_1, \dots, \sigma_n)))$ the class of all its maximal consistent extensions, existing by Lemma 3.1.

DEFINITION 4.3. For any $X \in \mathrm{ConExt}(\mathrm{Cn}(\mathbf{NKprob}(\sigma_1, \dots, \sigma_n)))$ we define

$$\models_{p^X} A[a, b] \quad \text{iff} \quad a \leqslant \max\{c | A[c, 1] \in X\} \quad \text{and} \quad b \geqslant \min\{c | A[0, c] \in X\};$$

p^X is called a emphcanonical model.

Obviously, such a definition provides that the mapping p^X, depending on X, has the adequate values.

For short, below we omit X from the denotation for p^X. Then, we also have:

LEMMA 4.5. *The canonical model is a model.*

PROOF. Let p be any mapping defined as above on an arbitrary maximal consistent set of any **NKprob**-theory. Then:

(1) We have that $p(A) = 1$, for each formula A which is classically provable, since $A[1, 1]$ is an axiom.

(2) If $p(A \wedge B) = 0$, $p(A) = a$ and $p(B) = b$, then immediately, by additivity rule, we infer $p(A \wedge B) = a + b$, for any formulae A and B.

(3) From Corollary 2.1 we have that if the formulae A and B are classically equivalent, then the formulae A and B have the same probabilities in the canonical model. □

LEMMA 4.6. $\models_{p^X} A[a, b]$ *iff* $A[a, b] \in X$.

PROOF. From $\models_{p^X} A[a, b]$, by the above definition and the monotonicity rule $(M\uparrow)$, we have $A[a, 1] \in X$. Similarly, $A[0, b] \in X$, wherefrom, by the monotonicity rule $(M\downarrow)$, we conclude $A[a, b] \in X$. Conversely, if $A[a, b] \in X$, then, as X is deductively closed and by the monotonicity rule $(M\uparrow)$, $A[a, 1] \in X$, so

$a \leqslant \max\{c|A[c,1] \in X\}$. Similarly, $A[0,b] \in X$, so $b \geqslant \min\{c|A[0,c] \in X\}$. Consequently, we have $\models_{p^X} A[a,b]$. □

Finally, as an immediate consequence of Lemmas 3.1, 4.5 and 4.6, we obtain the completeness of **NKprob**:

THEOREM 4.2. *Each consistent* **NKprob**-*theory has a model.*

PROOF. Bearing in mind that each consistent theory can be extended to a maximal consistent theory, and that such theory is described exactly as the canonical model, it suffices to prove that for each consistent theory there is a world of the canonical model satisfying it. □

5. Concluding remarks

Natural deduction systems introduced by Gentzen [**9**] and developed by Prawitz [**23**] were the antipode to the Hilbert style presenting logical systems in the sense that in the later case the axioms have a dominant role, while in the first one the inference rules dominate. Each one of these approaches to logical systems presentation has some advantages. From the proof-theoretical view point, the natural deductions facilitate immediate work with propositions by means of inference rules, and this was the reason that we develop a system making it easy to operate with probabilistic forms. Our approach, potentially applicable in the form of theories with probabilized propositions, has, as its main justification, soundness and completeness results respecting traditional Carnap's [**6**] and Popper's [**22**] type probability semantics.

Acknowlogment. The author thanks the anonymous referees of Publications de l'Institute Mathématique and the members of the Seminar for Probabilistic Logic of Mathematical Institute of SANU, Belgrade, and particularly professor Aleksandar Perović who suggested the inclusion of the rule $(I\emptyset)$ instead of a non-elegant form of axiom $A[0,1]$.

References

1. E. Adams, *The Logic of Conditionals*, Reidel, Dordrecht, 1975.
2. M. Boričić, *Hypothetical syllogism rule probabilized*, Bull. Symb. Log. **20**(3) (2014), 401–402, Abstract, Logic Colloquium 2012.
3. ———, *Models for the probabilistic sequent calculus*, Bull. Symb. Log. **21**(1) (2015), 60, Abstract, Logic Colloquium 2014.
4. ———, *Suppes–style sequent calculus for probability logic*, J. Log. Comput., (to appear)
5. G. Boole, *The Laws of Thought* Books, New York, 2003. (1st ed. 1854.)
6. R. Carnap, *Logical Foundations of Probability*, University of Chicago Press, Chicago, 1950.
7. M. Fréchet, *Généralisations du théoréme des probabilités totales*, Fundam. Math. **25** (1935), 379–387.
8. A. M. Frisch, P. Haddawy, *Anytime deduction for probabilistic logic*, Artif. Intell. **69** (1993), 93–122.
9. G. Gentzen, *Untersuchungen über das logische Schliessen*, Math. Z. **39** (1934-35), 176–210, 405–431. (or G. Gentzen, *Collected Papers*, (ed. M. E. Szabo), North–Holland, Amsterdam, 1969)

10. G. Gerla, R. Tortora, *Fuzzy natural deduction*, Z. Math. Logik Grundlagen Math. **36** (1990), 67–77.
11. G. Gerla, *Comparing fuzzy and crisp deduction systems*, Fuzzy Sets Syst. **67** (1994), 317–328.
12. T. Hailperin, *Best possible inequalities for the probability logical function of events*, Am. Math. Mon. **72** (1965), 343–359.
13. ———, *Probability logic*, Notre Dame J. Formal Logic **25** (1984), 198–212.
14. N. Ikodinović, *Craig interpolation theorem for classical propositional logic with some probability operators*, Publ. Inst. Math., Nouv. Sér. **69(83)** (2001), 27–33.
15. J. M. Keynes, *A Treatise on Probability*, Cambridge University Press, Cambridge, 2013. (1st ed. 1921)
16. H. Leblanc, B. C. van Fraassen, *On Carnap and Popper probability functions*, J. Symb. Log. **44** (1979), 369–373.
17. H. Leblanc, *Popper's 1955 axiomatization of absolute probability*, Pac. Philos. Quart. **69** (1982), 133–145.
18. ———, *Probability functions and their assumption sets — the singulary case*, J. Philos. Log. **12** (1983), 382–402.
19. Z. Ognjanović, M. Rašković, *A logic with higher order probabilities*, Publ. Inst. Math., Nouv. Sér. **60(74)** (1996), 1–4.
20. Z. Ognjanović, M. Rašković, Z. Marković, *Probability logics*, Z. Ognjanović (ed.); in: *Logic in Computer Science*, Zb. Rad. **12(20)**, Matematički institut SANU, Belgrade, 2009, 35–111.
21. J. Pavelka, *On fuzzy logic I: Many–valued rules of inference*, Z. Math. Logik Grundlagen Math. **25** (1979), 45–52.
22. K. R. Popper, *Two autonomous axiom systems for the calculus of probabilities*, Br. J. Philos. Sci. **6** (1955), 51–57, 176, 351.
23. D. Prawitz, *Natural Deduction. A Proof–theoretical Study*, Almquist and Wiksell, Stockholm, 1965.
24. M. Rašković, *Classical logic with some probability operators*, Publ. Inst. Math., Nouv. Sér. **53(67)** (1993), 1–3.
25. J. Venn, *The Logic of Chance*, Chelsea Publishing Company, New York, 1962. (1st ed. 1866.)
26. C. G. Wagner, *Modus tollens probabilized*, Br. J. Philos. Sci. **54**(4) (2004), 747Ű-753.

Faculty of Organizational Sciences
University of Belgrade
Belgrade
Serbia
marija.boricic@fon.bg.ac.rs

(Received 07 04 2015)
(Revised 30 06 2016)