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PRODUCT OF DIFFERENTIATION AND COMPOSITION OPERATORS ON BLOCH TYPE SPACES

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Abstract. We obtain some simple criteria for the boundedness and compactness of the product of differentiation and composition operator $C_\varphi D^m$ on Bloch type spaces.

1. Introduction

Denote by $H(\mathbb{D})$ the space of all analytic functions on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ in the complex plane. A function $f \in H(\mathbb{D})$ is said to belong to Bloch type spaces $\mathcal{B}^\alpha$ (or called α-Bloch spaces) if $\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)^\alpha < \infty$, $0 < \alpha < \infty$.

A function $f \in H(\mathbb{D})$ is said to belong to the little Bloch type space $\mathcal{B}^\alpha_0$ (or the little α-Bloch space) if $\lim_{|z|\to 1} |f'(z)|(1 - |z|^2)^\alpha = 0, \quad 0 < \alpha < \infty$. The classical Bloch space $\mathcal{B}$ is just $\mathcal{B}^1$. It is clear that $\mathcal{B}^\alpha$ is a Banach space with the norm $\|f\| = |f(0)| + \|f\|_{\mathcal{B}^\alpha}$. See [**24**] for the theory of Bloch type spaces.

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ is defined by $C_\varphi(f) = f \circ \varphi$, $f \in H(\mathbb{D})$. The differentiation operator D is defined by $Df = f'$, $f \in H(\mathbb{D})$. For a nonnegative integer $m \in \mathbb{N}$, we define $D^m f = f^{(m)}$, $f \in H(\mathbb{D})$. It is natural to define the product of differentiation and composition operators $C_\varphi D^m$ by $C_\varphi D^m f = f^{(m)} \circ \varphi$, $f \in H(\mathbb{D})$. A basic problem concerning concrete operators on various Banach spaces is to relate their operator theoretic properties to the function theoretic properties of the participating symbols, which attracted a lots of attention recently (the reader can refer to [**1**]–[**27**]).

It is a well-known consequence of the Schwarz–Pick Lemma that the composition operator is bounded on the Bloch space. See [**10, 11, 18–20, 23**] for the study of the compactness of composition operator on the Bloch space.

Product-type operators attracted considerable interest recently. The product of differentiation and composition operators has been studied, for example, in [**2**,

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4, **7**–**9**, **13**–**17**, **19**, **21**]. For some other product-type operators acting on Bloch-type spaces, see, for example [**3**, **5**, **6**, **12**, **22**, **26**, **27**] and the related references therein.

In [**19**], Wu and Wulan obtained two nice characterizations for the compactness of the product of differentiation and composition operators acting on the Bloch space as follows.

THEOREM 1.1. *Let φ be an analytic self-map of $\mathbb{D}$ and $m \in \mathbb{N}$. Then the following statements are equivalent.*

(i) $C_\varphi D^m : \mathcal{B} \to \mathcal{B}$ *is compact.*
(ii) $\lim_{n\to\infty} \big\|C_\varphi D^m(z^n)\big\|_{\mathcal{B}} = 0$.
(iii) $\lim_{|a|\to 1} \big\|C_\varphi D^m\big(\frac{a-z}{1-\bar{a}z}\big)\big\|_{\mathcal{B}} = 0$.

The condition (ii) was extended to the Bloch type spaces by Liang and Zhou in [**9**]. Among other results, they proved the following one.

THEOREM 1.2. *Let $0 < \alpha, \beta < \infty$, m a nonnegative integer, φ a self-map of the unit disk $\mathbb{D}$. Suppose that $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. Then $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if $\lim_{n\to\infty} n^{\alpha-1}\big\|C_\varphi D^m(z^n)\big\|_{\mathcal{B}^\beta} = 0$.*

We will extend condition (iii) in Theorem 1.1 to the Bloch type spaces.

Let X and Y be two Banach spaces. Recall that an operator $T : X \to Y$ is said to be bounded if there exists a constant $C > 0$ such that $\|T(f)\|_Y \leqslant C\|f\|_X$. Moreover, $T : X \to Y$ is said to be compact if it takes bounded sets in X to sets in Y which have compact closure.

In addition, we say that $A \lesssim B$ if there exists a constant C such that $A \leqslant CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Main results

To prove our main results, we need the following two auxiliary lemmas.

LEMMA 2.1. [**24**] *For $f \in H(\mathbb{D}), m \in \mathbb{N}$ and $\alpha > 0$. Then $f \in \mathcal{B}^\alpha$ if and only if $\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+m-1}|f^{(m)}(z)| < \infty$. Moreover,*

$$\|f\| \approx \sum_{j=0}^{m-1} \big|f^{(j)}(0)\big| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha+m-1}\big|f^{(m)}(z)\big|.$$

The following lemma can be proved in a standard way, see e.g. [**1**, Prop. 3.11].

LEMMA 2.2. *Let m be a positive integer and $0 < \alpha, \beta < \infty$. Let φ be a holomorphic self-map of $\mathbb{D}$. Then $C_\varphi D^m$ is compact if and only if $C_\varphi D^m : B^\alpha \to \mathcal{B}^\beta$ is bounded and for any bounded sequence $\{f_n\}$ in $\mathcal{B}^\alpha$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, then $\|C_\varphi D^m f\|_{\mathcal{B}^\beta} \to 0$ as $n \to \infty$.*

Since the boundedness of $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ implies that $\varphi \in \mathcal{B}^\beta$, we always assume that $\varphi \in \mathcal{B}^\beta$. We are now ready for the main results in this section.

THEOREM 2.1. *Let $0 < \alpha, \beta < \infty, m \in \mathbb{N}$ and φ be a self-map of $\mathbb{D}$ such that $\varphi \in \mathcal{B}^\beta$. Then the following statements are equivalent.*

(i) $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is bounded.*

(ii) $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{B}^\beta$ *is bounded.*
(iii) $\sup_{n\in\mathbb{N}} n^{\alpha-1}\|C_\varphi D^m(z^n)\|_{\mathcal{B}^\beta} < \infty$.
(iv) $\sup_{a\in\mathbb{D}} \big\|C_\varphi D^m\big(\frac{1-|a|^2}{(1-\bar{a}z)^\alpha}\big)\big\|_{\mathcal{B}^\beta} < \infty$.
(v) $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}|\varphi'(z)| < \infty$.

PROOF. (i)⇒(ii). This implication is obvious because of $\mathcal{B}_0^\alpha \subset \mathcal{B}^\alpha$.

(ii)⇒(iii). Assume that $\|C_\varphi D^m\|_{\mathcal{B}_0^\alpha\to\mathcal{B}^\beta} < \infty$. Note that the sequence $\{n^{\alpha-1}z^n\}$ is bounded in the little α-Bloch space $\mathcal{B}_0^\alpha$ (see, e.g. [**23**]). There exists a constant C such that $n^{\alpha-1}\big\|C_\varphi D^m(z^n)\big\|_{\mathcal{B}^\beta} \leqslant C\big\|C_\varphi D^m\big\|_{\mathcal{B}_0^\alpha\to\mathcal{B}^\beta}\big\|n^{\alpha-1}z^n\big\|_{\mathcal{B}^\alpha} < \infty$, for any $n \in \mathbb{N}$. This implies (iii).

(iii)⇒(iv). Assume that (iii) holds. The MacLaurin expansion of $\frac{1-|a|^2}{(1-\bar{a}z)^\alpha}$ is given by

$$\frac{1-|a|^2}{(1-\bar{a}z)^\alpha} = \big(1-|a|^2\big)\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}\bar{a}^n z^n.$$

By Stirling's formula, we have $\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} \approx n^{\alpha-1}$, as $n\to\infty$. This gives

$$\begin{aligned}\left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\bar{a}z)^\alpha}\right)\right\|_{\mathcal{B}^\beta} &\lesssim (1-|a|^2)\sum_{n=m+1}^{\infty} n^{\alpha-1}|a|^n\|C_\varphi D^m(z^n)\|_{\mathcal{B}^\beta}\\ &\leqslant (1-|a|^2)\Big(\sup_{n\in\mathbb{N}} n^{\alpha-1}\|C_\varphi^n D^m(z^n)\|_{\mathcal{B}^\beta}\Big)\sum_{n=m+1}^{\infty}|a|^n\\ &\leqslant 2\sup_{n\in\mathbb{N}} n^{\alpha-1}\|C_\varphi^n D^m(z^n)\|_{\mathcal{B}^\beta} < \infty,\end{aligned}$$

from which the implication follows.

(iv)⇒(v). If (iv) holds, then

$$\sup_{z\in\mathbb{D}}\alpha(\alpha+1)\cdots(\alpha+m)|a|^{m+1}\frac{1-|a|^2}{|1-\bar{a}\varphi(z)|^{\alpha+m+1}}|\varphi'(z)|(1-|z|^2)^\beta < \infty, \tag{2.1}$$

for any $a\in\mathbb{D}$. If $\varphi(z)=0$, then (v) automatically holds since $\varphi\in\mathcal{B}^\beta$. When $|\varphi(z)|>0$, taking $a=\varphi(z)$ in (2.1) we see that (v) holds.

(v)⇒(i). Assume that (v) holds. For any given $f\in\mathcal{B}^\alpha$, by Lemma 2.1 we have

$$\begin{aligned}\|C_\varphi D^m f\|_{\mathcal{B}^\beta} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f^{(m+1)}(\varphi(z))\varphi'(z)|\\ &= \sup_{z\in\mathbb{D}}|f^{(m+1)}(\varphi(z))|(1-|\varphi(z)|^2)^{\alpha+m}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}|\varphi'(z)|\\ &\lesssim \|f\|_{\mathcal{B}^\alpha}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}|\varphi'(z)|.\end{aligned}$$

This implies that $C_\varphi D^m : \mathcal{B}^\alpha\to\mathcal{B}^\beta$ is bounded. □

THEOREM 2.2. *Let $0<\alpha,\beta<\infty, m\in\mathbb{N}$ and φ be a self-map of $\mathbb{D}$ such that $\varphi\in\mathcal{B}^\beta$. Then the following statements are equivalent.*

(i) $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is compact.*
(ii) $C_\varphi D^m : \mathcal{B}_0^\alpha \to \mathcal{B}^\beta$ *is compact.*
(iii) $\lim_{n\to\infty} n^{\alpha-1}\|C_\varphi D^m(z^n)\|_{\mathcal{B}^\beta} = 0$.
(iv) $\lim_{|a|\to 1^-} \left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\bar{a}z)^\alpha}\right)\right\|_{\mathcal{B}^\beta} = 0$.
(v) $\lim_{|\varphi(z)|\to 1^-} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}|\varphi'(z)| = 0$.

PROOF. (i)$\Rightarrow$(ii). This implication is obvious because of $\mathcal{B}_0^\alpha \subset \mathcal{B}^\alpha$.

(ii)$\Rightarrow$(iii). Assume that (ii) holds. Note that the sequence $\{n^{\alpha-1}z^n\}$ is bounded in the little α-Bloch space $\mathcal{B}_0^\alpha$ and converges to 0 uniformly on a compact subset of $\mathbb{D}$. Lemma 2.2 implies that (iii) holds.

(iii)$\Rightarrow$(iv). We now assume that condition (iii) holds. By the argument as in the proof of Theorem 2.1, we have

$$\left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\bar{a}z)^\alpha}\right)\right\|_{\mathcal{B}^\beta} \lesssim (1-|a|^2)\sum_{n=m+1}^{\infty} n^{\alpha-1}|a|^n\|C_\varphi D^m(z^n)\|_{\mathcal{B}^\beta}.$$

If (iii) holds, then for any given $\epsilon > 0$ there exists a positive integer $N(N > m+1)$ such that $n^{\alpha-1}\|C_\varphi D^m(z^n)\|_{\mathcal{B}^\beta} < \epsilon$ for all $n > N$. Combining this with (iii) we obtain

$$\left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\bar{a}z)^\alpha}\right)\right\|_{\mathcal{B}^\beta} \lesssim (1-|a|^2)\sum_{n=m+1}^{N} n^{\alpha-1}|a|^n\|C_\varphi D^m(z^n)\|_{\mathcal{B}^\beta} + 2\epsilon. \tag{2.2}$$

Since $\{n^{\alpha-1}\|C_\varphi^n D^m(z^n)\|_{\mathcal{B}^\beta}\}$ is a bounded sequence, letting $|a| \to 1$ in (2.2) we obtain the following inequality.

$$\left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\bar{a}z)^\alpha}\right)\right\|_{\mathcal{B}^\beta} \lesssim 2\epsilon, \quad \text{as} \quad |a| \to 1.$$

Since ϵ is arbitrary, we conclude that (iv) holds.

(iv)$\Rightarrow$(v). If (iv) holds, then for any given $\epsilon > 0$, there is a $\delta \in (0,1)$ such that

$$\sup_{z\in\mathbb{D}} \alpha(\alpha+1)\cdots(\alpha+m)|a|^{m+1}\frac{1-|a|^2}{|1-\bar{a}\varphi(z)|^{\alpha+m+1}}|\varphi'(z)|(1-|z|^2)^\beta < \epsilon$$

whenever $\delta < |a| < 1$. In particular, if $w \in \mathbb{D}$ satisfies $|\varphi(w)| > \delta$, then we get

$$\sup_{z\in\mathbb{D}} \alpha(\alpha+1)\cdots(\alpha+m)|\varphi(w)|^{m+1}\frac{1-|\varphi(w)|^2}{|1-\overline{\varphi(w)}\varphi(z)|^{\alpha+m+1}}|\varphi'(z)|(1-|z|^2)^\beta < \epsilon,$$

which implies that

$$\sup_{z\in\mathbb{D}} \alpha(\alpha+1)\cdots(\alpha+m)\delta^{m+1}\frac{1-|\varphi(w)|^2}{|1-\overline{\varphi(w)}\varphi(z)|^{\alpha+m+1}}|\varphi'(z)|(1-|z|^2)^\beta < \epsilon.$$

Taking $z = w$ in the above inequality, we get

$$\alpha(\alpha+1)\cdots(\alpha+m)\delta^{m+1}\frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha+m}}|\varphi'(w)| < \epsilon$$

when $|\varphi(w)| > \delta$. This gives that (v) holds.

(v) $\Rightarrow$ (i). Assume that (v) holds. For any given $\epsilon > 0$ there exists a δ such that

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}|\varphi'(z)| < \epsilon, \tag{2.3}$$

whenever $\delta < |\varphi(z)| < 1$. This together with the fact that $\varphi \in \mathcal{B}^\beta$ imply (iii). Hence $C_\varphi D^m : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded by Theorem 2.1.

Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{B}^\alpha$ that converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then

$$\|C_\varphi D^m f_n\|_{\mathcal{B}^\beta} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |f_n^{(m+1)}(\varphi(z))\varphi'(z)| \leqslant I_1(n) + I_2(n),$$

where

$$I_1(n) = \sup_{|\varphi(z)|\leqslant\delta}(1-|z|^2)^\beta |f_n^{(m+1)}(\varphi(z))\varphi'(z)|,$$
$$I_2(n) = \sup_{\delta<|\varphi(z)|<1}(1-|z|^2)^\beta |f_n^{(m+1)}(\varphi(z))\varphi'(z)|.$$

It follows easily from Cauchy's formula that $f_n^{(m+1)}$ converges to 0 uniformly on a compact subset of $\mathbb{D}$. Since $\varphi \in \mathcal{B}^\beta$, we have $I_1 \to 0$ as $n \to \infty$.

To estimate $I_2(n)$, we note that $\sup_{z\in\mathbb{D}} |f_n^{(m+1)}(\varphi(z))|(1-|\varphi(z)|^2)^{\alpha+m} \lesssim \|f\|_{\mathcal{B}^\alpha}$ by Lemma 2.1. It follows from (2.3) that

$$I_2(n) \lesssim \|f_n\|_{\mathcal{B}^\alpha} \sup_{\delta<|\varphi(z)|<1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}|\varphi'(z)| \lesssim \epsilon\|f_n\|_{\mathcal{B}^\alpha}.$$

Since ϵ is arbitrary, we obtain $\lim_{n\to\infty}\|C_\varphi D^m f_n\|_{\mathcal{B}^\beta} = 0$. This shows that $C_\varphi D^m$ from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ is compact by Lemma 2.2. $\square$

Remark 2.1. When we take some special value for m, α and β in Theorems 2.1 and 2.2, we can get many results, which have appeared in [**4**, **9**, **11**, **19**–**21**]. We omit the details.

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