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UMBRAL INTERPOLATION

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ABSTRACT. A general linear interpolation problem is posed and solved. This problem is called *umbral interpolation problem* because its solution can be expressed by a basis of Sheffer polynomials. The truncation error and its bounds are considered. Some examples are discussed, in particular generalizations of Abel–Gontscharoff and central interpolation are studied. Numerical examples are given too.

1. Introduction

In [**5**–**7**] an application of Appell and Δ_h-Appell polynomials to linear interpolation problem for real functions has been given. In this note we will extend this approach to the more general so-called Sheffer polynomials [**8**,**9**,**12**–**14**,**16**,**24**]. For this purpose, let X be the linear space of real functions defined in the interval $[a,b]$ continuous and with continuous derivatives of all necessary orders; let $\mathcal{P}_n \subset X$, $n \in \mathbb{N}$, be the space of polynomials of degree less than or equal to n. Let Q be a δ-operator [**20**] on $\mathcal{P}_n$, and L a linear functional on X with $L(1) \neq 0$. Let be $XQ = \{f \in X : Q^i f \in X,\ i = 0,\dots,n, \text{ for all } n \in \mathbb{N}\}$, then for each $f \in XQ$ we want a polynomial $P_n[f]$, if it exists, of degree less than or equal to n such that

$$f = P_n[f] + R_n[f],$$

with

$$L(Q^i P_n[f]) = L(Q^i f), \quad i = 0,\dots,n. \tag{1.1}$$

If $Q = D$ or $Q = \Delta_h$, that is the differential operator or the finite difference operator respectively, we have that XQ is the set of analytic functions on $[a,b]$ and, respectively, the set of bounded functions in $[a,b]$. In these cases problem (1.1) admits a unique solution and it has been called *Appell interpolation problem* [**1**,**5**] or Δ_h-*Appell interpolation problem* [**6**,**7**], respectively.

In this note we want to address the general case and we give the solution of problem (1.1) only if XQ is known. We will call this problem *umbral interpolation problem*, because its solution can be expressed by a basis of Sheffer polynomials,

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also said *umbral basis*. Interpolation and approximation have been studied from an umbral point of view in several papers [**10**, **15**, **17**]. In [**18**, **19**] the authors study the sequence of polynomials, which solve the linear interpolation in $\mathcal{P}_n$, but not the general case, moreover no connection with real function is given. The paper is organized as follows: in Section 2 we give some preliminary definitions and results; in Section 3 we define the umbral interpolation and provide its solution; in Section 4 we give, as examples, generalizations of Abel–Gontscharoff [**11**] and central interpolation [**22**]; in Section 5, some numerical examples, which justify theoretical results, are given; finally, in Section 6 conclusions and further developments are announced.

2. Umbral basis for (L, Q)

In order to make the work self-contained we recall some basic notations of the umbral calculus [**8**, **20**, **21**]. Let Q be a δ-operator and $(p_n)_{n\in\mathbb{N}}$ be the associated sequence [**20**], that is, the sequence that satisfies $p_0(x) = 1, \quad p_n(0) = 0, \quad Qp_n = np_{n-1},\ n = 1, 2, \ldots$. It is known [**20**] that $(p_n(x))_{n\in\mathbb{N}}$ is of binomial type and it is a basis for $\mathcal{P}_n$. Let L be a linear functional on X with $L(1) \neq 0$ and let us set

$$\beta_n = L(p_n), \quad n = 0, 1, \ldots \tag{2.1}$$

and define the sequence of polynomials

$$\begin{cases} s_0(x) = \frac{1}{\beta_0} \\ s_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} p_0(x) & p_1(x) & p_2(x) & \cdots & \cdots & p_{n-1}(x) & p_n(x) \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & & & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \end{cases} \quad n = 1, 2, \ldots. \tag{2.2}$$

REMARK 2.1. [**8**] The sequence $(s_n(x))_{n\in\mathbb{N}}$ is a sequence of polynomials of degree less than or equal to n, for which we have

$$s_n(x) = \sum_{i=0}^{n} \binom{n}{i} \alpha_{n-i} p_i(x), \quad n = 0, 1, \ldots,$$

with

$$\alpha_0 = \frac{1}{\beta_0}, \quad \sum_{i=0}^{n} \binom{n}{i} \alpha_i \beta_{n-i} = 0, \quad n = 1, 2, \ldots.$$

REMARK 2.2 (Recurrence relation, [**8**]). For the polynomial sequence $(s_n(x))_{n\in\mathbb{N}}$ the following recurrence relation holds

$$s_n(x) = \frac{1}{\beta_0}\left(p_n(x) - \sum_{i=0}^{n-1} \binom{n}{i} \beta_{n-i} s_i(x)\right), \quad n = 1, 2, \ldots. \tag{2.3}$$

REMARK 2.3. [**8**] For the polynomial sequence $(s_n(x))_{n\in\mathbb{N}}$ we have

$$Qs_n(x) = ns_{n-1}(x), \quad n = 1, 2, \ldots.$$

REMARK 2.4. We have

$$L(Q^i s_n(x)) = i!\delta_{i,n}, \quad i = 0, \ldots, n, \tag{2.4}$$

where $\delta_{i,n}$ is the Kronecker symbol.

PROOF. It follows from Remark 2.3, from which we have

$$L(Q^i s_n(x)) = n(n-1)\ldots(n-i+1)L(s_{n-i}(x)) = i!\binom{n}{i}\delta_{i,n} = i!\delta_{i,n}. \qquad \square$$

DEFINITION 2.1. The sequence $(s_n(x))_{n\in\mathbb{N}}$ is a basis for $\mathcal{P}_n$, and we call it umbral basis for (L, Q). In the following, often, we will omit for (L, Q).

REMARK 2.5. We note explicitly that the polynomial sequence $(s_n(x))_{n\in\mathbb{N}}$ defined in (2.2) is the Sheffer sequence [**21**] for $(\beta_n, p_n(x))$ as defined in [**8**].

3. Umbral interpolation

Let L be a linear functional on X with $L(1) \neq 0$, Q a δ-operator on $\mathcal{P}_n$ and $\omega_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$; then the problem

$$L(Q^i P_n) = i!\omega_i, \quad i = 0, \ldots, n, \quad P_n \in \mathcal{P}_n,$$

is called *umbral interpolation problem* in $\mathcal{P}_n$.

THEOREM 3.1. *Let Q be a δ-operator on $\mathcal{P}_n$ and L be a linear functional on X with $L(1) \neq 0$. Let $(s_n(x))_{n\in\mathbb{N}}$ be the umbral basis for (L, Q) and $\omega_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$. The polynomial $P_n(x) = \sum_{i=0}^{n} \omega_i s_i(x)$, is the unique polynomial of degree less than or equal to n such that $L(Q^i P_n) = i!\omega_i$, $i = 0, \ldots, n$.*

PROOF. It is a straightforward consequence of (2.4) and of the linearity of Q. $\square$

COROLLARY 3.1. *For each $P_n(x) \in \mathcal{P}_n$ we have*

$$P_n(x) = \sum_{i=0}^{n} \frac{L(Q^i P_n)}{i!} s_i(x).$$

Let us consider a function $f \in XQ$. Then we have the following

THEOREM 3.2 (Main theorem). *The polynomial*

$$P_n[f](x) = \sum_{i=0}^{n} \frac{L(Q^i f)}{i!} s_i(x) \tag{3.1}$$

is the unique polynomial of degree less than or equal to n such that

$$L(Q^i P_n[f]) = L(Q^i f), \quad i = 0, \ldots, n.$$

PROOF. It follows from Theorem 3.1. $\square$

DEFINITION 3.1. The polynomial $P_n[f](x)$ is called umbral interpolation polynomial of the function f for (L, Q).

Therefore, it is interesting to consider the estimation of the remainder

$$R_n[f](x) = f(x) - P_n[f](x),\ \forall x \in [a,b]. \tag{3.2}$$

THEOREM 3.3. *For any $f(x) \in \mathcal{P}_n$ and $x \in [a,b]$,*

$$R_n[f](x) = 0, \quad \text{and} \quad R_n[p_{n+1}(x)] \neq 0.$$

PROOF. It follows from (2.1) and (2.3), noting that

$$L(Q^i p_n(x)) = n(n-1)\dots(n-i+1)L(p_{n-i}(x)) = i!\binom{n}{i}\beta_{n-i}. \qquad \square$$

For a fixed x we may consider the remainder $R_n[f](x)$ as a linear functional and, therefore, from Peano's theorem [**11**, p. 69], we have

THEOREM 3.4. *Let $f \in C^{n+1}[a,b]$. The following relation holds*

$$\forall x \in [a,b], \quad R_n[f](x) = \frac{1}{n!}\int_a^b K_n(x,t) f^{(n+1)}(t)\,dt, \tag{3.3}$$

where $K_n(x,t) = R_n[(x-t)_+^n] = (x-t)_+^n - \sum_{i=0}^{n} \frac{L_x[Q^i((x-t)_+^n)]}{i!} s_i(x).$

PROOF. It follows by Theorem 3.3 and Peano's theorem. $\square$

REMARK 3.1. By (3.3), if $f^{(n+1)} \in \mathcal{L}^p[a,b]$ and $K_n(x,t) \in \mathcal{L}^q[a,b]$ with $\frac{1}{p} + \frac{1}{q} = 1$, applying the Hölder's inequality, classical error bounds can be obtained.

Now, let us fix $z \in [a,b]$ and consider the polynomial

$$\overline{P}_n[f,z](x) \equiv f(z) + P_n[f](x) - P_n[f](z) = f(z) + \sum_{i=1}^{n} \frac{L(Q^i f)}{i!}(s_i(x) - s_i(z)). \tag{3.4}$$

In the following, to avoid encumbering the notation, we will denote it by $\overline{P}_n[f](x)$, omitting the dependence on z. Then we have the following

THEOREM 3.5. *The polynomial $\overline{P}_n[f](x)$ is an approximating polynomial of degree n for $f(x)$, i.e.,*

$$\forall x \in [a,b],\ f(x) = \overline{P}_n[f](x) + \overline{R}_n[f](x), \tag{3.5}$$

with $\overline{R}_n[p_i(x)] = 0$, $i = 0, \dots, n$ and $\overline{R}_n[p_{n+1}(x)] \neq 0$.

PROOF. For each $x \in [a,b]$, by (3.2) we get (3.5); the exactness follows from the exactness of the polynomial $P_n[f](x)$. $\square$

THEOREM 3.6. *The polynomial $\overline{P}_n[f](x)$ satisfies the interpolatory conditions*

$$\overline{P}_n[f](z) = f(z), \quad L(Q^i \overline{P}_n[f]) = L(Q^i f), \quad i = 1, \dots, n.$$

PROOF. It follows from (2.4). $\square$

We call $\overline{P}_n[f](x)$ *umbral interpolation polynomial of second kind.*

4. Examples

4.1. Abel–Sheffer interpolation. With the previous notation, let be f analytic in $[a,b]$ and $Qf = D_a f = f'(x+a)$, $a \in \mathbb{R}$, $a \neq 0$. Then the associated sequence is the Abel sequence [**20**]

$$A_0(x,a) = 1, \quad A_n(x,a) = x(x-an)^{n-1}, \quad n = 1,2,\dots.$$

Now, let L be a linear functional verifying $L(1) \neq 0$. Then the umbral interpolation polynomials (3.1) and (3.4) become

$$P_n[f](x) = L(f) + \sum_{i=1}^{n} \frac{L(f^{(i)}(x+ai))}{i!} s_i(x), \tag{4.1}$$

and, setting $z = 0$,

$$\overline{P}_n[f](x) = f(0) + \sum_{i=1}^{n} \frac{L(f^{(i)}(x+ai))}{i!} (s_i(x) - s_i(0)),$$

where $s_i(x)$ is the umbral basis for (L, D_a).

Abel–Gontscharoff interpolation. Let $L(f) = f(x_0)$, $x_0 \in [a,b]$. Then the umbral basis for (L, D_a) is the sequence

$$\widetilde{G}_0(x) = 1, \quad \widetilde{G}_n(x,a) = (x-x_0)(x-x_0-an)^{n-1}, \quad n = 1,2,\dots,$$

that is the classical Abel–Gontscharoff sequence [**11**] on the equidistant points $x_i = x_0 + ai$, $i = 0,\dots,n$. Interpolation polynomial (3.1) becomes

$$\widetilde{G}_n[f](x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0+ai)}{i!} \widetilde{G}_i(x,a),$$

i.e., the umbral interpolation is the well-known Abel–Gontscharoff interpolation [**11**] on the equidistant points $x_i = x_0 + ai$, $i = 0,\dots,n$. Therefore (4.1) can be seen as a generalization of Abel–Gontscharoff interpolation on the equidistant points. For the remainder $R_{L,n}[f](x) = \widetilde{G}_n[f](x) - f(x)$, for any $x \in [x_0, b]$, by Theorem 3.4, we have

$$R_{L,n}[f](x) = \frac{1}{n!} \int_{x_0}^{b} K_n(x,t) f^{(n+1)}(t)\, dt$$

where

$$K_n(x,t) = R_{L,n}[(x-t)_+^n] = (x-t)_+^n - \sum_{i=0}^{n} \binom{n}{i} (x_0+ai-t)_+^{n-i} \widetilde{G}_i(x,a).$$

REMARK 4.1. Abel–Gontscharoff interpolation, even in recent years, has been object of study [**23**]. In the future we will consider a comparison with previous works, especially as regards the error estimation.

Abel-Bernoulli interpolation. . Let $L(f)=\int_0^1 f(x)\,dx$. Then the umbral basis for (L,D_a) is the Bernoulli–Abel sequence $\widetilde{B}_n(x,a)$ [**8**].

Interpolation polynomials (3.1) and (3.4) become

$$\widetilde{B}_n[f](x)=\int_0^1 f(x)dx+\sum_{i=1}^{n}\frac{f^{(i-1)}(1+ai)-f^{(i-1)}(ai)}{i!}\widetilde{B}_i(x,a),$$

$$\overline{\widetilde{B}}_n[f](x)=f(0)+\sum_{i=1}^{n}\frac{f^{(i-1)}(1+ai)-f^{(i-1)}(ai)}{i!}(\widetilde{B}_i(x,a)-\widetilde{B}_i(0,a)). \tag{4.2}$$

For the remainder $R_{L,n}[f](x)=\widetilde{B}_n[f](x)-f(x)$, for any $x\in[0,b]$, by Theorem 3.4, we have

$$R_{L,n}[f](x)=\frac{1}{n!}\int_0^b K_n(x,t)f^{(n+1)}(t)\,dt$$

where

$$K_n(x,t)=R_{L,n}\big[(x-t)_+^n\big]=(x-t)_+^n-\int_0^1 (x-t)_+^n dx$$
$$-\sum_{i=1}^{n}\binom{n}{i-1}\frac{1}{i}\big[(1+ai-t)_+^{n-i+1}-(ai-t)_+^{n-i+1}\big]\widetilde{B}_i(x,a).$$

For the remainder $R_{L,n}[f](x)=\overline{\widetilde{B}}_n[f](x)-f(x)$, for any $x\in[0,b]$, by Theorem 3.4, we have

$$R_{L,n}[f](x)=\frac{1}{n!}\int_0^b K_n(x,t)f^{(n+1)}(t)\,dt$$

where

$$K_n(x,t)=R_{L,n}\big[(x-t)_+^n\big]=(x-t)_+^n$$
$$-\sum_{i=1}^{n}\binom{n}{i-1}\frac{1}{i}\big[(1+ai-t)_+^{n-i+1}-(ai-t)_+^{n-i+1}\big]\big(\widetilde{B}_i(x,a)-\widetilde{B}_i(0,a)\big).$$

Abel-Euler interpolation. We consider $L(f)=\frac{f(0)+f(1)}{2}$. Then the umbral basis for (L,D_a) is the Euler–Abel sequence $\widetilde{E}_n(x,a)$ [**8**].

Interpolation polynomials (3.1) and (3.4) become

$$\widetilde{E}_n[f](x)=\frac{f(0)+f(1)}{2}+\sum_{i=1}^{n}\frac{f^{(i)}(ai)+f^{(i)}(1+ai)}{2i!}\widetilde{E}_i(x,a), \tag{4.3}$$

$$\overline{\widetilde{E}}_n[f](x)=f(0)+\sum_{i=1}^{n}\frac{f^{(i)}(ai)+f^{(i)}(1+ai)}{2i!}\big(\widetilde{E}_i(x,a)-\widetilde{E}_i(0,a)\big).$$

For the remainder $R_{L,n}[f](x)=\widetilde{E}_n[f](x)-f(x)$, for any $x\in[0,b]$, we have

$$R_{L,n}[f](x)=\frac{1}{n!}\int_0^b K_n(x,t)f^{(n+1)}(t)\,dt$$

where

$$\begin{aligned}K_n(x,t) &= R_{L,n}\big[(x-t)_+^n\big]\\ &= (x-t)_+^n - \sum_{i=0}^{n}\binom{n}{i}\frac{1}{2}\big[(ai-t)_+^{n-i} + (1+ai-t)_+^{n-i}\big]\widetilde{E}_i(x,a).\end{aligned}$$

4.2. $\boldsymbol{\delta_h}$-Sheffer interpolation. Let be f bounded in $[a,b]$ and $Qf = \delta_h f(x) = \frac{f(x+h/2)-f(x-h/2)}{h}$. Moreover, let δ_h^{-1} be the inverse operator of δ_h, such that

$$\delta_h^{-1}\varphi(x) = f(x) \Leftrightarrow \delta_h f(x) = \varphi(x).$$

Then the associated sequence to δ_h is the sequence [**20**]

$$\begin{aligned}x^{[0]} = 1,\ x^{[n]} &\equiv x\Big(x + \Big(\frac{n}{2}-1\Big)h\Big)_{n-1}\\ &= x\Big(x + \Big(\frac{n}{2}-1\Big)h\Big)\cdots\Big(x + \Big(-\frac{n}{2}+1\Big)h\Big), \quad n = 1,2,\dots.\end{aligned}$$

Now, let L be a linear functional verifying $L(1) \neq 0$. Then umbral interpolation polynomials (3.1) and (3.4) become

$$P_n[f](x) = \sum_{i=0}^{n}\frac{L(\delta_h^i f)}{i!}s_i(x), \tag{4.4}$$

$$\overline{P}_n[f](x) = f(0) + \sum_{i=1}^{n}\frac{L(\delta_h^i f)}{i!}(s_i(x) - s_i(0)),$$

where $s_i(x)$ is the umbral basis for (L,δ_h).

As in the previous example we can consider the following cases:

δ_h*-central interpolation.* Let $L(f) = f(0)$. The umbral basis for (L,δ_h) is $s_n(x) = x^{[n]}$.

Interpolation polynomial (3.1) becomes

$$P_n[f](x) = f(0) + \sum_{i=1}^{n}\frac{\delta_h^i f(0)}{i!}x^{[i]}.$$

It is known as interpolation formula with central differences [**22**, p. 32], therefore (4.4) can be seen as generalization of central interpolation.

δ_h*-Bernoulli interpolation.* Let $L(f) = \big(D\delta_h^{-1}f\big)_{x=0}$. We call the umbral basis for (L,δ_h) δ_h-Bernoulli polynomial sequence $\widehat{B}_n(x)$. Interpolation polynomials (3.1) and (3.4) become

$$\widehat{B}_n[f](x) = \big(D\delta_h^{-1}f\big)_{x=0} + \sum_{i=1}^{n}\frac{\delta_h^{i-1}f'(0)}{i!}\widehat{B}_i(x),$$

$$\overline{\widehat{B}}_n[f](x) = f(0) + \sum_{i=1}^{n}\frac{\delta_h^{i-1}f'(0)}{i!}\big(\widehat{B}_i(x) - \widehat{B}_i(0)\big). \tag{4.5}$$

δ_h-Euler interpolation. Let $L(f) = (M_h f)_{x=0} = \frac{f\left(\frac{1}{2}h\right)+f\left(-\frac{1}{2}h\right)}{2}$. We call the umbral basis for (L, δ_h) δ_h-Euler polynomial sequence $\widehat{E}_n(x)$. Interpolation polynomials (3.1) and (3.4) become

$$\widehat{E}_n[f](x) = \frac{f(h/2) + f(-h/2)}{2} + \sum_{i=1}^{n} \frac{\delta_h^i\big(f(h/2) + f(-h/2)\big)}{2i!} \widehat{E}_i(x), \tag{4.6}$$

$$\overline{\widehat{E}}_n[f](x) = f(0) + \sum_{i=1}^{n} \frac{\delta_h^i\big(f(h/2) + f\big(-h/2\big)\big)}{2i!} \big(\widehat{E}_i(x) - \widehat{E}_i(0)\big). \tag{4.7}$$

5. Numerical examples

Now, we consider some interpolation test problems and report the numerical results obtained by using an ad hoc "Mathematica" code. We compare the error in approximating a given function with Appell, Abel–Sheffer, Δ_h-Appell and δ_h-Sheffer interpolation polynomials. In particular we compare numerical results obtained by applying:

- Abel-Bernoulli interpolation polynomial defined by (4.2) as follows

$$\overline{\widetilde{B}}_n[f](x) = f(0) + \sum_{i=1}^{n} \frac{f^{(i-1)}(1 + ai) - f^{(i-1)}(ai)}{i!} \big(\widetilde{B}_i(x, a) - \widetilde{B}_i(0, a)\big);$$

- Bernoulli interpolation polynomial defined in [**2**] as follows

$$B_n[f](x) = f(0) + \sum_{i=1}^{n} \frac{f^{(i-1)}(1) - f^{(i-1)}(0)}{i!} \big(B_i(x) - B_i(0)\big);$$

- δ_h-Bernoulli interpolation polynomial defined by (4.5) as follows

$$\overline{\widehat{B}}_n[f](x) = f(0) + \sum_{i=1}^{n} \frac{\delta_h^{i-1} f'(0)}{i!} \big(\widehat{B}_i(x) - \widehat{B}_i(0)\big);$$

- Bernoulli interpolation polynomial of second kind [**6**] defined as follows

$$\overline{B}_n^{II}[f](x) = f(0) + \sum_{i=0}^{n-1} f'(ih)\big(\mathcal{B}_i^{II}(x) - \mathcal{B}_i^{II}(0)\big).$$

- Abel-Euler interpolation polynomial defined by (4.3) as follows

$$\widetilde{E}_n[f](x) = \frac{f(0) + f(1)}{2} + \sum_{i=1}^{n} \frac{f^{(i)}(ai) + f^{(i)}(1 + ai)}{2i!} \widetilde{E}_i(x, a);$$

- Euler interpolation polynomial defined in [**5**] as follows

$$E_n[f](x) = \sum_{i=0}^{n} \frac{f^{(i)}(0) + f^{(i)}(1)}{2i!} E_i(x);$$

- δ_h-Euler interpolation polynomial defined by (4.6) as follows

$$\widehat{E}_n[f](x) = \frac{f(h/2) + f(-h/2)}{2} + \sum_{i=1}^{n} \frac{\delta_h^i\big(f(h/2) + f(-h/2)\big)}{2i!} \widehat{E}_i(x);$$

- Boole interpolation polynomial defined in [**6**] as follows

$$E_n^{II}[f](x) = \sum_{i=0}^{n} \frac{f(ih) + f((i+1)h)}{2} \mathcal{E}_i^{II}(x).$$

We emphasize that the compared polynomials of the same degree have the same degree of exactness.

EXAMPLE 5.1. Let us consider the function $f(x) = e^{(x+1)/2}$, $x \in [0,1]$. The interpolation error is reported in the following tables (numbers in parentheses indicate decimal exponents):

	$\overline{\overline{B}}_n[f](x)$	$\overline{B}_n[f](x)$	$\overline{\widehat{B}}_n[f](x)$	$\overline{B}_n^{II}[f](x)$
$n=5$	2.774(−6)	1.102(−6)	1.628(−5)	6.949(−7)
$n=6$	1.460(−7)	8.619(−8)	8.814(−7)	1.733(−8)
$n=7$	1.463(−8)	6.885(−9)	4.160(−8)	4.354(−10)
$n=8$	7.589(−10)	5.458(−10)	1.742(−9)	8.609(−12)

	$\widetilde{E}_n[f](x)$	$E_n[f](x)$	$\widehat{E}_n[f](x)$	$E_n^{II}[f](x)$
$n=5$	5.978(−5)	4.460(−5)	2.606(−6)	1.318(−7)
$n=6$	9.432(−6)	7.103(−6)	1.171(−7)	2.925(−9)
$n=7$	1.483(−6)	1.130(−6)	4.715(−9)	5.817(−11)
$n=8$	2.354(−7)	1.799(−7)	1.726(−10)	1.041(−12)

EXAMPLE 5.2. Let us consider the function $f(x) = \ln(x^2+10)$, $x \in [0,1]$. The interpolation error is reported in the following tables:

	$\overline{\overline{B}}_n[f](x)$	$\overline{B}_n[f](x)$	$\overline{\widehat{B}}_n[f](x)$	$\overline{B}_n^{II}[f](x)$
$n=5$	1.994(−6)	4.526(−6)	1.310(−4)	2.823(−6)
$n=6$	2.482(−6)	1.760(−6)	1.737(−6)	3.744(−7)
$n=7$	5.442(−7)	3.457(−7)	5.477(−6)	1.579(−8)
$n=8$	1.267(−7)	2.559(−7)	6.832(−8)	4.487(−9)

	$\widetilde{E}_n[f](x)$	$E_n[f](x)$	$\widehat{E}_n[f](x)$	$E_n^{II}[f](x)$
$n=5$	1.183(−4)	2.138(−4)	1.974(−4)	4.435(−7)
$n=6$	1.257(−4)	1.410(−4)	1.5591(−6)	6.225(−8)
$n=7$	6.482(−5)	8.666(−5)	5.843(−7)	1.624(−9)
$n=8$	5.736(−5)	7.829(−5)	4.823(−8)	1.041(−10)

EXAMPLE 5.3. Let us consider the function

$$f(x) = 10\cos(x) + \frac{\sin^2(x)}{10}, \quad x \in [0,1].$$

The interpolation error is reported in the following tables:

	$\overline{\overline{B}}_n[f](x)$	$\overline{B}_n[f](x)$	$\overline{\widehat{B}}_n[f](x)$	$\overline{B}_n^{II}[f](x)$
$n=5$	4.652(−4)	2.319(−4)	4.227(−3)	1.483(−4)
$n=6$	1.882(−5)	2.451(−6)	3.614(−6)	5.090(−7)
$n=7$	1.338(−5)	2.695(−6)	1.224(−5)	9.767(−8)
$n=8$	1.981(−6)	2.058(−6)	4.386(−7)	3.400(−8)

	$\widetilde{E}_n[f](x)$	$E_n[f](x)$	$\widehat{E}_n[f](x)$	$E_n^{II}[f](x)$
$n=5$	$9.016(-3)$	$9.265(-3)$	$6.613(-4)$	$2.737(-5)$
$n=6$	$1.881(-3)$	$3.195(-4)$	$2.770(-6)$	$6.299(-8)$
$n=7$	$1.490(-3)$	$7.061(-4)$	$1.201(-6)$	$1.661(-8)$
$n=8$	$6.494(-4)$	$6.746(-4)$	$3.200(-7)$	$4.111(-9)$

6. Conclusions

Let Q be a δ-operator on $\mathcal{P}_n$, X a linear space of functions such that $\mathcal{P}_n \subseteq X$ and L a linear functional on X, with $L(1) \neq 0$. Let be $XQ \subseteq X$ such that $Q^i f \in X$, then for all $f \in XQ$ we defined umbral interpolation for the couple (L, Q). In particular generalizations of Abel–Gontscharoff and central interpolation have been considered. Further developments are possible, for example the function series associated to the interpolation polynomial can be considered and its convergence radius can be studied. Furthermore, polynomial (4.7) seems to be of interest for applications on IVP. The multivariate case is interesting too.

References

1. P. Appell, *Sur une classe de polynomes*, Ann. Sci. École Norm. Sup. **2**(9) (1880), 119–144.
2. F. A. Costabile, *On expansion of a real function in Bernoulli polynomials and application*, Conf. Sem. Mat. Univ. Bari **273** (1999), 1–16.
3. F. A. Costabile, F. Dell'Accio, M. I. Gualtieri, *A new approach to Bernoulli polynomials*, Rend. Mat. Appl. **26**(1) (2006), 1–12.
4. F. A. Costabile, E. Longo, *A determinantal approach to Appell polynomials*, J. Comput. Appl. Math. **234**(5) (2010), 1528–1542.
5. ______, *The Appell interpolation problem*, J. Comput. Appl. Math. **236** (2011), 1024–1032.
6. ______, *Δ_h-Appell sequences and related interpolation problem*, Numer. Algorithms **63**(1) (2013), 165–186.
7. ______, *Algebraic theory of Appell polynomials with application to general linear interpolation problem*, in: H. A. Yasser (ed.), *Linear Algebra*, InTech Europe Publication, Rijeka, Croatia, 2012, 21–46.
8. ______, *An algebraic approach to Sheffer polynomial sequences*, Int. Trans. Spec. Funct. **25**(4) (2014), 295–311.
9. ______, *An algebraic exposition of umbral calculus with application to general linear interpolation problem: A survey*, Publ. Inst. Math., Nouv. Sér. **96**(110) (2014), 67–83.
10. M. Craciun, *Approximation operators constructed by means of Sheffer sequences*, Rev. Anal. Numer. Theor. Approx. **30** (2) (2001), 135–150.
11. P. J. Davis, *Interpolation & Approximation*, Dover Publication, Inc., New York, 1975.
12. R. Dere, Y. Simsek, *Genocchi polynomials associated with the Umbral algebra*, Appl. Math. Comput. **218** (2011), 756–761.
13. A. Di Bucchianico, *Probabilistic and analytical aspects of the umbral calculus*, vol. 119, *CWI Tract Series*, CWI, Amsterdam, 1997.
14. A. Di Bucchianico, D. Loeb, *A Selected Survey of Umbral Calculus*, Electron. J. Combin. Dynamic Surveys **DS3** (2000), 1–34.
15. M. E. H. Ismail, *Polynomials of binomial type and approximation theory*, J. Approx. Th. **23** (1978), 177–186.
16. S. Khan, M. Walid Al-Saad, G. Yasmin, *Some properties of Hermite-based Sheffer polynomials*, Appl. Math. Comput. **217** (2010), 2169–2183.
17. A. Lupas, *Approximation operators of binomial type*; in: M. W. Müller, M. D. Buhmann, D. H. Mache, M. Felten (eds), *New Developments in Approximation Theory*, Birkhäuser, Basel, 1999, 175–198.

18. M. Roman, *Polynomials, power series and interpolation*, J. Math. Anal. Appl. **80** (1981), 333–371.
19. S. Roman, *The theory of umbral calculus II*, J. Math. Anal. Appl. **89** (1982), 290–314.
20. S. Roman, G. Rota, *The Umbral Calculus*, Adv. Math. **27** (1978), 95–188.
21. I. M. Sheffer, *Some Properties of Polynomial Sets of Type Zero*, Duke Math. J. **5**(3) (1939), 590–622.
22. J. F. Steffensen, *Interpolation*, Courier Dover Publications, New York, 1950.
23. P. J. Y. Wong, R. P. Agarwal, *Abel–Gontscharoff interpolation error bounds for derivatives*, Proc. Roy. Soc. Edinburgh **119A** (1991), 367–372.
24. S. L. Yang, *Recurrence relations for the Sheffer sequences*, Linear Algebra Appl. **437**(12) (2012), 2986–2996.

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