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TRIGONOMETRIC MULTIPLE ORTHOGONAL POLYNOMIALS OF SEMI-INTEGER DEGREE AND THE CORRESPONDING QUADRATURE FORMULAS

Gradimir V. Milovanović, Marija P. Stanić, and Tatjana V. Tomović

Abstract. An optimal set of quadrature formulas with an odd number of nodes for trigonometric polynomials in Borges' sense [Numer. Math. 67 (1994), 271–288], as well as trigonometric multiple orthogonal polynomials of semi-integer degree are defined and studied. The main properties of such a kind of orthogonality are proved. Also, an optimal set of quadrature rules is characterized by trigonometric multiple orthogonal polynomials of semi-integer degree. Finally, theoretical results are illustrated by some numerical examples.

1. Introduction

Motivated by a problem that arises in the evaluation of computer graphics illumination models, Carlos Borges [**4**] examined a more abstract problem of numerically evaluating a set of p definite integrals taken with respect to p distinct weight functions, but related to a common integrand and the same interval of integration. For such a problem it is not efficient to use a set of p Gauss–Christoffel quadrature rules, because valuable information is wasted.

Borges in [**4**] introduced a performance ratio in the following way:

$$R = \frac{\text{Overall algebraic degree of precision} + 1}{\text{Number of integrand evaluation}}.$$

Taking the set of p Gauss–Christoffel quadrature rules, one has $R = 2/p$ and, hence, $R < 1$ for all $p > 2$.

If we select a set of n distinct nodes, common for all quadrature rules, then weight coefficients for each of p quadrature rules can be chosen in such a way that $R = 1$. Of course, an arbitrary selection of nodes does not lead us to the best possible quadrature rules. The optimal quadrature rules could be obtained by simulating the development of the Gauss–Christoffel quadrature rules. It was

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proved that the common nodes of an optimal set of quadrature rules are zeros of the corresponding type II multiple orthogonal polynomials, defined using orthogonality conditions spread out over p different measures [**12**,**13**]. A stable numerical method for construction of type II multiple orthogonal polynomials, as well as of optimal set of quadrature rules was given in [**12**].

Let us notice that multiple orthogonal polynomials arise naturally in the theory of simultaneous rational approximation, in particular in the Hermite–Padé approximation of a system of p Markov functions. When the Hermite–Padé approximation and multiple orthogonal polynomials are in question, we refer readers to the book by Nikishin and Sorokin [**14**, Chapter 4], the surveys by Aptekarev [**1**], de Bruin [**6**], and Milovanović and Stanić [**13**], as well as the papers by Piñeiro [**16**], Sorokin [**17**–**19**], Van Assche [**22**], Van Assche and Coussement [**23**], and Chapter 23 of Ismail's book [**7**].

Type II multiple orthogonal polynomials are also connected with the generalized Birkhoff–Young quadrature rules [**8**, **11**, **13**].

Here we investigate the optimal set of quadrature rules in Borges' sense for trigonometric polynomials. From the theory of Gaussian type quadrature rules, it is known that it is necessary to consider two different trigonometric orthogonal systems – standard trigonometric orthogonal polynomials and trigonometric orthogonal polynomials of semi-integer degree. As a matter of fact, for a quadrature rule with an even number of nodes and the maximal trigonometric degree of exactness one must consider orthogonality in the subspace of trigonometric polynomials, but in the case of an odd number of nodes and the maximal trigonometric degree of exactness orthogonality in subspace of trigonometric polynomials of semi-integer degree must be considered [**2**, **3**, **5**, **9**, **10**, **15**, **20**]. We restrict our attention to an optimal set of quadrature rules with an odd number of nodes. Therefore, here we introduce a concept of multiple orthogonality in the space of trigonometric polynomials of semi-integer degree. The paper is organized as follows. Trigonometric multiple orthogonal polynomials of semi-integer degree are defined in Section 2, where their main properties are proved. Section 3 is devoted to definition and characterization of an optimal sets of quadrature rules for trigonometric polynomials. Finally, numerical examples are presented in Section 4.

2. Trigonometric multiple orthogonal polynomials of semi-integer degree

For a nonnegative integer m and $\gamma \in \{0, 1/2\}$, we by $\mathcal{T}_m^\gamma$ denote the linear span of the set $\{\cos(k+\gamma)x, \sin(k+\gamma)x : k = 0, 1, \ldots, m\}$. Obviously, $\mathcal{T}_m^0 = \mathcal{T}_m$ is the linear space of all trigonometric polynomials of degree less than or equal to m, while $\mathcal{T}_m^{1/2}$ is the linear space of all trigonometric polynomials of semi-integer degree less than or equal to $m+1/2$. Of course, $\dim(\mathcal{T}_m) = 2m+1$ and $\dim(\mathcal{T}_m^{1/2}) = 2(m+1)$.

The trigonometric polynomial of semi-integer degree $m+1/2$ has the following form

$$A_m^{1/2}(x) = \sum_{\nu=0}^{m} \Big[c_\nu \cos\Big(\nu + \frac{1}{2}\Big)x + d_\nu \sin\Big(\nu + \frac{1}{2}\Big)x \Big], \tag{2.1}$$

where $c_\nu, d_\nu \in \mathbb{R}$, $|c_m| + |d_m| \neq 0$. Every trigonometric polynomial of semi-integer degree $m+1/2$ of the form (2.1) can be represented in the form (cf. [**21**, Lemma 1])

$$A_m^{1/2}(x) = A \prod_{k=0}^{2m} \sin \frac{x - x_k}{2} \quad (A \text{ is a nonzero constant}). \tag{2.2}$$

It is obvious that every $A_m^{1/2}(x)$ given by (2.2) is a trigonometric polynomial of semi-integer degree $m + 1/2$.

Remark 2.1. For the rest of this paper an important fact is that a trigonometric polynomial of semi-integer degree $m+1/2$ has exactly $2m+1$ zeros counting with their multiplicities and restricting to the strip $-\pi \leqslant \operatorname{Re} z < \pi$, as well as that the nonreal zeros appear in conjugate pairs [**3**, Theorem 1.1]. This means that the number of sign changes of trigonometric polynomials of semi-integer degree on $[-\pi, \pi)$ can not be even. The same is true also for any interval $[L, L + 2\pi)$, $L \in \mathbb{R}$, of length 2π.

Let p be a positive integer and $\mathbf{n} = (n_1, n_2, \ldots, n_p)$ a vector of p nonnegative integers, which is called a *multi-index* with length $|\mathbf{n}| = n_1 + n_2 + \cdots + n_p$. We introduce a partial order on multi-indices by

$$\mathbf{m} \preceq \mathbf{n} \;\Leftrightarrow\; m_\nu \leqslant n_\nu \quad \text{for every } \nu = 1, 2, \ldots, p. \tag{2.3}$$

Let $W = \{w_1, w_2, \ldots, w_p\}$ be a set of p weight functions, integrable and nonnegative on some interval E of length 2π, vanishing there only on a set of measure zero. In what follows, we always assume that the interval E is closed on the left and open on the right, i.e., that interval E is of the form $[L, 2\pi + L)$, for $L \in \mathbb{R}$.

Analogously with the multiple algebraic orthogonal polynomials, we define two types of trigonometric multiple orthogonal polynomials of semi-integer degree.

Definition 2.1. Let $\mathbf{n}$ be a multi-index. Type I multiple trigonometric orthogonal polynomials of semi-integer degree with respect to W are collected in a vector $(A_{\mathbf{n},1}^{1/2}, A_{\mathbf{n},2}^{1/2}, \ldots, A_{\mathbf{n},p}^{1/2})$ of trigonometric polynomials of semi-integer degree, where $A_{\mathbf{n},\nu}^{1/2}$ has semi-integer degree $n_\nu - 1/2$, $\nu = 1, 2, \ldots, p$, such that the following orthogonality conditions hold

$$\begin{aligned} &\sum_{\nu=1}^{p} \int_E A_{\mathbf{n},\nu}^{1/2} \cos\Big(k + \frac{1}{2}\Big) x\, w_\nu(x)\, dx = 0, \quad k = 0, 1, 2, \ldots, |\mathbf{n}| - 2, \\ &\sum_{\nu=1}^{p} \int_E A_{\mathbf{n},\nu}^{1/2} \sin\Big(k + \frac{1}{2}\Big) x\, w_\nu(x)\, dx = 0, \quad k = 0, 1, 2, \ldots, |\mathbf{n}| - 2, \end{aligned} \tag{2.4}$$

with the normalizations

$$\begin{aligned} &\sum_{\nu=1}^{p} \int_E A_{\mathbf{n},\nu}^{1/2} \cos\Big(|\mathbf{n}| - \frac{1}{2}\Big) x\, w_\nu(x)\, dx = 1, \\ &\sum_{\nu=1}^{p} \int_E A_{\mathbf{n},\nu}^{1/2} \sin\Big(|\mathbf{n}| - \frac{1}{2}\Big) x\, w_\nu(x)\, dx = 1. \end{aligned} \tag{2.5}$$

Conditions (2.4)–(2.5) give a linear system of $2|\mathbf{n}|$ equations for the $2|\mathbf{n}|$ unknown coefficients of the trigonometric polynomials of semi-integer degree $A_{\mathbf{n},\nu}^{1/2}$, $\nu = 1, 2, \ldots, p$. The multi-index $\mathbf{n}$ is normal for type I if system of equations (2.4)–(2.5) has a unique solution. For the type I multiple trigonometric orthogonal polynomials we define the following function

$$A_{\mathbf{n}}(x) = \sum_{\nu=1}^{p} A_{\mathbf{n},\nu}^{1/2} w_\nu(x). \tag{2.6}$$

Then, the orthogonality conditions (2.4) and the normalizations (2.5) become

$$\begin{aligned} &\int_E A_{\mathbf{n}}(x) \cos\Big(k+\frac{1}{2}\Big)x dx = 0, \quad k = 0,1,2,\ldots,|\mathbf{n}|-2, \\ &\int_E A_{\mathbf{n}}(x) \sin\Big(k+\frac{1}{2}\Big)x dx = 0, \quad k = 0,1,2,\ldots,|\mathbf{n}|-2, \end{aligned} \tag{2.7}$$

and

$$\int_E A_{\mathbf{n}}(x) \cos\Big(|\mathbf{n}|-\frac{1}{2}\Big)x dx = 1, \quad \int_E A_{\mathbf{n}}(x) \sin\Big(|\mathbf{n}|-\frac{1}{2}\Big)x dx = 1, \tag{2.8}$$

respectively.

Definition 2.2. Let $\mathbf{n}$ be a multi-index. Trigonometric polynomial of semi-integer degree $T_{\mathbf{n}}^{1/2}$ is a type II multiple trigonometric orthogonal polynomial of semi-integer degree with respect to W if it is of semi-integer degree $|\mathbf{n}| + 1/2$ and satisfies the following orthogonality conditions

$$\begin{aligned} &\int_E T_{\mathbf{n}}^{1/2}(x) \cos\Big(k_\nu+\frac{1}{2}\Big)x\, w_\nu(x)\, dx = 0, \quad k_\nu = 0,1,\ldots,n_\nu-1, \\ &\int_E T_{\mathbf{n}}^{1/2}(x) \sin\Big(k_\nu+\frac{1}{2}\Big)x\, w_\nu(x)\, dx = 0, \quad k_\nu = 0,1,\ldots,n_\nu-1, \end{aligned} \tag{2.9}$$

for $\nu = 1, 2, \ldots, p$.

Remark 2.2. Let us notice that if some $n_\nu = 0$, we do not have orthogonality conditions (2.9) for the corresponding weight w_ν.

For $p = 1$ we have the case of ordinary trigonometric orthogonal polynomials of semi-integer degree.

Orthogonality conditions (2.9) give a system of linear equations for the unknown coefficients of the trigonometric polynomial $T_{\mathbf{n}}^{1/2}$. Since

$$T_{\mathbf{n}}^{1/2}(x) = \sum_{k=0}^{|\mathbf{n}|} \Big(a_k \cos\Big(k+\frac{1}{2}\Big)x + b_k \sin\Big(k+\frac{1}{2}\Big)x\Big) \in \mathfrak{T}_{|\mathbf{n}|}^{1/2},$$

we have $2|\mathbf{n}|+2$ unknown coefficients a_k, b_k, $k = 0, 1, \ldots, |\mathbf{n}|$. Conditions (2.9) give $2(n_1+n_2+\cdots+n_p) = 2|\mathbf{n}|$ equations for the $2|\mathbf{n}|+2$ unknown coefficients of $T_{\mathbf{n}}^{1/2}$, so we have to fix two coefficients. We choose to fix in advance the leading coefficients

$a_{|\mathbf{n}|}$ and $b_{|\mathbf{n}|}$ (of course, $a_{|\mathbf{n}|}^2+b_{|\mathbf{n}|}^2\neq 0$). For the special choices of the leading coefficients, $(a_{|\mathbf{n}|},b_{|\mathbf{n}|})\in\{(1,0),(0,1)\}$, we introduce the following notations

$$T_{\mathbf{n}}^{C,1/2}(x)=\cos\Big(|\mathbf{n}|+\frac{1}{2}\Big)x+\sum_{k=0}^{|\mathbf{n}|-1}\Big(a_k\cos\Big(k+\frac{1}{2}\Big)x+b_k\sin\Big(k+\frac{1}{2}\Big)x\Big),$$
$$T_{\mathbf{n}}^{S,1/2}(x)=\sin\Big(|\mathbf{n}|+\frac{1}{2}\Big)x+\sum_{k=0}^{|\mathbf{n}|-1}\Big(a_k\cos\Big(k+\frac{1}{2}\Big)x+b_k\sin\Big(k+\frac{1}{2}\Big)x\Big).$$

We call $T_{\mathbf{n}}^{C,1/2}(x)$ and $T_{\mathbf{n}}^{S,1/2}(x)$ the monic cosine and the monic sine multiple orthogonal polynomial of semi-integer degree, respectively.

If system (2.9) has a unique solution, then the multi-index $\mathbf{n}$ is normal for type II.

Lemma 2.1. *A multi-index* $\mathbf{n}$ *is normal for type I if and only if it is normal for type II.*

Proof. For $\nu=1,\dots,p$ we introduce the following notations

$$I_{i,j}^{C,\nu}=\int_E\cos\Big(i+\frac{1}{2}\Big)x\cos\Big(j+\frac{1}{2}\Big)x\,w_\nu(x)\,dx,$$
$$I_{i,j}^{S,\nu}=\int_E\sin\Big(i+\frac{1}{2}\Big)x\sin\Big(j+\frac{1}{2}\Big)x\,w_\nu(x)\,dx,$$
$$I_{i,j}^{\nu}=\int_E\cos\Big(i+\frac{1}{2}\Big)x\sin\Big(j+\frac{1}{2}\Big)x\,w_\nu(x)\,dx,$$
$$m_{i,j}^{(\nu)}=\begin{bmatrix} I_{i,j}^{C,\nu} & I_{i,j}^{\nu}\\ I_{j,i}^{\nu} & I_{i,j}^{S,\nu}\end{bmatrix},$$

for $i,j=0,1,\dots$, and

$$M_\nu=\begin{bmatrix} m_{0,0}^{(\nu)} & m_{0,1}^{(\nu)} & \cdots & m_{0,|\mathbf{n}|-1}^{(\nu)}\\ m_{1,0}^{(\nu)} & m_{1,1}^{(\nu)} & \cdots & m_{1,|\mathbf{n}|-1}^{(\nu)}\\ \vdots & \vdots & \ddots & \vdots\\ m_{n_\nu-1,0}^{(\nu)} & m_{n_\nu-1,1}^{(\nu)} & \cdots & m_{n_\nu-1,|\mathbf{n}|-1}^{(\nu)}\end{bmatrix}.$$

Then, the matrix of system (2.4)–(2.5) is given by

$$M^I=\begin{bmatrix}M_1^T\ M_2^T\ \cdots\ M_p^T\end{bmatrix}_{2|\mathbf{n}|\times 2|\mathbf{n}|},$$

and the matrix of system (2.9) (the leading coefficients of $T_{\mathbf{n}}^{1/2}$ are fixed) is given by

$$M^{II}=\begin{bmatrix}M_1\\ M_2\\ \vdots\\ M_p\end{bmatrix}_{2|\mathbf{n}|\times 2|\mathbf{n}|}.$$

Obviously, M^{II} is the transpose of M^{I}, which means that determinants of systems (2.9) and (2.4)–(2.5) are equal, i.e., system (2.4)–(2.5) has a unique solution if and only if system (2.9) has a unique solution. $\square$

According to Lemma 2.1, we could just talk about *normal multi-indices*. If all multi-indices are normal, then we have a *perfect system*.

Since the matrix of coefficients of systems (2.9) can be singular, we need some additional conditions on the p weight functions to provide the uniqueness of multiple trigonometric orthogonal polynomials of semi-integer degree. It is easy to see that the uniqueness is guaranteed if the following set of functions

$$\Big\{w_\nu \cos\Big(k_\nu + \frac{1}{2}\Big)x,\ w_\nu \sin\Big(k_\nu + \frac{1}{2}\Big)x :\ \nu = 1, 2, \dots, p,\ k_\nu = 0, 1, \dots, n_\nu - 1\Big\},$$

form a Chebyshev system on E for the multi-index $\mathbf{n}$. We called such set $W = \{w_1, w_2, \dots, w_p\}$ trigonometric AT system (TAT system) of weight functions for multi-index $\mathbf{n}$.

The properties of zeros of type I and II multiple trigonometric orthogonal polynomials are given in the following two theorems.

THEOREM 2.1. *Suppose that* $\mathbf{n}$ *is a multi-index such that* $W = \{w_1, w_2, \dots, w_p\}$ *is a TAT system of weight functions for all multi-indices* $\mathbf{m}$ *such that* $\mathbf{m} \preceq \mathbf{n}$. *For the type I multiple trigonometric orthogonal polynomials of semi-integer degree with respect to* W *the function* $A_{\mathbf{n}}(x)$, *given by* (2.6), *has exactly* $2|\mathbf{n}| - 1$ *simple zeros on* E.

PROOF. It is easy to see that the type I function $A_{\mathbf{n}}(x)$ has at least one sign change on E, since if we assume the contrary, for $|\mathbf{n}| \geqslant 1$ the orthogonality

$$\int_E A_{\mathbf{n}}(x) \sin\frac{x - L}{2} dx = 0$$

would be impossible since $\sin(x - L)/2$ does not change its sign on $[L, 2\pi + L)$.

The function $A_{\mathbf{n}}(x)$ has at most $2|\mathbf{n}| - 1$ zeros on E since we are dealing with the TAT system. The number of its sign changes on E is odd (see Remark 2.1). Suppose it has $2m - 1$, $m < |\mathbf{n}|$, sign changes at the points $x_1, x_2, \dots, x_{2m-1} \in E$, and set

$$Q(x) = \prod_{i=1}^{2m-1} \sin\frac{x - x_i}{2}.$$

Then, $A_{\mathbf{n}}(x)Q(x)$ does not change sign on E and $\int_E A_{\mathbf{n}}(x)Q(x)\,dx \neq 0$, which is a contradiction with orthogonality conditions (2.4). Therefore, $m = |\mathbf{n}|$, which means that $A_{\mathbf{n}}(x)$ has exactly $2|\mathbf{n}| - 1$ simple zeros on E. $\square$

THEOREM 2.2. *Suppose that* $\mathbf{n}$ *is a multi-index such that* $W = \{w_1, w_2, \dots, w_p\}$ *is a TAT system of weight functions for all multi-indices* $\mathbf{m}$ *such that* $\mathbf{m} \preceq \mathbf{n}$. *Type II multiple trigonometric orthogonal polynomial of semi-integer degree* $T_{\mathbf{n}}^{1/2}(x)$ *with respect to* W *has exactly* $2|\mathbf{n}| + 1$ *simple zeros on* E.

Proof. In the similar way as in proof of Theorem 2.1 we conclude that $T_{\mathbf{n}}^{1/2}(x)$ has an odd number of sign changes on E for $|\mathbf{n}| \geqslant 1$.

Let us assume that the polynomial $T_{\mathbf{n}}^{1/2}(x)$ has $2m+1$ changes of sign on E at the points $x_0, x_1, \dots, x_{2m}$ and that $m < |\mathbf{n}|$. Let $\mathbf{m} = (m_1, m_2, \dots, m_p)$ be a multi-index such that $m = |\mathbf{m}|$, $\mathbf{m} \preceq \mathbf{n}$, and $m_j < n_j$ for at least one j. Now, we construct

$$Q(x) = \sum_{i=1}^{p} Q_{m_i}^{1/2}(x)\, w_i(x),$$

where each $Q_{m_i}^{1/2}$ is a trigonometric polynomial of semi-integer degree $m_i - 1/2$, for $i \neq j$, $Q_{m_j}^{1/2}$ is a trigonometric polynomial of semi-integer degree $m_j + 1/2$, satisfying the interpolation conditions $Q(x_k) = 0$, $k = 0, 1, \dots, 2m$, and $Q(x_{2m+1}) = 1$, for an additional point $x_{2m+1} \in E$. Since we have a Chebyshev system of $2m+2$ functions, this interpolation problem has a unique solution, and since the function Q already has $2m+1$ zeros it can have no additional sign changes. Of course, the function Q does not vanish identically since $Q(x_{2m+1}) \neq 0$. Obviously $T_{\mathbf{n}}^{1/2}(x)Q(x)$ does not change sign on E, so that

$$\int_E T_{\mathbf{n}}^{1/2}(x) Q(x)\, dx \neq 0,$$

but this is in contrast with the orthogonality relations (2.9). Hence, $T_{\mathbf{n}}^{1/2}(x)$ has exactly $2|\mathbf{n}| + 1$ simple zeros on E. □

We finish this section proving the following biorthogonality between the type II multiple trigonometric orthogonal polynomials of semi-integer degree $T_{\mathbf{n}}^{1/2}(x)$ and the type I functions $A_{\mathbf{m}}(x)$, given by (2.6).

Theorem 2.3. *Suppose that* $\mathbf{n}$ *and* $\mathbf{m}$ *are two multi-indices and that* $W = \{w_1, w_2, \dots, w_p\}$ *is a TAT system of weight functions for the both multi-indices* $\mathbf{n}$ *and* $\mathbf{m}$. *Then the following relation of biorthogonality*

$$\int_E T_{\mathbf{n}}^{1/2}(x) A_{\mathbf{m}}(x)\, dx = \begin{cases} 0, & \text{if } \mathbf{m} \preceq \mathbf{n}, \\ 0, & \text{if } |\mathbf{n}| \leqslant |\mathbf{m}| - 2, \\ a_{|\mathbf{n}|} + b_{|\mathbf{n}|}, & \text{if } |\mathbf{n}| = |\mathbf{m}| - 1, \end{cases}$$

holds, where $T_{\mathbf{n}}^{1/2}(x)$ *is the corresponding type II multiple trigonometric orthogonal polynomial of semi-integer degree with the leading coefficients* $a_{|\mathbf{n}|}$ *and* $b_{|\mathbf{n}|}$, *and* $A_{\mathbf{m}}(x)$ *the corresponding type I function, given by* (2.6).

Proof. Since $A_{\mathbf{m}}(x)$ is given by (2.6), we have

$$\begin{aligned} \int_E T_{\mathbf{n}}^{1/2}(x) A_{\mathbf{m}}(x)\, dx &= \int_E T_{\mathbf{n}}^{1/2}(x) \left(\sum_{\nu=1}^{p} A_{\mathbf{m},\nu}^{1/2} w_\nu(x) \right) dx \\ &= \sum_{\nu=1}^{p} \int_E T_{\mathbf{n}}^{1/2}(x) A_{\mathbf{m},\nu}^{1/2}\, w_\nu(x)\, dx. \end{aligned}$$

First we assume that $\mathbf{m} \preceq \mathbf{n}$. According to (2.3) and orthogonality conditions (2.9) for the $T_{\mathbf{n}}^{1/2}(x)$, we conclude that all integrals on the right hand side of the previous equality are equal to zero, i.e., $\int_E T_{\mathbf{n}}^{1/2}(x)A_{\mathbf{m}}(x)\,dx = 0$. Since $a_{|\mathbf{n}|}$ and $b_{|\mathbf{n}|}$ are the leading coefficients of $T_{\mathbf{n}}^{1/2}$, we have

$$\begin{aligned}\int_E T_{\mathbf{n}}^{1/2}(x)A_{\mathbf{m}}(x)\,dx = a_{|\mathbf{n}|}&\int_E A_{\mathbf{m}}(x)\,\cos\Big(|\mathbf{n}|+\frac{1}{2}\Big)xdx\\ &+ b_{|\mathbf{n}|}\int_E A_{\mathbf{m}}(x)\,\sin\Big(|\mathbf{n}|+\frac{1}{2}\Big)xdx\\ &+\sum_{k=0}^{|\mathbf{n}|-1} a_k\int_E A_{\mathbf{m}}(x)\,\cos\Big(k+\frac{1}{2}\Big)xdx\\ &+\sum_{k=0}^{|\mathbf{n}|-1} b_k\int_E A_{\mathbf{m}}(x)\,\sin\Big(k+\frac{1}{2}\Big)xdx.\end{aligned}\tag{2.10}$$

If $|\mathbf{n}| \leqslant |\mathbf{m}| - 2$, by using orthogonality conditions (2.7) for the $A_{\mathbf{m}}(x)$, we obtain that all integrals on the right hand side of the previous equality are equal to zero and hence $\int_E T_{\mathbf{n}}^{1/2}(x)A_{\mathbf{m}}(x)\,dx = 0$.

To finish the proof we consider the case $|\mathbf{n}| = |\mathbf{m}| - 1$. By using conditions (2.8) and (2.7), from (2.10) we get what is stated. □

3. Optimal set of quadrature rules for trigonometric polynomials

Let $\mathbf{n}$ be a multi-index and let $W = \{w_1, w_2, \ldots, w_p\}$ be a TAT system for $\mathbf{n}$ on interval E. In this section we consider evaluation of a set of p definite integrals over the same interval E, taken with respect to the weight functions from W and related to a common integrand, i.e., the set of integrals of the form

$$\int_E f(x)w_\nu(x)\,dx, \quad \nu = 1, 2, \ldots, p.$$

As it has been already said, we are interested only in quadrature rules with an odd number of nodes. We require that such a set of quadrature rules is optimal for trigonometric polynomials in Borges' sense. For that purpose, we start with definition of trigonometric degree of exactness.

Definition 3.1. For a weight function w, integrable and nonnegative on the interval E, vanishing there only on a set of a measure zero, a quadrature rule of the form

$$\int_E f(x)w(x)\,dx = \sum_{\nu=0}^{2n}\sigma_\nu f(\tau_\nu) + R_n(f)\tag{3.1}$$

has a *trigonometric degree of exactness* equal to d if $R_n(f) = 0$ for all $f \in \mathcal{T}_d$ and there exists $g \in \mathcal{T}_{d+1}$ such that $R_n(g) \neq 0$.

If $\tau_\nu \in E$, $\nu = 0, 1, \ldots, 2n$, are distinct points fixed in advance, then corresponding interpolatory quadrature rule (3.1) has a trigonometric degree of exactness equal to n. One can increase that trigonometric degree of exactness up to the maximal trigonometric degree of exactness, which is equal to $2n$, with appropriate choice of nodes. The maximal trigonometric degree of exactness is achieved when nodes are zeros of the corresponding trigonometric orthogonal polynomials of semi-integer degree $n+1/2$ [**5**, **9**, **21**]. In such a way we obtain the Gaussian quadrature rules for trigonometric polynomials.

Analogously with [**4**], we introduce the performance ratio with respect to trigonometric degree of exactness as follows

$$R_T = \frac{\text{Overall trigonometric degree of exactness} + 1}{\text{Number of integrand evaluation}}.$$

Taking distinct nodes for each of p quadrature rule, it is obvious that R_T will be maximal when we have the set of p Gaussian quadrature rules. In that case

$$R_T = \frac{2n+1}{p(2n+1)} = \frac{1}{p},$$

so that, $R_T < 1/2$ for all $p > 2$.

If we select a set of $2n+1$ distinct nodes, common for all quadrature rules, the weight coefficients for each of p quadrature rules can be chosen in such a way that $R_T > 1/2$, which will be shown in what follows.

Definition 3.2. Let $\mathbf{n}$ be a multi-index and let $W = \{w_1, w_2, \dots, w_p\}$ be a TAT system for $\mathbf{n}$ on the interval E. A set of quadrature rules of the form

$$\int_E f(x) w_\nu(x)\, dx \approx \sum_{k=0}^{2|\mathbf{n}|} A_{\nu,k} f(x_k), \quad \nu = 1, 2, \dots, p, \tag{3.2}$$

is an optimal set with respect to $(W, \mathbf{n})$ if and only if the weight coefficients $A_{\nu,k}$, $\nu = 1, 2, \dots, p$, $k = 0, 1, \dots, 2|\mathbf{n}|$, and the nodes x_k, $k = 0, 1, \dots, 2|\mathbf{n}|$, satisfy the following system of equations

$$\begin{aligned} \sum_{k=0}^{2|\mathbf{n}|} A_{\nu,k} &= \int_E w_\nu(x)\, dx, \\ \sum_{k=0}^{2|\mathbf{n}|} A_{\nu,k} \cos m_\nu x_k &= \int_E \cos m_\nu x \, w_\nu(x)\, dx, \quad m_\nu = 1, 2, \dots, |\mathbf{n}| + n_\nu, \\ \sum_{k=0}^{2|\mathbf{n}|} A_{\nu,k} \sin m_\nu x_k &= \int_E \sin m_\nu x \, w_\nu(x)\, dx, \quad m_\nu = 1, 2, \dots, |\mathbf{n}| + n_\nu, \end{aligned}$$

for $\nu = 1, 2, \dots, p$.

Remark 3.1. Let us notice that an optimal set of quadrature rules have $2|\mathbf{n}|+1$ distinct nodes, common for all quadrature rules, and overall trigonometric degree of exactness is equal to $|\mathbf{n}| + m$, where $m = \min\{n_1, n_2, \dots, n_p\}$. Therefore,

$$R_T = \frac{|\mathbf{n}| + m + 1}{2|\mathbf{n}| + 1} = \frac{1}{2} + \frac{m + 1/2}{2|\mathbf{n}| + 1} > 1/2.$$

In order to characterize the optimal set of quadrature rules, we prove the following representation of trigonometric polynomials. A special case of that representation for $m = n$ was proved by Turetzkii [**21**].

LEMMA 3.1. *Every trigonometric polynomial of degree $n + m$, $m \leqslant n$,*

$$B_{n+m}(x) = a_0 + \sum_{k=1}^{n+m} (a_k \cos kx + b_k \sin kx)$$

can be uniquely represented in the form

$$B_{n+m}(x) = A_n^{1/2}(x) S_{m-1}^{1/2}(x) + T_n(x), \tag{3.3}$$

where

$$A_n^{1/2}(x) = \sum_{k=0}^{n} \Big(c_k \cos\Big(k + \frac{1}{2}\Big)x + d_k \sin\Big(k + \frac{1}{2}\Big)x \Big)$$

is a certain trigonometric polynomial of semi-integer degree $n+\frac{1}{2}$, with $c_n+id_n \neq 0$, and

$$S_{m-1}^{1/2} = \sum_{\nu=0}^{m-1} \Big(\gamma_\nu \cos\Big(\nu + \frac{1}{2}\Big)x + \delta_\nu \sin\Big(\nu + \frac{1}{2}\Big)x \Big),$$

$$T_n(x) = u_0 + \sum_{k=1}^{n} (u_k \cos kx + v_k \sin kx)$$

are the required trigonometric polynomials of semi-integer degree $m - \frac{1}{2}$ and of degree n, respectively.

PROOF. Comparing the coefficients of $\cos \ell x$ and $\sin \ell x$, $\ell = n+1, n+2, \dots, n+m$, on both sides in (3.3), we obtain the following system of equations for determining the unknown coefficients γ_ν, δ_ν, $\nu = 0, 1, \dots, m-1$

$$\frac{1}{2} \sum_{k=\ell-m}^{n} (c_k \gamma_{\ell-k-1} - d_k \delta_{\ell-k-1}) = a_\ell, \quad \frac{1}{2} \sum_{k=\ell-m}^{n} (c_k \delta_{\ell-k-1} + d_k \gamma_{\ell-k-1}) = b_\ell, \tag{3.4}$$

for $\ell = n+1, n+2, \dots, n+m$. Since that system can be rewritten in the form

$$\sum_{k=\ell-m}^{n} (c_k + id_k)(\gamma_{\ell-k-1} + i\delta_{\ell-k-1}) = 2(a_\ell + ib_\ell), \quad \ell = n+1, \dots, n+m,$$

its determinant is equal to $(c_n+id_n)^m \neq 0$. Therefore, γ_ν and δ_ν, $\nu = 0, 1, \dots, m-1$, can be uniquely determined from (3.4).

Further, from the equation

$$\frac{1}{2} \sum_{k=0}^{m-1} (c_k \gamma_k + d_k \delta_k) + u_0 = a_0,$$

which is obtained by comparing free coefficients on both sides in (3.3), one can determine u_0.

Similarly, if we compare coefficients of $\cos \ell x$ and $\sin \ell x$, $\ell = 1, 2, \ldots, n$, on both sides in (3.3), we obtain the following system of equations

$$\begin{aligned}\frac{1}{2}\sum_{k=\ell}^{j}(c_k\gamma_{k-\ell}+d_k\delta_{k-\ell})+\frac{1}{2}\sum_{k=0}^{m-\ell-1}&(c_k\gamma_{k+\ell}+d_k\delta_{k+\ell})\\&+\frac{1}{2}\sum_{k=0}^{\ell-1}(c_k\gamma_{\ell-k-1}-d_k\delta_{\ell-k-1})+u_\ell=a_\ell,\\ \frac{1}{2}\sum_{k=\ell}^{j}(d_k\gamma_{k-\ell}-c_k\delta_{k-\ell})+\frac{1}{2}\sum_{k=0}^{m-\ell-1}&(c_k\delta_{k+\ell}-d_k\gamma_{k+\ell})\\&+\frac{1}{2}\sum_{k=0}^{\ell-1}(c_k\delta_{\ell-k-1}+d_k\gamma_{\ell-k-1})+v_\ell=b_\ell,\end{aligned}$$

where $\ell = 1, 2, \ldots, m-1$, $j = \min\{n, \ell+m-1\}$, and

$$\begin{aligned}\frac{1}{2}\sum_{k=\ell}^{j}(c_k\gamma_{k-\ell}+d_k\delta_{k-\ell})+\frac{1}{2}\sum_{k=\ell-m}^{\ell-1}(c_k\gamma_{\ell-k-1}-d_k\delta_{\ell-k-1})+u_\ell=a_\ell,\\ \frac{1}{2}\sum_{k=\ell}^{j}(d_k\gamma_{k-\ell}-c_k\delta_{k-\ell})+\frac{1}{2}\sum_{k=\ell-m}^{\ell-1}(c_k\delta_{\ell-k-1}+d_k\gamma_{\ell-k-1})+v_\ell=b_\ell,\end{aligned}$$

for $\ell = m, m+1, \ldots, n$ and $j = \min\{n, \ell+m-1\}$, from which one can uniquely obtain u_ℓ, v_ℓ, $\ell = 1, 2, \ldots, n$. $\square$

The characterization of an optimal set of quadrature rules for trigonometric polynomials is given by the following statement, which is the counterpart to the fundamental theorem of Gaussian quadratures.

THEOREM 3.1. *Let* $\mathbf{n}$ *be a multi-index and let* $W = \{w_1, w_2, \ldots, w_p\}$ *be a TAT system for* $\mathbf{n}$ *on interval* E. *A set of quadrature rules* (3.2) *is an optimal set with respect to* $(W, \mathbf{n})$ *if and only if*

1° *all rules are exact for all trigonometric polynomials from* $\mathcal{T}_{|\mathbf{n}|}$;

2° $T_{\mathbf{n}}^{1/2}(x) = \prod_{k=0}^{2|\mathbf{n}|} \sin \frac{x-x_k}{2}$ *is the type II multiple trigonometric orthogonal polynomial of semi-integer degree* $|\mathbf{n}| + 1/2$ *with respect to* $(W, \mathbf{n})$.

PROOF. Suppose first that quadrature rules (3.2) form an optimal set with respect to $(W, \mathbf{n})$.

For each $\nu = 1, 2, \ldots, p$, the corresponding quadrature rule with respect to the weight function w_ν is exact for all trigonometric polynomials of degree less than or equal to $|\mathbf{n}| + n_\nu$, hence, it is exact for those of degree less than or equal to $|\mathbf{n}|$. Thus, 1° is proved.

In order to prove 2° assume that $S_{m_\nu-1}^{1/2}(x)$, $\nu = 1, 2, \ldots, p$, is a trigonometric polynomial of semi-integer degree $m_\nu - 1/2$, where $m_\nu \leqslant n_\nu$, $\nu = 1, 2, \ldots, p$. Then $T_{\mathbf{n}}^{1/2}(x) S_{m_\nu-1}^{1/2}(x)$ is a trigonometric polynomial of degree less than or equal to $|\mathbf{n}| + n_\nu$, $\nu = 1, 2, \ldots, p$. Since the corresponding quadrature rule, with respect to the weight function w_ν, $\nu = 1,2,\dots,p$, is exact for all such trigonometric polynomials and $T_{\mathbf{n}}^{1/2}(x_k) = 0$, $k = 0,1,\dots,2|\mathbf{n}|$, it follows that

$$\int_E T_{\mathbf{n}}^{1/2}(x) S_{m_\nu-1}^{1/2}(x)\, w_\nu(x)\, dx = \sum_{k=0}^{2|\mathbf{n}|} A_{\nu,k} T_{\mathbf{n}}^{1/2}(x_k) S_{m_\nu-1}^{1/2}(x_k) = 0,$$

for $\nu = 1,2,\dots,p$, i.e., $T_{\mathbf{n}}^{1/2}(x)$ is the type II multiple trigonometric orthogonal polynomial of semi-integer degree $|\mathbf{n}| + 1/2$ with respect to W.

Let us now suppose that 1° and 2° hold for (3.2). Let $B_{|\mathbf{n}|+n_\nu}(x) \in \mathcal{T}_{|\mathbf{n}|+n_\nu}$, $\nu = 1,2,\dots,p$. According to Lemma 3.1, $B_{|\mathbf{n}|+n_\nu}(x)$ can be represented in the form

$$B_{|\mathbf{n}|+n_\nu}(x) = T_{\mathbf{n}}^{1/2}(x) S_{n_\nu-1}^{1/2}(x) + P_{|\mathbf{n}|}(x),$$

where $S_{n_\nu-1}^{1/2} \in \mathcal{T}_{n_\nu-1}^{1/2}$ and $P_{|\mathbf{n}|} \in \mathcal{T}_{|\mathbf{n}|}$. Now we have that

$$\begin{aligned}\int_E B_{|\mathbf{n}|+n_\nu}(x)\, w_\nu(x)\, dx &= \int_E [T_{\mathbf{n}}^{1/2}(x) S_{n_\nu-1}^{1/2}(x) + P_{|\mathbf{n}|}(x)]\, w_\nu(x)\, dx \\ &= \int_E T_{\mathbf{n}}^{1/2}(x) S_{n_\nu-1}^{1/2}(x)\, w_\nu(x)\, dx + \int_E P_{|\mathbf{n}|}(x)\, w_\nu(x)\, dx,\end{aligned}$$

$\nu = 1,2\dots,p$. From 2° it follows that

$$\int_E T_{\mathbf{n}}^{1/2}(x) S_{n_\nu-1}^{1/2}(x)\, w_\nu(x)\, dx = 0, \quad \nu = 1,2\dots,p,$$

and, since $P_{|\mathbf{n}|} \in \mathcal{T}_{|\mathbf{n}|}$, from 1° we have

$$\int_E P_{|\mathbf{n}|}(x)\, w_\nu(x)\, dx = \sum_{k=0}^{2|\mathbf{n}|} A_{\nu,k} P_{|\mathbf{n}|}(x_k), \quad \nu = 1,2\dots,p,$$

and hence

$$\int_E B_{|\mathbf{n}|+n_\nu}(x)\, w_\nu(x)\, dx = \sum_{k=0}^{2|\mathbf{n}|} A_{\nu,k} P_{|\mathbf{n}|}(x_k), \quad \nu = 1,2,\dots,p.$$

Finally, since $T_{\mathbf{n}}^{1/2}(x_k) = 0$, $k = 0,1,\dots,2|\mathbf{n}|$, it follows that $B_{|\mathbf{n}|+n_\nu}(x_k) = P_{|\mathbf{n}|}(x_k)$, $k = 0,1,\dots,2|\mathbf{n}|$, $\nu = 1,2,\dots,p$, and we have

$$\int_E B_{|\mathbf{n}|+n_\nu}(x)\, w_\nu(x)\, dx = \sum_{k=0}^{2|\mathbf{n}|} A_{\nu,k} B_{|\mathbf{n}|+n_\nu}(x_k), \quad \nu = 1,2,\dots,p,$$

i.e., the quadrature rule for the weight w_ν is exact for all trigonometric polynomials of degree less that or equal to $|\mathbf{n}|+n_\nu$, $\nu = 1,2,\dots,p$. Therefore, a set of quadrature rules (3.2) is an optimal set with respect to $(W, \mathbf{n})$. □

REMARK 3.2. When $p = 1$ the optimal set of quadrature rules reduces to the Gaussian quadrature rule for trigonometric polynomials.

4. Numerical examples

In this section we illustrate characterization of an optimal set of quadrature rules (3.2) by using Theorem 3.1.

EXAMPLE 4.1. Let us find parameters for the optimal set of quadrature rules on $E = [-\pi, \pi)$, for $p = 2$, $\mathbf{n} = (2, 1)$, with respect to the weight functions $w_1(x) = 1$ and $w_2(x) = 1 + \sin 2x$. First we check whether the following set of functions

$$\Big\{\cos\frac{x}{2}, \sin\frac{x}{2}, \cos\frac{3x}{2}, \sin\frac{3x}{2}, \cos\frac{x}{2}(1+\sin 2x), \sin\frac{x}{2}(1+\sin 2x)\Big\},$$

form a Chebyshev system on the interval $[-\pi, \pi)$. Let $-\pi \leqslant y_1 < y_2 \cdots < y_6 < \pi$ be arbitrary distinct points. By using an elementary transformation and properties of determinants one can easily obtain that the determinant

$$\begin{vmatrix}
\cos\frac{y_1}{2} & \sin\frac{y_1}{2} & \cos\frac{3y_1}{2} & \sin\frac{3y_1}{2} & \cos\frac{y_1}{2}(1+\sin 2y_1) & \sin\frac{y_1}{2}(1+\sin 2y_1)\\
\cos\frac{y_2}{2} & \sin\frac{y_2}{2} & \cos\frac{3y_2}{2} & \sin\frac{3y_2}{2} & \cos\frac{y_2}{2}(1+\sin 2y_2) & \sin\frac{y_2}{2}(1+\sin 2y_2)\\
\cos\frac{y_3}{2} & \sin\frac{y_3}{2} & \cos\frac{3y_3}{2} & \sin\frac{3y_3}{2} & \cos\frac{y_3}{2}(1+\sin 2y_3) & \sin\frac{y_3}{2}(1+\sin 2y_3)\\
\cos\frac{y_4}{2} & \sin\frac{y_4}{2} & \cos\frac{3y_4}{2} & \sin\frac{3y_4}{2} & \cos\frac{y_4}{2}(1+\sin 2y_4) & \sin\frac{y_4}{2}(1+\sin 2y_4)\\
\cos\frac{y_5}{2} & \sin\frac{y_5}{2} & \cos\frac{3y_5}{2} & \sin\frac{3y_5}{2} & \cos\frac{y_5}{2}(1+\sin 2y_5) & \sin\frac{y_5}{2}(1+\sin 2y_5)\\
\cos\frac{y_6}{2} & \sin\frac{y_6}{2} & \cos\frac{3y_6}{2} & \sin\frac{3y_6}{2} & \cos\frac{y_6}{2}(1+\sin 2y_6) & \sin\frac{y_6}{2}(1+\sin 2y_6)
\end{vmatrix}$$

is equal to

$$-1024 \prod_{\substack{i,j=1\\ i<j}}^{6} \sin\frac{y_i - y_j}{2} \neq 0.$$

Therefore, we have the Chebyshev set of functions, i.e., $W = \{w_1, w_2\}$ is a TAT system for the multi-index $\mathbf{n} = (2, 1)$.

Now, the type II multiple trigonometric orthogonal polynomial of semi-integer degree $|\mathbf{n}| + 1/2 = 3 + 1/2$ can be obtained from orthogonality conditions (2.9). We choose the leading coefficients $a_3 = b_3 = 1$, so that

$$T_{\mathbf{n}}^{1/2}(x) = \cos\frac{7x}{2} + \sin\frac{7x}{2} + \sum_{k=0}^{2}\Big[a_k \cos\Big(k+\frac{1}{2}\Big)x + b_k \sin\Big(k+\frac{1}{2}\Big)x\Big].$$

The solution of the corresponding system (2.9) is $a_0 = b_0 = a_1 = b_1 = a_2 = b_2 = 0$, i.e., $T_{\mathbf{n}}^{1/2}(x) = \cos\frac{7x}{2} + \sin\frac{7x}{2}$. The zeros of $T_{\mathbf{n}}^{1/2}(x)$, i.e., nodes of the quadrature rules x_i, $i = 0, 1, \ldots, 6$, are

$$x_0 = -\frac{13\pi}{14},\ x_1 = -\frac{9\pi}{14},\ x_2 = -\frac{5\pi}{14},\ x_3 = -\frac{\pi}{14},\ x_4 = \frac{3\pi}{14},\ x_5 = \frac{\pi}{2},\ x_6 = \frac{11\pi}{14}.$$

We obtain the weight coefficients $A_{\nu,k}$, $\nu = 1, 2$, $k = 0, 1, \ldots, 6$, by using the condition 1° of Theorem 3.1, i.e., from the conditions that the quadrature rules (3.2) are exact for all trigonometric polynomials from the space $\mathcal{T}_3$. The results are given in Table 1. The numbers in parentheses indicate decimal exponents.

TABLE 1. The weight coefficients $A_{\nu,k}$, $\nu = 1, 2$, $k = 0, 1, \ldots, 6$, of the optimal set of quadrature rules with respect to the set of functions $W = \{1, 1 + \sin 2x\}$ and $\mathbf{n} = (2, 1)$

k	$A_{1,k}$	$A_{2,k}$
0	0.897597901025655	1.28705103454674
1	0.897597901025655	1.59936819864474
2	0.897597901025655	1.95827603406575(−1)
3	0.897597901025655	5.08144767504572(−1)
4	0.897597901025655	1.77269114865139
5	0.897597901025655	8.97597901025655(−1)
6	0.897597901025655	2.25046533999260(−2)

EXAMPLE 4.2. Let us now construct the optimal set of quadrature rules on $E = [0, 2\pi)$, for $p = 3$, $\mathbf{n} = (1, 1, 1)$, with respect to the weight functions $w_1(x) = 3 - \cos 2x$, $w_2(x) = 1 + 2\sin x$, and $w_3(x) = 2 + \cos x$.

In this case we have to verify that the set of functions

$$\Big\{\cos\frac{x}{2}w_1(x),\ \sin\frac{x}{2}w_1(x),\ \cos\frac{x}{2}w_2(x),\ \sin\frac{x}{2}w_2(x),\ \cos\frac{x}{2}w_3(x),\ \sin\frac{x}{2}w_3(x)\Big\}$$

is a Chebyshev system on the interval $[0, 2\pi)$, which can be proved in the same way as in Example 4.1.

We consider the monic sine type II multiple orthogonal polynomial of semi-integer degree $T_{\mathbf{n}}^{S,1/2} \in \mathfrak{T}_3^{1/2}$. From orthogonality conditions (2.9) we get $T_{\mathbf{n}}^{S,1/2} = \sin 7x/2$. Then the nodes x_k, $k = 0, 1, \ldots, 6$, of the optimal set of quadrature rules are $x_k = 2k\pi/7$, $k = 0, 1, \ldots, 6$.

TABLE 2. The weight coefficients $A_{\nu,k}$, $\nu = 1, 2, 3$, $k = 0, 1, \ldots, 6$, of the optimal set of quadrature rules with respect to the set of functions $W = \{3 - \cos 2x,\ 1 + 2\sin x,\ 2 + \cos x\}$ and $\mathbf{n} = (1, 1, 1)$

k	$A_{1,k}$	$A_{2,k}$	$A_{3,k}$
0	1.79519580205131	8.97597901025655(−1)	2.69279370307697
1	2.89252802633042	2.30113849626382	2.35483893951061
2	3.50150146779564	2.64778439627711	1.59546147879785
3	2.13315056561766	1.67650416806782	9.86488037332638(−1)
4	2.13315056561766	1.186916339834889(−1)	9.86488037332638(−1)
5	3.50150146779564	−8.52588594225803(−1)	1.59546147879785
6	2.89252802633042	−5.05942694212505(−1)	2.35483893951061

For each $\nu = 1, 2, 3$, the corresponding weight coefficients $A_{\nu,k}$, $k = 0, 1, \ldots, 6$, given in Table 2, can be obtained by using condition 1° of Theorem 3.1.

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Mathematical Institute of the Serbian Academy of Sciences and Arts
Belgrade
Serbia
gvm@mi.sanu.ac.rs

Department of Mathematics and Informatics
Faculty of Science
University of Kragujevac
Kragujevac
Serbia
stanicm@kg.ac.rs
tomovict@kg.ac.rs