PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 96 (110) (2014), 193–210 DOI: 10.2298/PIM1410193M

GENERALIZED COHERENT PAIRS ON THE UNIT CIRCLE AND SOBOLEV ORTHOGONAL POLYNOMIALS

Francisco Marcellán and Natalia C. Pinzón-Cortés

Abstract. A pair of regular Hermitian linear functionals $(\mathcal{U},\mathcal{V})$ is said to be an (M,N)-*coherent pair of order m on the unit circle* if their corresponding sequences of monic orthogonal polynomials $\{\phi_n(z)\}_{n\geqslant 0}$ and $\{\psi_n(z)\}_{n\geqslant 0}$ satisfy

$$\sum_{i=0}^{M} a_{i,n}\phi^{(m)}_{n+m-i}(z)=\sum_{j=0}^{N} b_{j,n}\psi_{n-j}(z),\quad n\geqslant 0,$$

where $M,N,m\geqslant 0$, $a_{i,n}$ and $b_{j,n}$, for $0\leqslant i\leqslant M$, $0\leqslant j\leqslant N$, $n\geqslant 0$, are complex numbers such that $a_{M,n}\neq 0$, $n\geqslant M$, $b_{N,n}\neq 0$, $n\geqslant N$, and $a_{i,n}=b_{i,n}=0$, $i>n$. When $m=1$, $(\mathcal{U},\mathcal{V})$ is called a (M,N)-*coherent pair on the unit circle*.

We focus our attention on the Sobolev inner product

$$\left\langle p(z),q(z)\right\rangle_{\lambda}=\left\langle \mathcal{U},p(z)\overline{q}(1/z)\right\rangle+\lambda\left\langle \mathcal{V},p^{(m)}(z)\overline{q^{(m)}}(1/z)\right\rangle,\quad \lambda>0,\ m\in\mathbb{Z}^{+},$$

assuming that $\mathcal{U}$ and $\mathcal{V}$ is an (M,N)-coherent pair of order m on the unit circle. We generalize and extend several recent results of the framework of Sobolev orthogonal polynomials and their connections with coherent pairs. Besides, we analyze the cases $(M,N)=(1,1)$ and $(M,N)=(1,0)$ in detail. In particular, we illustrate the situation when $\mathcal{U}$ is the Lebesgue linear functional and $\mathcal{V}$ is the Bernstein–Szegő linear functional. Finally, a matrix interpretation of (M,N)-coherence is given.

1. Introduction

In the theory of orthogonal polynomials with respect to measures supported on the real line the notion of *coherent pair* (in our terminology, $(1,0)$-coherent pair of order 1) arose in the framework of Sobolev orthogonal polynomials and it

---

2010 *Mathematics Subject Classification:* 13B05; 42C05.

Key words and phrases: coherent pairs, Sobolev inner products, structure relations, Hermitian linear functionals, orthogonal polynomials on the unit circle, Sobolev orthogonal polynomials, Lebesgue linear functional, Bernstein–Szegő linear functional, Hessenberg matrices.

The work of the authors has been supported by Dirección General de Investigación, Desarrollo e Innovación, Ministerio de Economía y Competitividad of Spain, grant MTM2012-36732-C03-01.

To Professor Giuseppe Mastroianni on the occasion of his retirement.

was introduced by Iserles, Koch, Nørsett and Sanz-Serna [**4**]. From this pioneering contribution, coherent pairs and Sobolev orthogonal polynomials have been widely studied and extended in recent decades (for a historical summary, see e.g., the introductory sections in the recent papers [**6**] and [**8**] as well as [**9**]. Lately, de Jesus, Marcellán, Petronilho, and Pinzón-Cortés [**5**] generalized all those works focussing the attention on the Sobolev inner product

$$\langle p(x), r(x)\rangle_\lambda = \int_{\mathbb{R}} p(x)\, r(x)\, d\mu_0 + \lambda \int_{\mathbb{R}} p^{(m)}(x)\, r^{(m)}(x)\, d\mu_1, \quad \lambda > 0,\ m \in \mathbb{Z}^+,$$

where $p(x)$ and $r(x)$ are polynomials with real coefficients. The positive definite Borel measures μ_0 and μ_1 supported on the real line are said to be a (M,N)-*coherent pair of order* m if their corresponding sequences of monic orthogonal polynomials (SMOP) $\{P_n(x)\}_{n\geqslant 0}$ and $\{Q_n(x)\}_{n\geqslant 0}$, satisfy an algebraic relation

$$\sum_{i=0}^{M} a_{i,n} P_{n+m-i}^{(m)}(x) = \sum_{i=0}^{N} b_{i,n} Q_{n-i}(x), \quad n \geqslant 0, \tag{1.1}$$

where M, N are fixed non-negative integers, $\{a_{i,n}\}_{n\geqslant 0}, \{b_{i,n}\}_{n\geqslant 0} \subset \mathbb{C}$, $a_{M,n} \neq 0$ if $n \geqslant M$, $b_{N,n} \neq 0$ if $n \geqslant N$, $a_{i,n} = b_{i,n} = 0$ when $i > n$.

Similarly, in the framework of orthogonal polynomials of a discrete variable, Álvarez-Nodarse, Petronilho, Pinzón-Cortés, and Sevinik-Adıgüzel [**1**] analyzed the case when μ_0 and μ_1 are discrete measures supported either on a uniform lattice or on a q-lattice, and they are an (M,N)-D_ν-coherent pair of order m, $\nu = \omega \in \mathbb{C}\setminus\{0\}$ or $\nu = q \in \mathbb{C} \setminus \{0,1\}$, when in (1.1) instead of the standard derivative operator you consider either $D_\omega p(x) = \frac{p(x+\omega)-p(x)}{\omega}$ or $D_q p(x) = \frac{p(qx)-p(x)}{(q-1)x}$, the difference and the q-difference operator, respectively.

In this work we will deal with the analysis of coherent pairs in the framework of the theory of orthogonal polynomials on the unit circle. The structure of the manuscript is as follows. In Section 2, we will state the basic definitions, notations and results which will be useful in the forthcoming sections. In Section 3, we will introduce the Sobolev inner product

$$\big\langle p(z), q(z)\big\rangle_\lambda = \big\langle \mathcal{U}, p(z)\, \overline{q}(1/z)\big\rangle + \lambda \big\langle \mathcal{V}, p^{(m)}(z)\, \overline{q^{(m)}}(1/z)\big\rangle, \quad \lambda > 0,\ m \in \mathbb{Z}^+, \tag{1.2}$$

and we will study its corresponding sequence of monic orthogonal polynomials when the regular Hermitian linear functionals $\mathcal{U}$ and $\mathcal{V}$ form an (M,N)-coherent pair of order m. In this way, we will highlight the cases $(M,N) = (1,1)$ and $(M,N) = (1,0)$. These results generalize those given by Branquinho, Foulquié-Moreno, Marcellán, and Rebocho in [**2**]. Finally, as an example, we will consider the cases when $\mathcal{U}$ is the Lebesgue linear functional and $\mathcal{V}$ is the Bernstein-Szegő linear functional. In Section 4, we will analyze (M,N)-coherent pairs of hermitian linear functionals on the unit circle from a matrix point of view, showing an interesting relation between the Hessenberg matrices associated with the regular Hermitian linear functionals that constitute such an (M,N)-coherent pair of order 1 (see [**7**] for the case of the linear functionals on the real line). As a special case, we will study the case when $\mathcal{U}$ is the linear functional associated with the Lebesgue measure on the unit circle. Furthermore, when $(\mathcal{U},\mathcal{V})$ is an (M,N)-coherent pair of order

m on the unit circle, we will give a matrix representation of the multiplication operator by z in terms of the basis of Sobolev polynomials orthogonal with respect to (1.2), involving a matrix similar to the Hessenberg matrix associated with $\mathcal{U}$.

2. Preliminaries

Let us consider the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the linear space of Laurent polynomials with complex coefficients $\Lambda = \operatorname{span}_{\mathbb{C}}\{z^n : n \in \mathbb{Z}\}$, and a linear functional $\mathcal{U} : \Lambda \longrightarrow \mathbb{C}$. We can associate with $\mathcal{U}$ a *sequence of moments* $\{c_n\}_{n\in\mathbb{Z}}$ defined by $c_n = \langle \mathcal{U}, z^n\rangle$, $n \in \mathbb{Z}$, and a bilinear form $\langle p(z), q(z)\rangle_{\mathcal{U}} = \langle \mathcal{U}, p(z)\overline{q}(1/z)\rangle$, where $p, q \in \mathbb{P}$, the linear space of polynomials with complex coefficients, and $\langle \mathcal{U}, f(z)\rangle$ denotes the image of $f(z) \in \Lambda$ by the linear functional $\mathcal{U}$. Notice that

$$\big\langle p(z), q(z)\big\rangle_{\mathcal{U}} = \big\langle p(z)\overline{q}(1/z), 1\big\rangle_{\mathcal{U}} = \big\langle 1, \overline{p}(1/z)q(z)\big\rangle_{\mathcal{U}} = \big\langle \overline{q}(1/z), \overline{p}(1/z)\big\rangle_{\mathcal{U}}, \quad p, q \in \mathbb{P}.$$

When $c_{-n} = \overline{c}_n$, $n \geqslant 0$, the linear functional $\mathcal{U}$ is said to be *Hermitian.* In this way, an Hermitian linear functional $\mathcal{U}$ is said to be *quasi-definite or regular* (resp. *positive definite*) if $\det(T_n) \neq 0$ (resp. > 0) for $n \geqslant 0$, where $T_n = [c_{j-k}]_{k,j=0}^{n}$, $n \geqslant 0$. Notice that in this case $\mathcal{U}$ is an Hermitian bilinear form. Besides, an Hermitian linear functional $\mathcal{U}$ is regular if and only if there exists a (unique) *sequence of monic orthogonal polynomials on the unit circle* (OPUC, in short) with respect to such a linear functional, i.e., if there exists a sequence of monic polynomials $\{\phi_n(z)\}_{n\geqslant 0}$ such that $\deg(\phi_n(z)) = n$, $n \geqslant 0$, and $\langle \phi_m(z), \phi_n(z)\rangle_{\mathcal{U}} = \kappa_n\delta_{m,n}$, $\kappa_n \neq 0$. This sequence of monic OPUC $\{\phi_n(z)\}_{n\geqslant 0}$ satisfies the *forward Szegő recurrence relation*

$$\phi_n(z) = z\phi_{n-1}(z) + \alpha_n\phi_{n-1}^*(z), \quad n \geqslant 1, \quad \phi_0(z) = 1, \tag{2.1}$$

where $\phi_n^*(z)$ denotes the *reversed polynomial* of $\phi_n(z)$ defined by $\phi_n^*(z) = z^n\overline{\phi}_n(1/z)$, $n \geqslant 0$, and $\alpha_n = \phi_n(0)$, $n \geqslant 0$, are said to be the *Verblunsky coefficients* of $\mathcal{U}$. The sequence of monic OPUC $\{\phi_n(z)\}_{n\geqslant 0}$ is completely determined by the sequence of Verblunsky coefficients $\{\alpha_n\}_{n\geqslant 0}$ which satisfy $|\alpha_n| \neq 1$ (resp. < 1) for $n \geqslant 1$ when $\mathcal{U}$ is regular (resp. positive definite) [**10, 11**].

On the other hand, for every positive definite Hermitian linear functional $\mathcal{U}$ we have an integral representation of the associated inner product as

$$\big\langle p(z), q(z)\big\rangle_{\mathcal{U}} = \big\langle \mathcal{U}, p(z)\,\overline{q}(1/z)\big\rangle = \frac{1}{2\pi}\int_0^{2\pi} p(z)\,\overline{q}(1/z)\,d\mu(\theta), \quad z = e^{i\theta},\ p, q \in \mathbb{P},$$

where μ is a nontrivial probability measure supported on an infinite subset of $\mathbb{T}$ [**10, 11**]. In this way, the *Bernstein–Szegő linear functional with parameter* $-C$ is a well known and elementary positive definite linear functional associated with the measure $d\mu(\theta) = \frac{1-|C|^2}{|1+Ce^{i\theta}|^2}\frac{d\theta}{2\pi}$, $C \in \mathbb{C}$, $|C| < 1$, such that the corresponding monic OPUC and moments are $\phi_0(z) = 1$, $\phi_n(z) = z^{n-1}(z + C)$, $n \geqslant 1$, and $c_n = (-C)^n$, $n \geqslant 0$, respectively. Notice that for $C = 0$, you recover the *Lebesgue linear functional.*

From (2.1), it is easy to check that the multiplication operator by z in terms of the (monic orthogonal) basis $\{\phi_n(z)\}_{n\geqslant 0}$ is given by

$$z\phi_n(z) = \phi_{n+1}(z) - \alpha_{n+1}\big\langle \phi_n(z), \phi_n(z)\big\rangle_{\mathcal{U}} \sum_{j=0}^{n} \frac{\overline{\alpha}_j}{\big\langle \phi_j(z), \phi_j(z)\big\rangle_{\mathcal{U}}}\, \phi_j(z), \quad n \geqslant 0,$$

which can be written in a matrix form as

$$z\Phi(z) = \mathcal{H}_\phi \Phi(z), \tag{2.2}$$

where $\Phi(z)$ is an infinite vector and $\mathcal{H}_\phi$ is an infinite lower Hessenberg matrix given by $\Phi(z) = \big[\phi_0(z),\ \phi_1(z),\ \cdots\big]^T$, and

$$\mathcal{H}_\phi = \big[h_{k,j}\big]_{k,j\geqslant 0} = \begin{cases} 1, & \text{if } j = k+1, \\ -\frac{\langle \phi_n(z), \phi_n(z)\rangle_{\mathcal{U}}}{\langle \phi_j(z), \phi_j(z)\rangle_{\mathcal{U}}}\, \alpha_{n+1}\overline{\alpha}_j, & \text{if } j \leqslant k, \\ 0, & \text{if } j > k+1. \end{cases}$$

On the other hand, for fixed $m, n \geqslant 0$, $\phi_n^{[m]}(z) = \phi_{n+m}^{(m)}(z)/(n+1)_m$ will denote the monic polynomial of degree n, that is the mth derivative of the monic polynomial of degree $n+m$, $\phi_{n+m}(z)$. Here $(n+1)_m$ is the *Pochhammer symbol* defined by $(a)_n = a(a+1)\cdots(a+n-1)$, $n \geqslant 1$, and $(a)_0 = 1$.

3. Sobolev orthogonal polynomials and (M, N)-coherent pairs of order m on the unit circle

Let us consider the Sobolev inner product

$$\big\langle p(z), q(z)\big\rangle_\lambda = \int_{\mathbb{T}} p(z)\, \overline{q}(1/z)\, d\mu_0(z) + \lambda \int_{\mathbb{T}} p^{(m)}(z)\, \overline{q^{(m)}}(1/z)\, d\mu_1(z),\ \lambda > 0,\ m \in \mathbb{Z}^+,$$

where $p, q \in \mathbb{P}$, μ_0, μ_1 are nontrivial probability measures supported on an infinite subset of the unit circle $\mathbb{T}$. These measures are associated with positive definite Hermitian linear functionals $\mathcal{U}$ and $\mathcal{V}$, respectively, defined on Λ, and whose sequences of monic OPUC are $\{\phi_n(z)\}_{n\geqslant 0}$ and $\{\psi_n(z)\}_{n\geqslant 0}$, respectively. Thus, the Sobolev inner product can also be written as

$$\begin{aligned} \big\langle p(z), q(z)\big\rangle_\lambda &= \big\langle \mathcal{U}, p(z)\, \overline{q}(1/z)\big\rangle + \lambda\big\langle \mathcal{V}, p^{(m)}(z)\, \overline{q^{(m)}}(1/z)\big\rangle \\ &= \big\langle p(z), q(z)\big\rangle_{\mathcal{U}} + \lambda\big\langle p^{(m)}(z), q^{(m)}(z)\big\rangle_{\mathcal{V}},\ \ p, q \in \mathbb{P},\ \lambda > 0,\ m \in \mathbb{Z}^+, \end{aligned} \tag{3.1}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{U}}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{V}}$ are the inner products associated with $\mathcal{U}$ and $\mathcal{V}$, respectively. Moreover, using this notation, we can also consider the case when $\mathcal{U}$ and $\mathcal{V}$ are regular Hermitian linear functionals.

On the other hand, the sequence of monic Sobolev OPUC with respect to the inner product $\langle\cdot,\cdot\rangle_\lambda$ will be denoted by $\{S_n(z;\lambda)\}_{n\geqslant 0}$.

Every monic Sobolev OPUC $S_n(z;\lambda)$, $n \geqslant 0$, can be written as

$$S_n(z;\lambda) = \frac{1}{\det\big([w_{k,j}]_{k,j=0}^{n-1}\big)} \begin{vmatrix} w_{0,0} & \cdots & w_{0,n-1} & w_{0,n} \\ \vdots & \ddots & \vdots & \vdots \\ w_{n-1,0} & \cdots & w_{n-1,n-1} & w_{n-1,n} \\ 1 & \cdots & z^{n-1} & z^n \end{vmatrix}, \quad n \geqslant 1,$$

$S_0(z;\lambda) = 1$, where $w_{k,j} = \big\langle z^k, z^j\big\rangle_\lambda = u_{k-j} + \lambda(k-m+1)_m(j-m+1)_m v_{k-j}$, $k, j \geqslant 0$. Thus, every coefficient of $S_n(z;\lambda)$ is a rational function of λ such that its numerator and denominator are polynomials of the same degree. Hence, there exist the monic limit polynomials

$$T_n(z) = \lim_{\lambda\to\infty} S_n(z;\lambda), \quad n \geqslant 0. \tag{3.2}$$

In this way,

$$\left\langle T_n(z), z^j\right\rangle_{\mathcal{U}} = 0, \;\; j < \min\{n, m\}, \quad \text{and} \quad \left\langle T_n^{(m)}(z), z^j\right\rangle_{\mathcal{V}} = 0, \;\; j < n - m. \tag{3.3}$$

As a consequence,

$$T_n(z) = \sum_{j=0}^{n} \frac{\left\langle T_n(z), \phi_j(z)\right\rangle_{\mathcal{U}}}{\left\langle \phi_j(z), \phi_j(z)\right\rangle_{\mathcal{U}}} \phi_j(z) = \sum_{j=0}^{n-m} \frac{\left\langle T_n(z), \phi_{j+m}(z)\right\rangle_{\mathcal{U}}}{\left\langle \phi_{j+m}(z), \phi_{j+m}(z)\right\rangle_{\mathcal{U}}} \phi_{j+m}(z), \tag{3.4}$$

for $n \geqslant m$, as well as

$$\frac{T^{(m)}_{n+m}(z)}{(n+1)_m} = \sum_{j=0}^{n} \frac{\left\langle T^{(m)}_{n+m}(z)/(n+1)_m, \psi_j(z)\right\rangle_{\mathcal{V}}}{\left\langle \psi_j(z), \psi_j(z)\right\rangle_{\mathcal{V}}} \psi_j(z) = \psi_n(z), \tag{3.5}$$

for $n \geqslant 0$. Thus,

$$\psi_n(z) = \phi^{[m]}_n(z) + \sum_{j=0}^{n-1} \frac{(j+1)_m}{(n+1)_m} \frac{\left\langle T_{n+m}(z), \phi_{j+m}(z)\right\rangle_{\mathcal{U}}}{\left\langle \phi_{j+m}(z), \phi_{j+m}(z)\right\rangle_{\mathcal{U}}} \phi^{[m]}_j(z), \quad n \geqslant 0.$$

From (3.3), for $n \geqslant 0$ we get

$$T_n(z) = \sum_{j=0}^{n} \frac{\left\langle T_n(z), S_j(z;\lambda)\right\rangle_{\lambda} S_j(z;\lambda)}{\left\langle S_j(z;\lambda), S_j(z;\lambda)\right\rangle_{\lambda}} = S_n(z;\lambda) + \sum_{j=m}^{n-1} \frac{\left\langle T_n(z), S_j(z;\lambda)\right\rangle_{\mathcal{U}} S_j(z;\lambda)}{\left\langle S_j(z;\lambda), S_j(z;\lambda)\right\rangle_{\lambda}},$$

and together with (3.4), for $n \geqslant m$,

$$S_n(z;\lambda) + \sum_{j=m}^{n-1} \frac{\left\langle T_n(z), S_j(z;\lambda)\right\rangle_{\mathcal{U}}}{\left\langle S_j(z;\lambda), S_j(z;\lambda)\right\rangle_{\lambda}} S_j(z;\lambda) = \phi_n(z) + \sum_{j=m}^{n-1} \frac{\left\langle T_n(z), \phi_j(z)\right\rangle_{\mathcal{U}}}{\left\langle \phi_j(z), \phi_j(z)\right\rangle_{\mathcal{U}}} \phi_j(z).$$

Finally, from (3.1), we obtain that $\left\langle \phi_n(z), z^j\right\rangle_{\lambda} = 0$, for $j < n < m$, and hence $S_n(z;\lambda) = \phi_n(z)$ for $n < m$, (since the uniqueness of the sequence of monic OPUC).

So, we have proved the following result.

Proposition 3.1. *Given the Sobolev inner product* (3.1)*, it follows that*

$$\begin{aligned} \psi_n(z) &= \phi^{[m]}_n(z) + \sum_{j=0}^{n-1} \frac{(j+1)_m}{(n+1)_m} \frac{\left\langle T_{n+m}(z), \phi_{j+m}(z)\right\rangle_{\mathcal{U}}}{\left\langle \phi_{j+m}(z), \phi_{j+m}(z)\right\rangle_{\mathcal{U}}} \phi^{[m]}_j(z), \quad n \geqslant 0, \\ S_n(z;\lambda) + \sum_{j=m}^{n-1} \frac{\left\langle T_n(z), S_j(z;\lambda)\right\rangle_{\mathcal{U}}}{\left\langle S_j(z;\lambda), S_j(z;\lambda)\right\rangle_{\lambda}} S_j(z;\lambda) &= \phi_n(z) + \sum_{j=m}^{n-1} \frac{\left\langle T_n(z), \phi_j(z)\right\rangle_{\mathcal{U}}}{\left\langle \phi_j(z), \phi_j(z)\right\rangle_{\mathcal{U}}} \phi_j(z), \end{aligned} \tag{3.6}$$

$n \geqslant m$, and $S_n(z;\lambda) = \phi_n(z)$, $n \leqslant m$, where the monic polynomials $T_n(z)$, $n \geqslant 0$, are given by (3.2).

Now, we are interested in the case when the regular Hermitian linear functionals $\mathcal{U}$ and $\mathcal{V}$ form a (M,N)-*coherent pair of order* m *on the unit circle*, which means that their corresponding sequences of monic OPUC $\{\phi_n(z)\}_{n\geqslant 0}$ and $\{\psi_n(z)\}_{n\geqslant 0}$ satisfy the following algebraic relation

$$\phi_n^{[m]}(z)+\sum_{i=1}^{M} a_{i,n}\phi_{n-i}^{[m]}(z)=\psi_n(z)+\sum_{i=1}^{N} b_{i,n}\psi_{n-i}(z),\quad n\geqslant 0, \tag{3.7}$$

where M,N are nonnegative integers, and the sequences $\{a_{i,n}\}_{n\geqslant 0},\{b_{i,n}\}_{n\geqslant 0}\subset\mathbb{C}$ are such that $a_{M,n}\neq 0$, $n\geqslant M$, $b_{N,n}\neq 0$, $n\geqslant N$, and $a_{i,n}=b_{i,n}=0$, $i>n$.

In this case, what does (3.6) become? Our aim now is to answer this question. If we substitute (3.5) in (3.7) and then we integrate m times the resulting equation, (3.7) becomes, for $n\geqslant 0$,

$$\frac{\phi_{n+m}(z)}{(n+1)_m}+\sum_{i=1}^{M} a_{i,n}\frac{\phi_{n-i+m}(z)}{(n-i+1)_m}=\frac{T_{n+m}(z)}{(n+1)_m}+\sum_{i=1}^{N} b_{i,n}\frac{T_{n-i+m}(z)}{(n-i+1)_m}+\sum_{j=0}^{m-1}\kappa_{n,j}z^j.$$

Consequently, if we apply $\left\langle\,\cdot\,,z^k\right\rangle_{\mathcal{U}}$ for $k=0,1,\dots,m-1$, to both sides in previous equation, and using (3.3), it follows that $\sum_{j=0}^{m-1}\kappa_{n,j}u_{j-k}=0$, $k=0,\dots,m-1$, which is a linear system with a unique solution since $\det\left([u_{j-k}]_{k,j=0}^{m-1}\right)\neq 0$. Therefore, $\kappa_{n,j}=0$, $j=0,\dots,m-1$, $n\geqslant 0$. Thus,

$$\frac{\phi_{n+m}(z)}{(n+1)_m}+\sum_{i=1}^{M} a_{i,n}\frac{\phi_{n-i+m}(z)}{(n-i+1)_m}=\frac{T_{n+m}(z)}{(n+1)_m}+\sum_{i=1}^{N} b_{i,n}\frac{T_{n-i+m}(z)}{(n-i+1)_m},\quad n\geqslant 0. \tag{3.8}$$

Furthermore,

$$\frac{T_{n+m}(z)}{(n+1)_m}+\sum_{i=1}^{N} b_{i,n}\frac{T_{n-i+m}(z)}{(n-i+1)_m}=\frac{S_{n+m}(z;\lambda)}{(n+1)_m}+\sum_{j=1}^{n+m}\frac{c_{j,n,\lambda}}{(n+1)_m}S_{n-j+m}(z;\lambda),\ n\geqslant 0, \tag{3.9}$$

where from (3.8) and (3.5), the coefficients $c_{j,n,\lambda}$, $1\leqslant j\leqslant n+m$, $n\geqslant 0$, are

$$\begin{aligned}&\langle S_{n-j+m}(z;\lambda),S_{n-j+m}(z;\lambda)\rangle_\lambda\frac{c_{j,n,\lambda}}{(n+1)_m}\\&\quad=\sum_{i=1}^{M}\frac{a_{i,n}\left\langle\phi_{n-i+m}(z),S_{n-j+m}(z;\lambda)\right\rangle_{\mathcal{U}}}{(n-i+1)_m}+\lambda\sum_{i=1}^{N} b_{i,n}\left\langle\psi_{n-i}(z),S_{n-j+m}^{(m)}(z;\lambda)\right\rangle_{\mathcal{V}},\end{aligned}$$

and, as a consequence, $c_{j,n,\lambda}=0$ for $j>i$ or $j>K=\max\{M,N\}$ or $j>n$. So, from (3.8) and (3.9) we get, for $n\geqslant 0$,

$$\phi_{n+m}(z)+\sum_{i=1}^{M}\frac{(n+1)_m}{(n-i+1)_m}a_{i,n}\phi_{n-i+m}(z)=S_{n+m}(z;\lambda)+\sum_{j=1}^{K}c_{j,n,\lambda}S_{n-j+m}(z;\lambda).$$

Besides, from the previous proposition $S_n(z;\lambda)=\phi_n(z)$ for $n\leqslant m$.

Conversely, if the previous equation holds, we can apply $\langle \cdot, p(z)\rangle_\lambda$ on both sides of this equation, for any $p \in \mathbb{P}_{n-K+m-1}$. As a consequence,

$$\lambda \left\langle \left(\phi^{(m)}_{n+m}(z) + \sum_{i=1}^{M} \frac{(n+1)_m}{(n-i+1)_m}\, a_{i,n} \phi^{(m)}_{n-i+m}(z) \right), p^{(m)}(z) \right\rangle_{\mathcal{V}} = 0.$$

In other words,

$$\left\langle \left(\phi^{[m]}_{n}(z) + \sum_{i=1}^{M} a_{i,n} \phi^{[m]}_{n-i}(z) \right), q(z) \right\rangle_{\mathcal{V}} = 0, \quad q \in \mathbb{P}_{n-K-1}.$$

Since

$$\phi^{[m]}_{n}(z) + \sum_{i=1}^{M} a_{i,n} \phi^{[m]}_{n-i}(z) = \psi_n(z) + \sum_{j=1}^{n} b_{j,n} \psi_{n-j}(z), \quad n \geqslant 0,$$

then $b_{j,n} = 0$, $n-j \leqslant n-K-1$, which is a (M,K)-coherence relation of order m.

Summarizing, we generalize the relation between Sobolev orthogonal polynomials and $(1,0)$-coherent pairs on the unit circle stated in [**2**].

Theorem 3.1. *If $(\mathcal{U},\mathcal{V})$ is an (M,N)-coherent pair of order m of Hermitian linear functionals on the unit circle given by* (3.7)*, then $S_n(z;\lambda) = \phi_n(z)$, $n \leqslant m$, and*

$$\tag{3.10} \begin{aligned} \phi_{n+m}(z) + \sum_{i=1}^{M} \frac{(n+1)_m}{(n-i+1)_m}\, a_{i,n} \phi_{n-i+m}(z) \\ = S_{n+m}(z;\lambda) + \sum_{j=1}^{K} c_{j,n,\lambda} S_{n-j+m}(z;\lambda), \quad n \geqslant 0, \end{aligned}$$

where $K = \max\{M,N\}$, $c_{j,n,\lambda} = 0$ for $n < j \leqslant K$, and

(3.11)

$$\begin{aligned} c_{j,n,\lambda} = \frac{(n+1)_m}{\left\langle S_{n-j+m}(z;\lambda), S_{n-j+m}(z;\lambda)\right\rangle_\lambda} \Bigg[\sum_{i=j}^{M} \frac{a_{i,n} \left\langle \phi_{n-i+m}(z), S_{n-j+m}(z;\lambda)\right\rangle_{\mathcal{U}}}{(n-i+1)_m} \\ + \lambda \sum_{i=j}^{N} b_{i,n} \left\langle \psi_{n-i}(z), S^{(m)}_{n-j+m}(z;\lambda)\right\rangle_{\mathcal{V}} \Bigg], \quad 1 \leqslant j \leqslant K. \end{aligned}$$

Conversely, if there exist sequences of complex numbers $\{a_{i,n}\}_{n\geqslant 0}$, $1 \leqslant i \leqslant M$, and $\{c_{j,n,\lambda}\}_{n\geqslant 0}$, $1 \leqslant j \leqslant K$, such that (3.10) *holds with $a_{i,n} = 0$ if $n-i+m < 0$, and $c_{j,n,\lambda} = 0$ if $n-j+m < 0$, then*

$$\phi^{[m]}_{n}(z) + \sum_{i=1}^{M} a_{i,n} \phi^{[m]}_{n-i}(z) = \psi_n(z) + \sum_{j=1}^{K} b_{j,n} \psi_{n-j}(z), \quad n \geqslant 0,$$

i.e., $(\mathcal{U},\mathcal{V})$ is a (M,K)-coherent pair of order m on the unit circle (assuming $b_{K,n} \neq 0$, $n \geqslant K$), where $b_{j,n} = 0$ for $n < j \leqslant K$, and

$$b_{j,n} = \frac{\left\langle \left(\phi^{[m]}_{n}(z) + \sum_{i=1}^{M} a_{i,n} \phi^{[m]}_{n-i}(z)\right), \psi_{n-j}(z)\right\rangle_{\mathcal{V}}}{\langle \psi_{n-j}(z), \psi_{n-j}(z)\rangle_{\mathcal{V}}}, \quad 1 \leqslant j \leqslant \min\{K,n\},\ n \geqslant 0.$$

REMARK 3.1. For $j = K$ and $n \geqslant K$ (3.11) reads

$$c_{K,n,\lambda} = \frac{(n+1)_m}{\left\langle S_{n-K+m}(z;\lambda), S_{n-K+m}(z;\lambda)\right\rangle_\lambda}\Bigg[\frac{a_{M,n}}{(n-M+1)_m}\big\langle \phi_{n-M+m}(z),$$
$$\phi_{n-M+m}(z)\big\rangle_{\mathcal{U}}\delta_{M,K} + \lambda(n-N+1)_m b_{N,n}\big\langle \psi_{n-N}(z), \psi_{n-N}(z)\big\rangle_{\mathcal{V}}\delta_{N,K}\Bigg].$$

Thus, for every $n \geqslant K$ we will claim that

- if $M > N$ and $a_{M,n} \neq 0$, then $c_{K,n,\lambda} \neq 0$,
- if $M < N$ and $b_{N,n} \neq 0$, then $c_{K,n,\lambda} \neq 0$,
- if $M = N (= K)$ and $a_{M,n}b_{N,n} \neq 0$, then $c_{K,n,\lambda} \neq 0$ holds if and only if $a_{K,n}\langle \phi_{n-K+m}(z), \phi_{n-K+m}(z)\rangle_{\mathcal{U}} + \lambda(n-K+1)_m^2 b_{K,n}\langle \psi_{n-K}(z), \psi_{n-K}(z)\rangle_{\mathcal{V}} \neq 0$.

Applying Theorem 3.1, we obtain the monic Sobolev OPUC $S_n(z;\lambda)$, $n \geqslant 0$, as well as the coefficients $c_{j,n,\lambda}$, $1 \leqslant j \leqslant K$, $n \geqslant 0$, assuming that $(\mathcal{U}, \mathcal{V})$ is an (M, N)-coherent pair of order m of Hermitian linear functionals on the unit circle. Nevertheless, we can compute the sequences $\{\langle S_n(z;\lambda), S_n(z;\lambda)\rangle_\lambda\}_{n\geqslant 0}$ and $\{c_{j,n,\lambda}\}_{n\geqslant 0}$, $1 \leqslant j \leqslant K$, without knowing the explicit expressions of the monic Sobolev OPUC $S_n(z;\lambda)$, $n \geqslant 0$. Of course, if we also want to obtain these monic Sobolev OPUC polynomials, then we can use (3.10).

Indeed, let

(3.12)
$$s_n = \big\langle S_n(z;\lambda), S_n(z;\lambda)\big\rangle_\lambda,\ \widetilde{a}_{i,n} = \frac{(n+1)_m}{(n-i+1)_m}\, a_{i,n},\ \widetilde{b}_{i,n} = (n+1)_m b_{i,n},\quad n \geqslant 0,$$

where $\widetilde{a}_{i,n} = \widetilde{b}_{i,n} = 0$ for $i > n$, $\widetilde{a}_{0,n} = 1$ and $\widetilde{b}_{0,n} = (n+1)_m$, for $n \geqslant 0$.

Since (3.10) and (3.11) hold setting $c_{0,n,\lambda} = 1$ for $n \geqslant 0$, from (3.7) and (3.10), it follows that (3.11) becomes, for $n \geqslant j$ and $0 \leqslant j \leqslant K$,

$$\begin{aligned} s_{n-j+m}c_{j,n,\lambda} = {} & \sum_{i=j}^{M}\sum_{\ell=0}^{M} \widetilde{a}_{i,n}\overline{\widetilde{a}}_{\ell,n-j}\big\langle \phi_{n-i+m}(z), \phi_{n-j-\ell+m}(z)\big\rangle_{\mathcal{U}} \\ & - \sum_{i=j}^{M}\sum_{\ell=1}^{K} \widetilde{a}_{i,n}\overline{c}_{\ell,n-j,\lambda}\big\langle \phi_{n-i+m}(z), S_{n-j-\ell+m}(z;\lambda)\big\rangle_{\mathcal{U}} \\ & + \lambda\sum_{i=j}^{N}\sum_{\ell=0}^{N} \widetilde{b}_{i,n}\overline{\widetilde{b}}_{\ell,n-j}\big\langle \psi_{n-i}(z), \psi_{n-j-\ell}(z)\big\rangle_{\mathcal{V}} \\ & - \lambda\sum_{i=j}^{N}\sum_{\ell=1}^{K} \widetilde{b}_{i,n}\overline{c}_{\ell,n-j,\lambda}\big\langle \psi_{n-i}(z), S^{(m)}_{n-j-\ell+m}(z;\lambda)\big\rangle_{\mathcal{V}}, \end{aligned}$$

where $\langle \phi_{n-i+m}(z), S_{n-j-\ell+m}(z;\lambda)\rangle_{\mathcal{U}} = 0$, and $\langle \psi_{n-i}(z), S^{(m)}_{n-j-\ell+m}(z;\lambda)\rangle = 0$, for $i < j+\ell$ or $j+\ell > K$ $(\geqslant M, N)$. Consequently, for $n \geqslant j$ and $0 \leqslant j \leqslant K$, we get

$$\begin{aligned}
s_{n-j+m}c_{j,n,\lambda} = {} & \sum_{i=j}^{M} \widetilde{a}_{i,n}\overline{\overline{a}}_{i-j,n-j}\big\langle \phi_{n-i+m}(z), \phi_{n-i+m}(z)\big\rangle_{\mathcal{U}} \\
& + \lambda \sum_{i=j}^{N} \widetilde{b}_{i,n}\overline{\overline{b}}_{i-j,n-j}\big\langle \psi_{n-i}(z), \psi_{n-i}(z)\big\rangle_{\mathcal{V}} \\
& - \sum_{\ell=1}^{K-j} \overline{c}_{\ell,n-j,\lambda} \sum_{i=j+\ell}^{M} \widetilde{a}_{i,n}\big\langle \phi_{n-i+m}(z), S_{n-j-\ell+m}(z;\lambda)\big\rangle_{\mathcal{U}} \\
& - \lambda \sum_{\ell=1}^{K-j} \overline{c}_{\ell,n-j,\lambda} \sum_{i=j+\ell}^{N} \widetilde{b}_{i,n}\big\langle \psi_{n-i}(z), S^{(m)}_{n-j-\ell+m}(z;\lambda)\big\rangle_{\mathcal{V}}.
\end{aligned}$$

Notice that, according to (3.11), the sum of the last two terms is equal to $-\sum_{\ell=1}^{K-j} \overline{c}_{\ell,n-j,\lambda}s_{n-j-\ell+m}c_{j+\ell,n,\lambda}$. Therefore, replacing n by $n+j$, we have got the recurrence relation stated in the following theorem.

THEOREM 3.2. *If $(\mathcal{U},\mathcal{V})$ is an (M,N)-coherent pair of order m on the unit circle given by* (3.7)*, then*

$$s_{n+m}c_{j,n+j,\lambda} = \zeta_{j,n,\lambda} - \sum_{\ell=1}^{K-j} \overline{c}_{\ell,n,\lambda}c_{j+\ell,n+j,\lambda}s_{n-\ell+m}, \quad 0 \leqslant j \leqslant K,\ n \geqslant 0, \tag{3.13}$$

$s_n = \langle \phi_n(z), \phi_n(z)\rangle_{\mathcal{U}}$, *$n < m$, $c_{j,n,\lambda} = 0$, $n < j \leqslant K$, $c_{0,n,\lambda} = 1$, $n \geqslant 0$, where*

$$\begin{aligned}
\zeta_{j,n,\lambda} = {} & \sum_{i=j}^{M} \widetilde{a}_{i,n+j}\,\overline{\overline{a}}_{i-j,n}\big\langle \phi_{n+j-i+m}(z), \phi_{n+j-i+m}(z)\big\rangle_{\mathcal{U}} \\
& + \lambda \sum_{i=j}^{N} \widetilde{b}_{i,n+j}\,\overline{\overline{b}}_{i-j,n}\big\langle \psi_{n+j-i}(z), \psi_{n+j-i}(z)\big\rangle_{\mathcal{V}}, \quad 0 \leqslant j \leqslant K,
\end{aligned}$$

with s_n, $\widetilde{a}_{i,n}$ and $\widetilde{b}_{i,n}$, $n \geqslant 0$, given by (3.12)*, and $K = \max\{M,N\}$.*

REMARK 3.2. • For the recurrence relation appearing in (3.13), we can associate the following matrix with $K+1$ rows and infinitely many columns

$$\begin{bmatrix}
s_m & s_{m+1} & s_{m+2} & \cdots & & & & \\
0 & c_{1,1,\lambda} & c_{1,2,\lambda} & c_{1,3,\lambda} & \cdots & & & \\
0 & 0 & c_{2,2,\lambda} & c_{2,3,\lambda} & c_{2,4,\lambda} & \cdots & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
0 & \cdots & 0 & 0 & c_{K,K,\lambda} & c_{K,K+1,\lambda} & c_{K,K+2,\lambda} & \cdots
\end{bmatrix},$$

which indicates the order for the computation of the sequences $\{s_{m+n}\}_{n\geqslant 0}$ and $\{c_{j,n+j,\lambda}\}_{n\geqslant 0}$, $1 \leqslant j \leqslant K$, through its decreasing diagonals.

• For $j = 0$, (3.13) becomes the following non-homogeneous linear difference equation of order K satisfied by $\{s_n\}_{n\geqslant 0}$

$$s_{n+m} + \sum_{\ell=1}^{K} |c_{\ell,n,\lambda}|^2 s_{n-\ell+m} = \zeta_{0,n,\lambda}, \quad n \geqslant 0,$$

$s_n = \langle \phi_n(z), \phi_n(z) \rangle_{\mathcal{U}}$, $n < m$, and $c_{j,n,\lambda} = 0$, $n < j \leqslant K$, where for $n \geqslant 0$,

$$\zeta_{0,n,\lambda} = \sum_{i=0}^{M} |\widetilde{a}_{i,n}|^2 \langle \phi_{n-i+m}(z), \phi_{n-i+m}(z) \rangle_{\mathcal{U}} + \lambda \sum_{i=0}^{N} |\widetilde{b}_{i,n}|^2 \langle \psi_{n-i}(z), \psi_{n-i}(z) \rangle_{\mathcal{V}}.$$

The sequences $\{s_n = \langle S_n(z;\lambda), S_n(z;\lambda) \rangle_\lambda\}_{n\geqslant 0}$ and $\{c_{j,n,\lambda}\}_{n\geqslant 0}$, $1 \leqslant j \leqslant K$, fulfill some additional properties in the cases when $\mathcal{U}$ and $\mathcal{V}$ constitute a $(1,1)$-coherent or $(1,0)$-coherent pair of order m of Hermitian linear functionals on the unit circle. This will be the topic to be studied in the next two subsections.

3.1. $(1,1)$-coherent pairs of order m on the unit circle. Let $(\mathcal{U}, \mathcal{V})$ be a $(1,1)$-coherent pair of order m of regular Hermitian linear functionals on the unit circle such that

$$\phi_n^{[m]}(z) + a_{1,n}\phi_{n-1}^{[m]}(z) = \psi_n(z) + b_{1,n}\psi_{n-1}(z), \quad n \geqslant 0, \tag{3.14}$$

with $a_{1,0} = b_{1,0} = 0$. Let $\{S_n(z;\lambda)\}_{n\geqslant 0}$ be the sequence of monic Sobolev OPUC with respect to inner product (3.1) and let $s_n = \langle S_n(z;\lambda), S_n(z;\lambda) \rangle_\lambda$, $n \geqslant 0$, denotes their square norm in terms of the Sobolev inner product. Then,

i. From Theorem 3.1, the monic Sobolev OPUC $S_n(z;\lambda)$, $n \geqslant 0$, satisfy

$$\phi_{m+n}(z) + \frac{n+m}{n} a_{1,n}\phi_{m+n-1}(z) = S_{m+n}(z;\lambda) + c_{1,n,\lambda} S_{m+n-1}(z;\lambda), \quad n \geqslant 0,$$

and $S_n(z;\lambda) = \phi_n(z)$, $n \leqslant m$, where

$$\begin{aligned} c_{1,n,\lambda} = \frac{1}{s_{m+n-1}} \Big[& \frac{n+m}{n} a_{1,n} \langle \phi_{m+n-1}(z), \phi_{m+n-1}(z) \rangle_{\mathcal{U}} \\ & + \lambda (n)_m (n+1)_m b_{1,n} \langle \psi_{n-1}(z), \psi_{n-1}(z) \rangle_{\mathcal{V}} \Big]. \end{aligned}$$

In particular, $c_{1,0,\lambda} = 0$.

ii. From Theorem 3.2, the sequences $\{s_n\}_{n\geqslant 0}$ and $\{c_{1,n,\lambda}\}_{n\geqslant 0}$ satisfy

$$s_{n+m} c_{1,n+1,\lambda} = \zeta_{1,n,\lambda} \quad \text{and} \quad s_{n+m} = \zeta_{0,n,\lambda} - |c_{1,n,\lambda}|^2 s_{n+m-1}, \quad n \geqslant 0,$$

where

$$\begin{aligned} \zeta_{1,n,\lambda} &= \frac{n+m+1}{n+1} a_{1,n+1} \langle \phi_{m+n}(z), \phi_{m+n}(z) \rangle_{\mathcal{U}} \\ &\qquad + \lambda (n+1)_m (n+2)_m b_{1,n+1} \langle \psi_n(z), \psi_n(z) \rangle_{\mathcal{V}}, \\ \zeta_{0,n,\lambda} &= \langle \phi_{n+m}(z), \phi_{n+m}(z) \rangle_{\mathcal{U}} + \frac{(n+m)^2}{n^2} |a_{1,n}|^2 \langle \phi_{n+m-1}(z), \phi_{n+m-1}(z) \rangle_{\mathcal{U}} \\ &\quad + \lambda (n+1)_m^2 \big[\langle \psi_n(z), \psi_n(z) \rangle_{\mathcal{V}} + |b_{1,n}|^2 \langle \psi_{n-1}(z), \psi_{n-1}(z) \rangle_{\mathcal{V}} \big]. \end{aligned} \tag{3.15}$$

Therefore, if $\zeta_{1,n,\lambda} \neq 0$ for $n \geqslant 0$, then s_n and $c_{1,n,\lambda}$, $n \geqslant 0$, are given by

$$s_{m+n+1} = \zeta_{0,n+1,\lambda} - \frac{|\zeta_{1,n,\lambda}|^2}{\overline{s}_{m+n}}, \quad \frac{\zeta_{1,n,\lambda}}{c_{1,n+1,\lambda}} = \zeta_{0,n,\lambda} - \zeta_{1,n-1,\lambda}\overline{c}_{1,n,\lambda}, \quad n \geqslant 0, \tag{3.16}$$

with $c_{1,0,\lambda} = 0$, $s_m = \zeta_{0,0,\lambda}$, $s_n = \langle \phi_n(z), \phi_n(z) \rangle_{\mathcal{U}}$, $n < m$.

iii. Rewriting the equations in (3.16) we get $c_{1,1,\lambda} = \zeta_{1,0,\lambda}/\zeta_{0,0,\lambda}$, and

$$\overline{c}_{1,n,\lambda} = \frac{\zeta_{0,n,\lambda}}{\zeta_{1,n-1,\lambda}} - \frac{\frac{\zeta_{1,n,\lambda}}{\zeta_{1,n-1,\lambda}}}{c_{1,n+1,\lambda}}, \quad n \geqslant 1, \quad \overline{s}_{m+n} = \frac{|\zeta_{1,n,\lambda}|^2}{\zeta_{0,n+1,\lambda} - s_{m+n+1}}, \quad n \geqslant 0.$$

As a consequence, when $\zeta_{1,n,\lambda} \neq 0$ for $n \geqslant 0$, every constant s_{m+n}, $n \geqslant 0$, and $c_{1,n,\lambda}$, $n \geqslant 1$, can be represented by continued fractions as

$$\overline{s}_{m+n} = \frac{|\zeta_{1,n,\lambda}|^2}{\big[\zeta_{0,n+1,\lambda}} - \frac{|\zeta_{1,n+1,\lambda}|^2\big]}{\big[\zeta_{0,n+2,\lambda}} - \cdots, \quad n \geqslant 0,$$

$$\overline{c}_{1,n,\lambda} = \frac{\zeta_{0,n,\lambda}}{\zeta_{1,n-1,\lambda}} - \frac{\frac{\zeta_{1,n,\lambda}}{\zeta_{1,n-1,\lambda}}}{\Big[\frac{\zeta_{0,n+1,\lambda}}{\zeta_{1,n,\lambda}}} - \frac{\frac{\zeta_{1,n+1,\lambda}}{\zeta_{1,n,\lambda}}\Big]}{\Big[\frac{\zeta_{0,n+2,\lambda}}{\zeta_{1,n+1,\lambda}}} - \cdots, \quad n \geqslant 1.$$

iv. From the theory of continued fractions, it is possible to define a sequence $\{\varpi_{n,\lambda}\}_{n\geqslant 0}$ as follows

$$\varpi_{0,\lambda} = 1 \quad \text{and} \quad \varpi_{n+1,\lambda} = s_{m+n}\varpi_{n,\lambda}, \ \ n \geqslant 0. \tag{3.17}$$

So, when $s_n \in \mathbb{R}$, $n \geqslant 0$, (for instance, when $\mathcal{U}$ and $\mathcal{V}$ are positive definite Hermitian linear functionals), then the first equation in (3.16) becomes

$$\varpi_{n+2,\lambda} = \zeta_{0,n+1,\lambda}\varpi_{n+1,\lambda} - |\zeta_{1,n,\lambda}|^2\varpi_{n,\lambda}, \ n \geqslant 0, \quad \varpi_{1,\lambda} = \zeta_{0,0,\lambda}, \quad \varpi_{0,\lambda} = 1.$$

As a consequence, since $\zeta_{1,n,\lambda} \neq 0$, $n \geqslant 0$, let us consider the sequence of monic polynomials $\{\varpi_n(x;\lambda)\}_{n\geqslant 0}$ such that $\varpi_n(0;\lambda) = \varpi_{n,\lambda}$, $n \geqslant 0$, satisfying the three-term recurrence relation

$$\begin{gathered} \varpi_{n+1}(x;\lambda) = (x + \zeta_{0,n,\lambda})\varpi_n(x;\lambda) - |\zeta_{1,n-1,\lambda}|^2\varpi_{n-1}(x;\lambda), \quad n \geqslant 0, \\ \varpi_0(x;\lambda) = 1, \quad \varpi_{-1}(x;\lambda) = 0. \end{gathered} \tag{3.18}$$

From Favard's theorem, the sequence of monic polynomials $\{\varpi_n(x;\lambda)\}_{n\geqslant 0}$ is orthogonal with respect to some regular linear functional on $\mathbb{P}$, which will be positive definite when $\zeta_{0,n,\lambda} \in \mathbb{R}$. Then, this sequence will be orthogonal with respect to some positive Borel measure supported on the real line. Therefore, the sequences $\{s_n\}_{n\geqslant 0}$ and $\{c_{1,n,\lambda}\}_{n\geqslant 0}$ fulfill

$$s_{m+n} = \frac{\varpi_{n+1}(0;\lambda)}{\varpi_n(0;\lambda)}, \quad c_{1,n+1,\lambda} = \zeta_{1,n,\lambda}\frac{\varpi_n(0;\lambda)}{\varpi_{n+1}(0;\lambda)}, \quad n \geqslant 0.$$

Remark 3.3. If $c_{1,n,\lambda} \in \mathbb{R}$ and $\zeta_{1,n,\lambda} \neq 0$ for $n \geqslant 0$, then the same analysis done in the previous item **iv** holds for the second equation in (3.16). In fact, using

$$\theta_{n+1,\lambda} = \frac{\zeta_{1,n,\lambda}/\zeta_{1,n-1,\lambda}}{c_{1,n+1,\lambda}}\theta_{n,\lambda}, \ n \geqslant 1, \quad \theta_{1,\lambda} = \frac{\zeta_{1,0,\lambda}}{c_{1,1,\lambda}}\theta_{0,\lambda}, \quad \theta_{0,\lambda} = 1,$$

instead of (3.17), we get the existence of a SMOP $\{\theta_n(x;\lambda)\}_{n\geqslant 0}$, such that $\theta_n(0;\lambda) = \theta_{n,\lambda}$ for $n \geqslant 0$, which satisfies the three-term recurrence relation

$$\begin{aligned}
\theta_{n+1}(x;\lambda) &= \Big(x + \frac{\zeta_{0,n,\lambda}}{\zeta_{1,n-1,\lambda}}\Big)\theta_n(x;\lambda) - \frac{\zeta_{1,n-1,\lambda}}{\zeta_{1,n-2,\lambda}}\theta_{n-1}(x;\lambda), \quad n \geqslant 2,\\
\theta_2(x;\lambda) &= \Big(x + \frac{\zeta_{0,1,\lambda}}{\zeta_{1,0,\lambda}}\Big)\theta_1(x;\lambda) - \zeta_{1,0,\lambda}\theta_0(x;\lambda),\\
\theta_1(x;\lambda) &= x + \zeta_{0,0,\lambda}, \quad \theta_0(x;\lambda) = 1.
\end{aligned} \tag{3.19}$$

Additionally, if $\zeta_{0,n,\lambda}, \zeta_{1,n,\lambda} \in \mathbb{R}$ and $\zeta_{1,n,\lambda} > 0$ for $n \geqslant 0$, then $\{\theta_n(x;\lambda)\}_{n\geqslant 0}$ is orthogonal with respect to some positive Borel measure supported on the real line.

Finally, in this case, the sequences $\{s_n\}_{n\geqslant 0}$ and $\{c_{1,n,\lambda}\}_{n\geqslant 0}$ satisfy $c_{1,1,\lambda} = \zeta_{1,0,\lambda}\frac{\theta_0(0;\lambda)}{\theta_1(0;\lambda)}$ and

$$c_{1,n+1,\lambda} = \frac{\zeta_{1,n,\lambda}}{\zeta_{1,n-1,\lambda}}\frac{\theta_n(0;\lambda)}{\theta_{n+1}(0;\lambda)}, \quad s_{m+n} = \zeta_{1,n-1,\lambda}\frac{\theta_{n+1}(0;\lambda)}{\theta_n(0;\lambda)}, \quad n \geqslant 1.$$

EXAMPLE 3.1. In [**3**, Section 4] it was proved that if $(\mathcal{U},\mathcal{V})$ is a $(1,1)$-coherent pair of order m of Hermitian linear functionals on the unit circle given by (3.14), such that $\mathcal{U}$ is the Lebesgue linear functional, $\mathcal{V}$ is normalized, i.e., $v_0 = 1$, $|b_{1,1} - a_{1,1}| < 1$, and, either $a_{1,2} = a_{1,1} - b_{1,1}$ (i.e., $b_{1,2} = 0$) or $b_{1,N} = 0$ for some $N \geqslant 2$, then $a_{1,n} = a_{1,1} - b_{1,1}$ as well as $b_{1,n} = 0$ for $n \geqslant 2$, and $\mathcal{V}$ is the Bernstein–Szegő linear functional with parameter $-a_{1,2} = b_{1,1} - a_{1,1}$. In this way, $\langle\phi_n(z),\phi_n(z)\rangle_{\mathcal{U}} = \langle z^n, z^n\rangle_{\mathcal{U}} = 1$, $n \geqslant 0$, $\langle\psi_n(z),\psi_n(z)\rangle_{\mathcal{V}} = \langle z^n + a_{1,2}z^{n-1}, z^n + a_{1,2}z^{n-1}\rangle_{\mathcal{V}} = 1 - |a_{1,2}|^2$, $n \geqslant 1$, and $\langle\psi_0(z),\psi_0(z)\rangle_{\mathcal{V}} = 1$. Thus, from (3.15), the coefficients $\zeta_{0,n,\lambda}$ and $|\zeta_{1,n,\lambda}|^2$, $n \geqslant 0$, appearing in the three-term recurrence relation (3.18) satisfied by the associated SMOP $\{\varpi_n(x;\lambda)\}_{n\geqslant 0}$, are $\zeta_{0,0,\lambda} = 1 + \lambda(m!)^2 \in \mathbb{R}$,

$$\begin{aligned}
\zeta_{0,1,\lambda} &= 1 + (m+1)^2|a_{1,1}|^2 + \lambda[(m+1)!]^2\big(1 - |a_{1,2}|^2 + |b_{1,1}|^2\big) \in \mathbb{R},\\
\zeta_{0,n,\lambda} &= 1 + \frac{(n+m)^2}{n^2}|a_{1,2}|^2 + \lambda(n+1)_m^2\big(1 - |a_{1,2}|^2\big) \in \mathbb{R}, \quad n \geqslant 2,\\
\zeta_{1,0,\lambda} &= (m+1)a_{1,1} + \lambda(m!)^2(m+1)b_{1,1}, \quad \zeta_{1,n,\lambda} = \frac{n+m+1}{n+1}\,a_{1,2}, \;\; n \geqslant 1,
\end{aligned}$$

where $a_{1,2} = a_{1,1} - b_{1,1}$. Additionally, from previous equations, the coefficients of the three-term recurrence relation (3.19) satisfied by the SMOP $\{\theta_n(x;\lambda)\}_{n\geqslant 0}$ are

$$\begin{aligned}
\frac{\zeta_{0,n,\lambda}}{\zeta_{1,n-1,\lambda}} &= \frac{1 + \frac{(n+m)^2}{n^2}|a_{1,2}|^2 + \lambda(n+1)_m^2\big(1 - |a_{1,2}|^2\big)}{\frac{n+m}{n}\,a_{1,2}}, \quad n \geqslant 2,\\
\frac{\zeta_{0,1,\lambda}}{\zeta_{1,0,\lambda}} &= \frac{1 + (m+1)^2|a_{1,1}|^2 + \lambda[(m+1)!]^2\big(1 - |a_{1,2}|^2 + |b_{1,1}|^2\big)}{(m+1)a_{1,1} + \lambda(m!)^2(m+1)b_{1,1}},\\
\frac{\zeta_{1,n,\lambda}}{\zeta_{1,n-1,\lambda}} &= \frac{n(n+m+1)}{(n+1)(n+m)}, \; n \geqslant 2, \quad \frac{\zeta_{1,1,\lambda}}{\zeta_{1,0,\lambda}} = \frac{\frac{m+2}{2}\,a_{1,2}}{(m+1)a_{1,1} + \lambda(m!)^2(m+1)b_{1,1}}.
\end{aligned}$$

3.2. $(1,0)$-coherent pairs of order m on the unit circle. When the regular Hermitian linear functionals $\mathcal{U}$ and $\mathcal{V}$ form a $(1,0)$-coherent pair of order m

on the unit circle, i.e., (3.14) reads

$$\phi_n^{[m]}(z) + a_{1,n}\phi_{n-1}^{[m]}(z) = \psi_n(z),\ n \geqslant 0, \quad a_{1,0} = 0, a_{1,n} \neq 0,\ n \geqslant 1,$$

all the results obtained in the previous subsection hold taking $b_{1,n} = 0$, for $n \geqslant 0$.

In particular, the coefficients $\zeta_{1,n,\lambda}$ and $\zeta_{0,n,\lambda}$, $n \geqslant 0$, become

$$\zeta_{1,n,\lambda} = \frac{n+m+1}{n+1}\, a_{1,n+1}\big\langle \phi_{m+n}(z), \phi_{m+n}(z)\big\rangle_{\mathcal{U}},$$

$$\zeta_{0,n,\lambda} = \big\langle \phi_{n+m}(z), \phi_{n+m}(z)\big\rangle_{\mathcal{U}} + \frac{(n+m)^2}{n^2}|a_{1,n}|^2\big\langle \phi_{n+m-1}(z), \phi_{n+m-1}(z)\big\rangle_{\mathcal{U}} + \lambda(n+1)_m^2\big\langle \psi_n(z), \psi_n(z)\big\rangle_{\mathcal{V}},$$

i.e., constants and polynomials in the variable λ of degree 1. In this way, in the item **iv**, it is possible to show that $\varpi_n(0;\lambda)$ is a polynomial in λ of degree n with leading coefficient $\prod_{j=0}^{n-1}(j+1)_m^2\big\langle \psi_j(z), \psi_j(z)\big\rangle_{\mathcal{V}}$, $n \geqslant 1$, using induction on n and (3.18). Consequently, since $a_{1,n} \neq 0$ for $n \geqslant 1$, from (3.18), the monic polynomials $\widetilde{\varpi}_n(\lambda) = \varpi_n(0;\lambda)/(\prod_{j=0}^{n-1}(j+1)_m^2\langle \psi_j(z), \psi_j(z)\rangle_{\mathcal{V}})$, $n \geqslant 1$, are a SMOP in the variable λ because they satisfy the following three-term recurrence relation

$$\widetilde{\varpi}_{n+1}(\lambda) = (\lambda + \alpha_n)\widetilde{\varpi}_n(\lambda) - \beta_n\widetilde{\varpi}_{n-1}(\lambda),\ \ n \geqslant 0,\ \widetilde{\varpi}_0(\lambda) = 1, \tag{3.20}$$

where $\alpha_0 = \frac{\langle \phi_m(z), \phi_m(z)\rangle_{\mathcal{U}}}{(1)_m^2\langle \psi_0(z), \psi_0(z)\rangle_{\mathcal{V}}}$, $\beta_0 = 0$,

$$\alpha_n = \frac{\big\langle \phi_{n+m}(z), \phi_{n+m}(z)\big\rangle_{\mathcal{U}} + \frac{(n+m)^2}{n^2}|a_{1,n}|^2\big\langle \phi_{n+m-1}(z), \phi_{n+m-1}(z)\big\rangle_{\mathcal{U}}}{(n+1)_m^2\big\langle \psi_n(z), \psi_n(z)\big\rangle_{\mathcal{V}}}, \quad n \geqslant 1,$$

$$\beta_n = \frac{|a_{1,n}|^2|\big\langle \phi_{m+n-1}(z), \phi_{m+n-1}(z)\big\rangle_{\mathcal{U}}|^2}{(n)_m^4\big\langle \psi_n(z), \psi_n(z)\big\rangle_{\mathcal{V}}\big\langle \psi_{n-1}(z), \psi_{n-1}(z)\big\rangle_{\mathcal{V}}}, \quad n \geqslant 1.$$

As above, $\{\widetilde{\varpi}_{n+1}(\lambda)\}_{n\geqslant 0}$ is the SMOP with respect to some positive Borel measure supported on the real line if $\alpha_n \in \mathbb{R}$ and $\beta_{n+1} > 0$ for $n \geqslant 0$ (for instance, when $\mathcal{U}$ and $\mathcal{V}$ are positive definite Hermitian linear functionals).

Finally, the sequences $\{s_n\}_{n\geqslant 0}$ and $\{c_{1,n,\lambda}\}_{n\geqslant 0}$ satisfy

$$s_{m+n} = \kappa_n\frac{\widetilde{\varpi}_{n+1}(\lambda)}{\widetilde{\varpi}_n(\lambda)},\ n \geqslant 0, \quad c_{1,n,\lambda} = \widetilde{\kappa}_n\frac{\widetilde{\varpi}_{n-1}(\lambda)}{\widetilde{\varpi}_n(\lambda)},\ n \geqslant 1,$$

with

$$\kappa_n = (n+1)_m^2\langle \psi_n(z), \psi_n(z)\rangle_{\mathcal{V}}, \quad \widetilde{\kappa}_n = a_{1,n}\,\frac{(n+1)_m}{(n)_m^3}\frac{\langle \phi_{m+n-1}(z), \phi_{m+n-1}(z)\rangle_{\mathcal{U}}}{\langle \psi_{n-1}(z), \psi_{n-1}(z)\rangle_{\mathcal{V}}}.$$

Example 3.2. In a similar way as in Example 3.1, taking into account that $b_{1,n} = 0$, $n \geqslant 0$, let us consider a $(1,0)$-coherent pair of order m, $(\mathcal{U}, \mathcal{V})$, such that $\mathcal{U}$ is the Lebesgue linear functional and $\mathcal{V}$ is normalized, i.e., $v_0 = 1$. If $|a_{1,1}| < 1$, then $a_{1,n} = a_{1,1}$, $n \geqslant 2$, and $\mathcal{V}$ is the Bernstein–Szegő linear functional with parameter $-a_{1,1}$. Hence, $\langle \phi_n(z), \phi_n(z)\rangle_{\mathcal{U}} = 1$, $n \geqslant 0$, $\langle \psi_n(z), \psi_n(z)\rangle_{\mathcal{V}} = 1 - |a_{1,1}|^2$, $n \geqslant 1$, $\langle \psi_0(z), \psi_0(z)\rangle_{\mathcal{V}} = 1$. As a consequence, $\zeta_{0,0,\lambda} = 1 + \lambda(m!)^2$, and for $n \geqslant 1$,

$$\zeta_{1,n-1,\lambda} = \frac{(n+m)a_{1,1}}{n}, \quad \zeta_{0,n,\lambda} = 1 + \frac{(n+m)^2|a_{1,1}|^2}{n^2} + \lambda(n+1)_m^2\big(1 - |a_{1,1}|^2\big).$$

Besides, $\alpha_0 = \frac{1}{(1)_m^2}$, $\beta_0 = 0$, $\beta_1 = \frac{|a_{1,1}|^2}{(1)_m^4(1-|a_{1,1}|^2)} > 0$,

$$\alpha_n = \frac{1 + \frac{(n+m)^2}{n^2}|a_{1,1}|^2}{(n+1)_m^2\left(1-|a_{1,1}|^2\right)} \in \mathbb{R},\ n \geqslant 1, \quad \beta_n = \frac{|a_{1,1}|^2}{(n)_m^4\left(1-|a_{1,1}|^2\right)^2} > 0,\ n \geqslant 2,$$

are the coefficients of the three-term recurrence relation (3.20) satisfied by the corresponding SMOP $\{\widetilde{\varpi}_n(\lambda)\}_{n\geqslant 0}$.

4. A matrix interpretation of (M, N)-coherence on the unit circle

Let $\mathcal{U}$ and $\mathcal{V}$ be two regular Hermitian linear functionals defined on Λ, and let $\{\phi_n(z)\}_{n\geqslant 0}$ and $\{\psi_n(z)\}_{n\geqslant 0}$ be their corresponding sequences of monic OPUC. $(\mathcal{U}, \mathcal{V})$ is said to be a (M, N)-*coherent pair on the unit circle* if

$$\frac{\phi'_{n+1}(z)}{n+1} + \sum_{i=1}^{M} a_{i,n}\frac{\phi'_{n-i+1}(z)}{n-i+1} = \psi_n(z) + \sum_{i=1}^{N} b_{i,n}\psi_{n-i}(z), \quad n \geqslant 0, \tag{4.1}$$

holds, where M, N are non-negative integers, $a_{i,n}$, $b_{i,n}$, $n \geqslant 0$ are complex numbers such that $a_{M,n} \neq 0$ for $n \geqslant M$, $b_{N,n} \neq 0$ for $n \geqslant N$, and $a_{i,n} = b_{i,n} = 0$ for $i > n$.

If we consider the infinite dimensional vectors

$$\begin{gathered} \Phi(z) = [\phi_0(z), \phi_1(z), \cdots]^T, \quad \Phi_1(z) = [\phi_1(z), \phi_2(z), \cdots]^T \\ \Psi(z) = [\psi_0(z), \psi_1(z), \cdots]^T, \end{gathered} \tag{4.2}$$

then we can formulate the (M, N)-coherence relation (4.1) in a matrix form as

$$\mathcal{A}_1\Phi'_1(z) = \mathcal{B}\Psi(z), \tag{4.3}$$

$$\mathcal{A}\Phi'(z) = \mathcal{B}\Psi(z), \tag{4.4}$$

where $\mathcal{A}$, $\mathcal{A}_1$, and $\mathcal{B}$ are infinite matrices whose 0th rows (counting the rows from zero) are $[1\ 1\ 0\ \cdots]$, $[1\ 0\ \cdots]$, and $[1\ 0\ \cdots]$, respectively, and their corresponding nth rows, for $n \geqslant 1$, are

$$\begin{array}{c} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow n\text{th position} \\ \Big[\underbrace{0 \ \cdots \ 0}_{n-M+1 \text{ zeros}} \ \ \frac{a_{M,n}}{n-M+1} \ \ \cdots \ \ \frac{a_{2,n}}{n-1} \ \ \frac{a_{1,n}}{n} \ \ \frac{1}{n+1} \ \ 0 \ \ \cdots\Big] \end{array}$$

$$\Big[\underbrace{0 \ \cdots \ 0}_{n-M \text{ zeros}} \ \ \frac{a_{M,n}}{n-M+1} \ \ \cdots \ \ \frac{a_{1,n}}{n} \ \ \frac{1}{n+1} \ \ 0 \ \ \cdots\Big]$$

$$\Big[\underbrace{0 \ \cdots \ 0}_{n-N \text{ zeros}} \ \ b_{N,n} \ \ \cdots \ \ b_{1,n} \ \ 1 \ \ 0 \ \ \cdots\Big]$$

respectively. Thus, $\mathcal{A}$ is a lower Hessenberg matrix with $M+1$ nonzero diagonals, whose entries of its superdiagonal are $\frac{1}{n+1}$, $n \geqslant 0$, and the entries of its main diagonal are 1 and $\frac{a_{1,n}}{n}$, $n \geqslant 1$, and $\mathcal{A}_1$ and $\mathcal{B}$ are nonsingular lower triangular matrices with $M+1$ and $N+1$ nonzero diagonals, respectively, such that the entries of their main diagonals are $\frac{1}{n+1}$, $n \geqslant 0$, and 1's, respectively.

Also, we can consider the multiplication operators by z in terms of the bases $\{\phi_n(z)\}_{n\geqslant 0}$ and $\{\psi_n(z)\}_{n\geqslant 0}$ given in (2.2),

$$z\Phi(z) = \mathcal{H}_\phi \Phi(z) \quad \text{and} \quad z\Psi(z) = \mathcal{H}_\psi \Psi(z), \tag{4.5}$$

where $\mathcal{H}_\phi$ and $\mathcal{H}_\psi$ are the infinite lower Hessenberg matrices associated with $\mathcal{U}$ and $\mathcal{V}$, respectively. So, in this context, first we state a general result and, as a consequence, we apply it to coherent pairs .

LEMMA 4.1. *If the sequences of monic OPUC $\{\phi_n(z)\}_{n\geqslant 0}$, $\{\psi_n(z)\}_{n\geqslant 0}$ satisfy*

$$\Phi'(z) = \mathcal{M}\Psi(z), \tag{4.6}$$

where $\Phi(z)$ and $\Psi(z)$ are given by (4.2) and $\mathcal{M}$ is an infinite matrix (such that its 0th row is zero since $\phi_0'(x) = 0$), then

$$\mathcal{H}_\phi^2 \mathcal{M} - 2\mathcal{H}_\phi \mathcal{M} \mathcal{H}_\psi + \mathcal{M}\mathcal{H}_\psi^2 = 0, \tag{4.7}$$

where $\mathcal{H}_\phi$ and $\mathcal{H}_\psi$ are the Hessenberg matrices associated with $\{\phi_n(z)\}_{n\geqslant 0}$ and $\{\psi_n(z)\}_{n\geqslant 0}$, respectively.

PROOF. We have that

$$\mathcal{M}\mathcal{H}_\psi \Psi(z) + \Phi(z) \underset{(4.5)}{\overset{(4.6)}{=}} z\Phi'(z) + \Phi(z) = \big(z\Phi(z)\big)' \overset{(4.5)}{=} \mathcal{H}_\phi \Phi' e(z) \overset{(4.6)}{=} \mathcal{H}_\phi \mathcal{M}\Psi(z),$$

or, equivalently,

$$\Phi(z) = (\mathcal{H}_\phi \mathcal{M} - \mathcal{M}\mathcal{H}_\psi)\Psi(z). \tag{4.8}$$

Hence,

$$\mathcal{H}_\phi(\mathcal{H}_\phi \mathcal{M} - \mathcal{M}\mathcal{H}_\psi)\Psi(z) \overset{(4.8)}{=} \mathcal{H}_\phi \Phi(z) \underset{(4.5)}{\overset{(4.8)}{=}} (\mathcal{H}_\phi \mathcal{M} - \mathcal{M}\mathcal{H}_\psi)\mathcal{H}_\psi \Psi(z). \qquad \square$$

PROPOSITION 4.1. *If $(\mathcal{U}, \mathcal{V})$ is an (M,N)-coherent pair of regular Hermitian linear functionals on the unit circle given by (4.3), then*

$$\mathcal{H}_\phi^2 \begin{bmatrix} \mathbf{0} \\ \mathcal{A}_1^{-1}\mathcal{B} \end{bmatrix} - 2\mathcal{H}_\phi \begin{bmatrix} \mathbf{0} \\ \mathcal{A}_1^{-1}\mathcal{B} \end{bmatrix} \mathcal{H}_\psi + \begin{bmatrix} \mathbf{0} \\ \mathcal{A}_1^{-1}\mathcal{B} \end{bmatrix} \mathcal{H}_\psi^2 = 0,$$

where $\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots \end{bmatrix}$ is the zero row.

PROOF. The result is a straightforward consequence of Lemma 4.1 taking as $\mathcal{M}$ the matrix obtained from $\mathcal{A}_1^{-1}\mathcal{B}$ by shifting the matrix one position downward, i.e., adding a zero row to top. $\square$

PROPOSITION 4.2. *Let $(\mathcal{U}, \mathcal{V})$ be an (M,N)-coherent pair of regular Hermitian linear functionals on the unit circle given by (4.4), such that the matrix $\mathcal{A}$ is a nonsingular matrix (e.g., when $M = 1$ and $N \geqslant 0$, we have that $a_{1,n} \neq 0$, $n \geqslant 1$, and, as a consequence, $\mathcal{A}$ is a nonsingular upper bidiagonal matrix). Let consider the matrices $\mathcal{M}_\phi$ and $\mathcal{M}_\psi$, which are similar to the corresponding Hessenberg matrices of $\mathcal{U}$ and $\mathcal{V}$, respectively, such that*

$$\mathcal{M}_\phi = \mathcal{A}\mathcal{H}_\phi \mathcal{A}^{-1} \quad \text{and} \quad \mathcal{M}_\psi = \mathcal{B}\mathcal{H}_\psi \mathcal{B}^{-1}. \tag{4.9}$$

Then,

$$(\mathcal{M}_\phi - \mathcal{M}_\psi)^2 = [\mathcal{M}_\phi, \mathcal{M}_\psi],$$

where $[\mathcal{M}_\phi, \mathcal{M}_\psi]$ *is the commutator of* $\mathcal{M}_\phi$ *and* $\mathcal{M}_\psi$ *defined by* $[\mathcal{M}_\phi, \mathcal{M}_\psi] = \mathcal{M}_\phi\mathcal{M}_\psi - \mathcal{M}_\psi\mathcal{M}_\phi$.

Proof. From (4.4) and (4.7), $\mathcal{H}_\phi^2\mathcal{A}^{-1}\mathcal{B} - 2\mathcal{H}_\phi\mathcal{A}^{-1}\mathcal{B}\mathcal{H}_\psi + \mathcal{A}^{-1}\mathcal{B}\mathcal{H}_\psi^2 = 0$ holds. Therefore, multiplying on the left by $\mathcal{A}$ and on the right by $\mathcal{B}^{-1}$ in both sides of the previous equation and taking into account (4.9), the previous equation becomes $0 = \mathcal{M}_\phi^2 - 2\mathcal{M}_\phi\mathcal{M}_\psi + \mathcal{M}_\psi^2 = (\mathcal{M}_\phi - \mathcal{M}_\psi)^2 - [\mathcal{M}_\phi, \mathcal{M}_\psi]$. □

Remark 4.1. When $\mathcal{U}$ is the Lebesgue linear functional we get $\phi_n^{[m]}(z) = z^n = \phi_n(z)$, $m, n \geqslant 0$. So, if $(\mathcal{U}, \mathcal{V})$ is an (M,N)-coherent pair of order m, $m \geqslant 0$, and $\mathcal{U}$ is the Lebesgue linear functional, then (3.7) becomes

$$\phi_n(z) + \sum_{i=1}^{M} a_{i,n}\phi_{n-i}(z) = \psi_n(z) + \sum_{i=1}^{N} b_{i,n}\psi_{n-i}(z), \quad n \geqslant 0,$$

which can be written in a matrix form as

$$\widehat{\mathcal{A}}_1\Phi(z) = \mathcal{B}\Psi(z), \tag{4.10}$$

where $\Phi(z)$, $\Psi(z)$, $\mathcal{B}$ are given as in (4.4), and the infinite matrix $\widehat{\mathcal{A}}_1$ is as $\mathcal{B}$, i.e., it is a lower triangular matrix with $M+1$ nonzero diagonals, whose entries of its main diagonal are all 1's and its nth row for $n \geqslant 1$, counting the rows from zero, is

$$\begin{bmatrix} \underbrace{0 \ \cdots \ 0}_{n-M \text{ zeros}} & a_{M,n} & \cdots & a_{1,n} & 1 & 0 & \cdots \end{bmatrix}$$

Consequently, $\widehat{\mathcal{A}}_1$ is nonsingular as $\mathcal{B}$. Hence,

$$\widehat{\mathcal{A}}_1\mathcal{H}_\phi\widehat{\mathcal{A}}_1^{-1}\mathcal{B}\Psi(z) \overset{(4.10)}{=} \widehat{\mathcal{A}}_1\mathcal{H}_\phi\Phi(z) \overset{(4.5)}{=} z\widehat{\mathcal{A}}_1\Phi(z) \overset{(4.10)}{=} z\mathcal{B}\Psi(z) \overset{(4.5)}{=} \mathcal{B}\mathcal{H}_\psi\Psi(z),$$

from which we can conclude that $\widehat{\mathcal{A}}_1\mathcal{H}_\phi\widehat{\mathcal{A}}_1^{-1} = \widehat{\mathcal{M}}_\phi = \mathcal{M}_\psi = \mathcal{B}\mathcal{H}_\psi\mathcal{B}^{-1}$, where $\mathcal{H}_\phi$ and $\mathcal{H}_\psi$ are the Hessenberg matrices associated with $\mathcal{U}$ and $\mathcal{V}$, respectively. Therefore, $\mathcal{H}_\phi$ and $\mathcal{H}_\psi$ are similar matrices.

4.1. A matrix interpretation of Sobolev orthogonal polynomials and (M,N)-coherence of order m on the unit circle. Let us recall that there is a close relation between Sobolev orthogonal polynomials (M,N)-coherent pairs of order m of regular Hermitian linear functionals on the unit circle, $m \geqslant 1$. More precisely, in Theorem 3.1 we showed that the (M,N)-coherence relation of order m on the unit circle (3.7) satisfied by the sequences of monic OPUC associated with the regular Hermitian linear functionals $\mathcal{U}$ and $\mathcal{V}$ implies (3.10), i.e.,

$$\phi_{n+m}(z) + \sum_{i=1}^{M} \widetilde{a}_{i,n}\phi_{n-i+m}(z) = S_{n+m}(z;\lambda) + \sum_{j=1}^{K} c_{j,n,\lambda}S_{n-j+m}(z;\lambda), \quad n \geqslant 0,$$

and $S_n(z;\lambda) = \phi_n(z)$, $n \leqslant m$, where $K = \max\{M,N\}$, $c_{j,n,\lambda} = 0$, $n < j \leqslant K$, $\widetilde{a}_{i,n} = \frac{(n+1)_m}{(n-i+1)_m} a_{i,n}$, $n \geqslant 0$, $\widetilde{a}_{i,n} = 0$, $i > n$, $\widetilde{a}_{0,n} = 1$, $n \geqslant 0$. Here $S_n(z;\lambda)$, $n \geqslant 0$, are the monic Sobolev polynomials on the unit circle orthogonal with respect to inner product (3.1). In this way, if we write these algebraic relations as

$$\widetilde{\mathcal{A}}\Phi(z) = \mathcal{C}\mathbf{s}(z;\lambda),$$

where $\Phi(z) = [\phi_0(z),\ \phi_1(z),\ \cdots]^T$, $\mathbf{s}(z;\lambda) = [S_0(z;\lambda),\ S_1(z;\lambda),\ \cdots]^T$, and

$$\begin{array}{ccccccccc} & & & & & \downarrow^{(n+m)\text{th position}} & & & \\ \big[& \underbrace{0 \ \ \cdots \ \ 0}_{n-M+m \text{ zeros}} & \widetilde{a}_{M,n} & \cdots & \widetilde{a}_{1,n} & 1 & 0 & \cdots & \big] \end{array}$$

$$\big[\ \underbrace{0 \ \ \cdots \ \ 0}_{n-K+m \text{ zeros}} \ \ c_{K,n,\lambda} \ \ \cdots \ \ c_{1,n,\lambda} \ \ 1 \ \ 0 \ \ \cdots\ \big]$$

are the $(n+m)$th rows, respectively, $n \geqslant 0$, and $[0\ \cdots\ 0\ 1\ 0\ \cdots]$ are the first m rows, counting from zero, of the lower triangular matrices $\widetilde{\mathcal{A}}$ and $\mathcal{C}$ (notice that these matrices have $M+1$ and $K+1$ nonzero diagonals, respectively, and 1's as entries of their main diagonal), then we obtain the following result.

Proposition 4.3. *If $(\mathcal{U},\mathcal{V})$ is an (M,N)-coherent pair of order m, $m \geqslant 1$, of regular Hermitian linear functionals on the unit circle, then the matrix representation of the multiplication operator by z in terms of $\{S_n(z;\lambda)\}_{n\geqslant 0}$, the basis of monic Sobolev OPUC with respect to the inner product* (3.1)*, is given by*

$$z\mathbf{s}(z;\lambda) = \mathcal{H}_{S,\phi,\lambda}\mathbf{s}(z;\lambda),$$

where $\mathcal{H}_{S,\phi,\lambda}$ is a lower Hessenberg matrix similar to the Hessenberg matrix associated with the linear functional $\mathcal{U}$.

Proof. Since

$$z\mathbf{s}(z;\lambda) \overset{(4.1)}{=} \mathcal{C}^{-1}\widetilde{\mathcal{A}}z\Phi(z) \overset{(2.2)}{=} \mathcal{C}^{-1}\widetilde{\mathcal{A}}\mathcal{H}_\phi\Phi(z) \overset{(4.1)}{=} \mathcal{C}^{-1}\widetilde{\mathcal{A}}\mathcal{H}_\phi\widetilde{\mathcal{A}}^{-1}\mathcal{C}\mathbf{s}(z;\lambda),$$

and if $\mathcal{H}_{S,\phi,\lambda} = \mathcal{C}^{-1}\widetilde{\mathcal{A}}\mathcal{H}_\phi\widetilde{\mathcal{A}}^{-1}\mathcal{C} = \big[\mathcal{C}^{-1}\widetilde{\mathcal{A}}\big]\mathcal{H}_\phi\big[\mathcal{C}^{-1}\widetilde{\mathcal{A}}\big]^{-1}$, then the proof is complete. $\square$

References

1. R. Álvarez-Nodarse, J. Petronilho, N.C. Pinzón-Cortés, R. Sevinik-Adıgüzel, *On linearly related sequences of difference derivatives of discrete orthogonal polynomials*, J. Comput. Appl. Math., in press
2. A. Branquinho, A. Foulquié-Moreno, F. Marcellán, M.N. Rebocho, *Coherent pairs of linear functionals on the unit circle*, J. Approx. Theory **153** (2008), 122–137.
3. L. Garza, F. Marcellán, N.C. Pinzón-Cortés, $(1,1)$-*Coherent pairs on the unit circle*, Abstract Appl. Anal. (2013), Article ID 307974, 8 pages
4. A. Iserles, P.E. Koch, S.P. Nørsett, J.M. Sanz-Serna, *On polynomials orthogonal with respect to certain Sobolev inner products*, J. Approx. Theory **65** (1991), 151–175.
5. M.N. de Jesus, F. Marcellán, J. Petronilho, N.C. Pinzón-Cortés, (M,N)-*coherent pairs of order* (m,k) *and Sobolev orthogonal polynomials*, J. Comput. Appl. Math. **256** (2014), 16–35.
6. M.N. de Jesus, J. Petronilho, *Sobolev orthogonal polynomials and* (m,n)-*coherent pairs of measures*, J. Comput. Appl. Math. **237** (2013), 83–101.
7. F. Marcellán, N.C. Pinzón-Cortés, (M,N)-*oherent pairs of linear functionals and Jacobi matrices*, Appl. Math. Comput. **232** (2014), 76–83.
8. ———, *Higher order coherent pairs*, Acta Appl. Math. **121** (2012), 105–135.

9. H. G. Meijer, *A short history of orthogonal polynomials in a Sobolev space, I. The nondiscrete case*, Nieuw Arch. Wisk. **14** (1996), 93–112.
10. B. Simon, *Orthogonal Polynomials on the Unit Circle*, 2 vols. Am. Math. Soc. Colloq. Publ. Series. **54**, Am. Math. Soc., Providence, Rhode Island, 2005.
11. G. Szegő, *Orthogonal Polynomials*, (fourth edition) Am. Math. Soc. Colloq. Publ. Series **23**, Am. Math. Soc., Providence, Rhode Island, 1975.

Departamento de Matemáticas
Universidad Carlos III de Madrid
Avenida de la Universidad 30
28911 Leganés, Spain
and
Instituto de Ciencias Matemáticas (ICMAT)
Calle Nicolás Cabrera 13–15
Campus de Cantoblanco
28049 Madrid, Spain
pacomarc@ing.uc3m.es

Departamento de Matemáticas
Pontificia Universidad Javeriana
Carrera 7, No 43–82, Bogotá
Bogotá, D.C. 1100-1000 Colombia
npinzon@math.uc3m.es