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A FAST ALGORITHM FOR THE NUMERICAL SOLUTION OF AN INTEGRAL EQUATION WITH LOGARITHMIC KERNEL

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Abstract. We describe an algorithm for the numerical solution of an integral equation of the form

$$-\frac{1}{\pi}\int_{-1}^{1}[(y-x)\ln|y-x|-h(x,y)]\,\frac{u(y)\,dy}{\sqrt{1-y^2}}=f(x),\quad -1<x<1,$$

which is based on a collocation-quadrature method and which has the same convergence rate as this method, but only $O(n\log n)$ complexity. This integral equation turns out to be an ill-posed problem in (the best possible choice of) a pair of non-periodic Sobolev-like spaces. The present paper presents the technique, how to overcome this peculiarity in the investigation of the fast algorithm.

1. Introduction

Integral equations in which the kernel function of the integral operator contains a logarithmic part like $\ln|y-x|$ occur in various situations, in particular in applying boundary integral methods to boundary value problems for partial differential equations. For example, the single layer potential with density $f(x)$ for the Laplace equation in two dimensions is defined by $-\frac{1}{2\pi}\int_\Gamma \ln|y-x|f(y)\,d\Gamma_y$, where Γ denotes the boundary of the region under consideration. In [**7**] kernel functions of the form $(y-x)^\kappa\ln|y-x|$, $(x,y)\in[-1,1]^2$ with $\kappa=1$ and $\kappa=2$ are considered (cf. (2.1)), motivated by the problem of determining the shear force when a two-dimensional plate is supported along a line segment.

In [**10**, eq. (11.1)] kernels of the form

$$b_0(x,y)\ln|y-x|+b_1(x,y),\quad (x,y)\in[-1,1]^2 \tag{1.1}$$

are investigated, where the essential assumption is that the C^∞-function $b_0(x,y)$ satisfies $b_0(x,x)\neq 0$ for all $x\in[-1,1]$. This is not the case under consideration

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here, which leads to the fact that the respective operator is not longer a Fredholm operator in pairs of Banach or Hilbert spaces which seem to be suitable chosen for the problem under consideration (cf. [**10**, p. 326] with Proposition 2.1 of the present paper). Here, the image of the operator is not closed (and not dense), which means that the associated operator equation is an ill-posed problem. In [**6**] it is shown how it is possible to handle this peculiarity in proving the applicability of a collocation-quadrature method to such an equation. It is well known that discretization can imply also a kind of regularization (see, for example, [**9**, Chapter 17] and the references given there). The aim of the present paper is to use the results of [**6**] in order to realize a known idea for the construction of a fast algorithm also in the present situation of an operator with not closed and not dense image, where we concentrate on the case $\kappa = 1$.

The outline of the paper is as follows. In Section 2 we collect the results from [**7**] on the mapping properties needed here for the involved operator. In Section 3 the collocation and the quadrature method and respective results from [**6**] are presented, which are necessary for designing and investigating the fast algorithm studied in Section 4. Finally, in Section 5 we present some numerical results confirming the theoretical ones.

2. Preliminary results

The paper [**7**] is devoted to solvability properties of integral equations of the form

$$-\frac{1}{\pi}\int_{-1}^{1}\left[(y-x)^{\kappa}\ln|y-x|-h(x,y)\right]\frac{u(y)\,dy}{\sqrt{1-y^2}}=f(x),\ -1<x<1, \tag{2.1}$$

where $\kappa = 0, 1, 2$, while [**6**] deals with their numerical solution by collocation and collocation-quadrature methods. Of course, the case $\kappa = 0$ is well understood in both analytic and numerical sense (see, for example, [**1**, **4**, **8**, **11**, **12**]). To examine the existence and uniqueness of solutions of equations like (2.1), in [**7**] integral operators of the form

$$(\mathcal{A}_\kappa u)(x) = -\frac{1}{\pi}\int_{-1}^{1}(y-x)^{\kappa}\ln|y-x|\,\frac{u(y)\,dy}{\sqrt{1-y^2}}, \quad -1<x<1,$$

are considered in weighted Sobolev spaces $\mathbf{L}^{2,s}_\sigma$. Here, for $s \geqslant 0$, $\mathbf{L}^{2,s}_\sigma$ denotes the Hilbert space (a Sobolev-like space) of all $f : (-1,1) \to \mathbb{C}$ satisfying

$$\|f\|_{\sigma,s} := \left(\sum_{n=0}^{\infty}(n+1)^{2s}\left|\langle f, p^\sigma_n\rangle_\sigma\right|^2\right)^{1/2} < \infty$$

equipped with the inner product $\langle f,g\rangle_{\sigma,s} = \sum_{n=0}^{\infty}(n+1)^{2s}\langle f,p^\sigma_n\rangle_\sigma\overline{\langle g,p^\sigma_n\rangle_\sigma}$, where $\sigma(x) = (1-x^2)^{-\frac{1}{2}}$, $\langle f,g\rangle_\sigma = \int_{-1}^{1} f(x)\overline{g(x)}\sigma(x)\,dx$, and $p^\sigma_n(x)$ denotes the nth normalized Chebyshev polynomial of first kind, $p^\sigma_0(x) = \frac{1}{\sqrt{\pi}}$, $p^\sigma_n(\cos s) = \sqrt{2/\pi}\,\cos(ns)$. Spaces of this kind were introduced in [**1**, **3**, **5**, **11**] for investigating numerical methods for different classes of singular integral equations. So, in [**11**] there is shown that the operator $V : \mathbf{L}^{2,s}_{\varphi} \to \mathbf{L}^{2,s+1}_{\sigma}$ with $\varphi(x) = \sqrt{1-x^2}$ and

$$(Vu)(x) = -\frac{1}{\pi}\int_{-1}^{1} \ln|y-x|\, u(y)\, dy, \quad -1 < x < 1$$

is an isomorphism (cf. also [**12**]). In [**10**, Chapter 11] there are considered operators with kernels of the form (1.1), provided that $b_0(x,x) \neq 0$ for all $x \in [-1,1]$, in the pair of spaces $\left(\mathbf{L}^2_{\varphi}, \mathbf{H}^1_{\varphi}\right)$ with $\mathbf{L}^2_{\varphi} := \mathbf{L}^{2,0}_{\varphi}$, where the respective Sobolev-like space is constructed in another way. Namely, the space $\mathbf{H}^1_{\varphi}$ is defined as the set of all continuous functions $f : [-1,1] \to \mathbb{C}$ having a distributional derivative $f' \in \mathbf{L}^2_{\varphi}$ equipped with the norm $\|f\|_{\mathbf{H}^1_{\varphi}} = \sqrt{\|f\|^2_{\mathbf{L}^2_{\sigma}} + \|f'\|^2_{\mathbf{L}^2_{\varphi}}}$. That $\mathbf{H}^1_{\varphi}$ equals $\mathbf{L}^{2,1}_{\sigma}$ is proved in [**1**, Theorems 2.8 and 2.17]. We prefer to use the above description of the spaces $\mathbf{L}^{2,s}_{\sigma}$, since in this way one can easily see that these spaces are isometrically isomorphic to weighted sequence spaces (see below) with the help of which we can describe the mapping properties of the involved operators very precisely (cf. Proposition 2.1 together with Remark 2.1, Lemma 2.1 and Remark 2.2).

For $s < 0$, $\mathbf{L}^{2,s}_{\sigma}$ is defined as the dual space $\left(\mathbf{L}^{2,-s}_{\sigma}\right)^*$. In addition to these spaces we use the sequence spaces $\ell^2_s = \left\{\xi = (\xi_n)_{n=0}^{\infty} : \sum_{n=0}^{\infty} |\xi_n|^2 (n+1)^{2s} < \infty\right\}$, $s \in \mathbb{R}$, equipped with the inner product $\langle \xi, \eta\rangle_s = \sum_{n=0}^{\infty} (n+1)^{2s} \xi_n \overline{\eta_n}$ and the norm $\|\xi\|_s = \sqrt{\langle \xi, \xi\rangle_s}$. Clearly, for $s \geqslant 0$, the operator $\mathcal{J} : \mathbf{L}^{2,s}_{\sigma} \to \ell^2_s$, $f \mapsto (\langle f, p^{\sigma}_n\rangle_{\sigma})_{n=0}^{\infty}$ is an isometrical isomorphism. Moreover, the dual space $\left(\mathbf{L}^{2,s}_{\sigma}\right)^*$ is isometric isomorphic to ℓ^2_{-s} via the map $\mathcal{J} : \left(\mathbf{L}^{2,s}_{\sigma}\right)^* \to \ell^2_{-s}$, $v \mapsto (v(p^{\sigma}_n))_{n=0}^{\infty}$ (cf. [**7**, Section 5]). This allows us to consider the operators $\mathcal{A}_{\kappa}$ in all spaces $\mathbf{L}^{2,s}_{\sigma}$, $s \in \mathbb{R}$ by identifying them with the operators $\mathcal{J}^{-1}\mathbf{A}_{\kappa}\mathcal{J}$, where $\mathbf{A}_{\kappa}\xi$ is defined for any complex number sequence $\xi = (\xi_n)_{n=0}^{\infty}$ by the condition $\mathbf{A}_{\kappa}\xi = \mathcal{J}\mathcal{A}_{\kappa}\mathcal{J}^{-1}\xi$ for all $\xi \in \ell^2$ (see also Proposition 2.1).

In what follows, for simplicity we will restrict to the case $\kappa = 1$ and set $\mathcal{B} := \mathcal{A}_1$ as well as $\mathbf{B} := \mathbf{A}_1$. Hence, we are interested in designing a fast algorithm for the numerical solution of an integral equation of the form

$$-\frac{1}{\pi}\int_{-1}^{1} \left[(y-x)\ln|y-x| - h(x,y)\right] \frac{u(y)\, dy}{\sqrt{1-y^2}} = f(x), \quad -1 < x < 1. \tag{2.2}$$

Let us summarize some results from [**6**,**7**] which are important for our investigations here.

Proposition 2.1. [**7**, Corollaries 5.2, 5.3 and Remark 5.4] *The operator $\mathcal{B} : \mathbf{L}^{2,s}_{\sigma} \to \mathbf{L}^{2,s+2}_{\sigma}$ is linear and bounded. Its image is not closed in $\mathbf{L}^{2,s+2}_{\sigma}$. Moreover,*

$$\mathbf{B}\xi = (\beta_n\xi_{n+1} - \beta_{n-1}\xi_{n-1})_{n=0}^{\infty} \tag{2.3}$$

with $\xi_{-1} := 0$, $\beta_{-1} := 0$, $\beta_0 = \frac{\ln 2 - 1}{\sqrt{2}}$, $\beta_n = \frac{1}{2n(n+1)}$, $n \geqslant 1$. If $f \in \mathbf{L}^{2,s}_{\sigma}$ for some $s > \tau + 3$ and $\tau \geqslant -1$, then the equation

$$\mathcal{B}u = f + \zeta_0 p^{\sigma}_0 \tag{2.4}$$

has a unique solution $(u, \zeta_0) \in \mathbf{L}^{2,\tau}_{\sigma} \times \mathbb{C}$. Furthermore, for $f \equiv 0$, equation (2.4) has only the trivial solution in $\mathbf{L}^{2,-\frac{3}{2}}_{\sigma} \times \mathbb{C}$.

Let us note that the assumption $f \in \mathbf{L}^{2,s}_\sigma$ with $s > \tau + 3$ is sharp in the sense that in case $s < \tau + 3$ the existence of a solution in $\mathbf{L}^{2,\tau}_\sigma \times \mathbb{C}$ is not guaranteed (cf. [**7**, Cor. 5.7]).

Remark 2.1. [**7**, p. 2819] For $\eta \in \ell^2_s$, $s > \tau + 3$, $\tau \geqslant -1$, the solution of $\mathbf{B}\xi = \eta + (\zeta_0, 0, \dots)$ is given by

$$\zeta_0 = -4\beta_0 \sum_{m=1}^{\infty} m\,\eta_{2m} - \eta_0, \quad \xi_0 = -\frac{1}{\beta_0}\sum_{m=0}^{\infty}(2m+1)\eta_{2m+1},$$

and, for $j = 0, 1, \dots$, by

$$\xi_{2j+1} = -4(2j+1)\sum_{m=j+1}^{\infty} m\,\eta_{2m}, \quad \xi_{2(j+1)} = -4(j+1)\sum_{m=j+1}^{\infty}(2m+1)\eta_{2m+1}.$$

For given $\eta \in \ell^2_s$ with $s > 2$, define $\mathbf{B}_0\eta = (\xi, \zeta_0)$, where $\xi = [\xi_j]_{j=0}^\infty$ and $\zeta_0 \in \mathbb{C}$ are given by Remark 2.1.

Lemma 2.1. [**6**, Lemma 3.1] *Let $2 < t + 3 < s$. Then the operator $\mathcal{B}_0 : \mathbf{L}^{2,s}_\sigma \to \mathbf{L}^{2,t}_\sigma \times \mathbb{C}$ defined by $\widehat{\mathcal{J}}^{-1}\mathbf{B}_0\mathcal{J}$, where $\widehat{\mathcal{J}}(u, \zeta_0) = (\mathcal{J}u, \zeta_0)$, is linear and bounded and satisfies*

$$\mathcal{B}_0(\mathcal{B}u - \zeta_0 p^\sigma_0) = (u, \zeta_0) \quad \forall\, (u, \zeta_0) \in \mathbf{L}^{2,-1}_\sigma \times \mathbb{C} : \mathcal{B}u \in \mathbf{L}^{2,s}_\sigma.$$

As norm of (u, ζ) in $\widetilde{\mathbf{L}}^{2,t}_\sigma := \mathbf{L}^{2,t}_\sigma \times \mathbb{C}$ we take $\|(u, \zeta)\|_{\sigma,t,\sim} := \sqrt{\|u\|^2_{\sigma,t} + |\zeta|^2}$.

Remark 2.2. From the proof of [**6**, Lemma 3.1] one can see that there is a constant c such that $\|\mathcal{B}_0\|_{\mathbf{L}^{2,s}_\sigma \to \widetilde{\mathbf{L}}^{2,t}_\sigma} \leqslant c\sqrt{\sum_{n=0}^\infty (n+1)^{2(t-s)+5}} =: \rho_{s-t}$.

3. Collocation for the dominant equation and a quadrature method

In this section we describe the numerical method which is the basis for the construction of the fast algorithm in Section 4. Having in mind Prop. 2.1 we are interested in the numerical approximation of the solution $(u, \zeta_0) \in \mathbf{L}^2_\sigma \times \mathbb{C}$ of the modified (in comparison to (2.2)) equation

$$-\frac{1}{\pi}\int_{-1}^{1} [(y - x)\ln|y - x| - h(x, y)]\,\frac{u(y)\,dy}{\sqrt{1 - y^2}} = f(x) + \frac{\zeta_0}{\sqrt{\pi}}, \quad -1 < x < 1,$$

which we write shortly as

$$\mathcal{B}u = f + \zeta_0 p^\sigma_0 \tag{3.1}$$

and which is equivalent to the equation

$$\mathbf{B}\xi = \eta + (\zeta_0, 0, 0, \dots), \quad \eta = \mathcal{J}f,\ \xi = \mathcal{J}u. \tag{3.2}$$

We look for $u_n \in \operatorname{im}\mathcal{P}_{n-1}$, where $\mathcal{P}_n : \mathbf{L}^2_\sigma \to \mathbf{L}^2_\sigma$ denotes the orthoprojection

$$\mathcal{P}_n f = \sum_{k=0}^{n-1} \langle f, p^\sigma_k\rangle_\sigma\, p^\sigma_k,$$

by solving the collocation equations

$$(3.3) \qquad (\mathcal{B}u_n)\,(x^\sigma_{nj}) = f(x^\sigma_{nj}) + \frac{\zeta_0^n}{\sqrt{\pi}}, \quad j = 1, \ldots, n,$$

with the zeros $x^\sigma_{nj} = \cos\frac{2j-1}{2n}\pi$ of p^σ_n. Denoting by $\mathcal{L}^\sigma_n$ the Lagrange interpolating operator with respect to the nodes x^σ_{nj}, $j = 1, \ldots, n$, the system (3.3) can be written equivalently as

$$(3.4) \qquad \mathcal{B}u_n = \mathcal{L}^\sigma_n f + \zeta_0^n p_0^\sigma,$$

taking into account $\mathcal{B}u_n \in \operatorname{im}\mathcal{P}_n$ for all $u_n \in \operatorname{im}\mathcal{P}_{n-1}$ (cf., (2.3)), or as

$$(3.5) \qquad \mathbf{B}\xi^n = (\eta_0^n + \zeta_0^n, \eta_1^n, \ldots, \eta_{n-1}^n, 0, 0, \ldots), \quad \xi^n = \mathcal{J}u_n \in \operatorname{im}\mathbf{P}_{n-1},$$

with η_k^n given by

$$(3.6) \quad \eta_k^n = \frac{\pi}{n}\sum_{j=1}^n f(x^\sigma_{nj})p^\sigma_k(x^\sigma_{nj}) = \begin{cases} \frac{\sqrt{\pi}}{n}\sum\limits_{j=1}^n f(x^\sigma_{nj}) & k = 0, \\ \frac{\sqrt{2\pi}}{n}\sum\limits_{j=1}^n \cos\frac{k(2j-1)\pi}{2n}\, f(x^\sigma_{nj}) & k = 1, \ldots, n-1. \end{cases}$$

Moreover, $\mathbf{P}_n : \ell^2 \to \ell^2$, $\xi = (\xi_n)_{n=0}^\infty \mapsto (\xi_0, \ldots, \xi_{n-1}, 0, \ldots)$.

By $c \neq c(n, f, \ldots)$ we will mean that the constant c does not depend on $n, f, \ldots$, where c can have different values at different places. We remark that (see, for example, [**1**, Theorem 3.4]), for $s > \frac{1}{2}$,

$$(3.7) \qquad \|f - \mathcal{L}^\sigma_n f\|_{\sigma,t} \leqslant c\,n^{t-s}\|f\|_{\sigma,s}, \quad 0 \leqslant t \leqslant s, \;\; f \in \mathbf{L}^{2,s}_\sigma,$$

with $c \neq c(n, f, t)$. Note that (3.7) implies the uniform boundedness of $\mathcal{L}^\sigma_n$ in $\mathbf{L}^{2,s}_\sigma$ for $s > \frac{1}{2}$.

Proposition 3.1. [**6**, Prop. 2.1] *Assume that $f \in \mathbf{L}^{2,s}_\sigma$ for some $s > \tau + 3$, $\tau \geqslant 0$. Then the unique solution (u_n, ζ_0^n) of equation* (3.4) *converges in $\mathbf{L}^2_\sigma \times \mathbb{C}$ to the unique solution $(u, \zeta_0) \in \mathbf{L}^2_\sigma \times \mathbb{C}$ of equation* (3.1)*, where*

$$|\zeta_0^n - \zeta_0| \leqslant c\,n^{\frac{3}{2}-s}\|f\|_{\sigma,s} \quad \textit{and} \quad \|u_n - u\|_{\sigma,t} \leqslant c\,n^{t-\tau}\|f\|_{\sigma,s}, \quad 0 \leqslant t \leqslant \tau,$$

with $c \neq c(n, f, t)$.

Now, assume that $s_0 > 3$ and that the continuous function $h : [-1, 1]^2 \to \mathbb{C}$ satisfies

$$(\mathrm{A}) \qquad h(\cdot, y) \in \mathbf{L}^{2,s_0}_\sigma \text{ uniformly w.r.t. } y \in [-1, 1].$$

Define $\mathcal{H} : \mathbf{L}^2_\sigma \to \mathbf{L}^2_\sigma$ by

$$(\mathcal{H}u)(x) := \frac{1}{\pi}\int_{-1}^1 h(x, y)\,\frac{u(y)\,dy}{\sqrt{1 - y^2}}, \quad -1 < x < 1.$$

Then condition (A) ensures that $\mathcal{H} \in \mathcal{L}(\mathbf{L}^2_\sigma, \mathbf{L}^{2,s_0}_\sigma)$ (cf. [**1**, Lemma 4.2]). We look for an approximate solution $(u_n, \zeta_0^n) \in \operatorname{im}\mathcal{P}_{n-1} \times \mathbb{C}$ of the equation

$$(3.8) \qquad (\mathcal{B} + \mathcal{H})u = f + \zeta_0 p_0^\sigma,$$

where we assume that

(B) Equation (3.8) with $f \equiv 0$ has only the trivial solution in $\mathbf{L}^2_\sigma \times \mathbb{C}$.

For $u \in \mathbf{L}^2_\sigma$ we define $\mathcal{H}^0_n u$ by $\left(\mathcal{H}^0_n u\right)(x) = \left(\mathcal{L}^\sigma_n g_n\right)(x)$ with

$$g_n(x) = \frac{1}{\pi}\int_{-1}^{1} \left[\mathcal{L}^\sigma_{n-1} h(x,.)\right](y)\, \frac{u(y)\,dy}{\sqrt{1-y^2}}, \quad -1 < x < 1. \tag{3.9}$$

We look for an approximate solution $(u_n, \zeta^n_0) \in \operatorname{im}\mathcal{P}_{n-1} \times \mathbb{C}$ of equation (3.8) by solving

$$(\mathcal{B} + \mathcal{H}^0_n)u_n = \mathcal{L}^\sigma_n f + \zeta^n_0 p^\sigma_0. \tag{3.10}$$

Taking into account the algebraic accuracy of the Gauss–Chebyshev quadrature, we can write equation (3.10) as the system

$$-\frac{1}{\pi}\int_{-1}^{1} (y - x^\sigma_{nj}) \ln|y - x^\sigma_{nj}|\, \frac{u_n(y)\,dy}{\sqrt{1-y^2}} + \frac{1}{n-1}\sum_{k=1}^{n-1} h(x^\sigma_{nj}, x^\sigma_{n-1,k})\chi^n_k = f(x^\sigma_{nj}) + \frac{\zeta^n_0}{\sqrt{\pi}},$$

$j = 1, \ldots, n$, with $\chi^n_k = u_n(x^\sigma_{n-1,k})$. Assume that

(C) $\quad h(x,.) \in \mathbf{L}^{2,s_0}_\sigma$ uniformly w.r.t. $x \in [-1,1]$.

Lemma 3.1. [**6**, Lemma 3.2] *Assume that $3 \leqslant t + 3 < s < s_0$ and that conditions* (A) *and* (B) *are fulfilled. There exists a constant $\gamma_{ts} > 0$ such that, for all sufficiently large n,*

$$\|(\mathcal{B} + \mathcal{L}^\sigma_n \mathcal{H})u - \zeta_0 p^\sigma_0\|_{\sigma,s} \geqslant \gamma_{ts}\, \|(u,\zeta_0)\|_{\sigma,t,\sim} \quad \forall\,(u,\zeta_0) \in \mathbf{L}^2_\sigma \times \mathbb{C} : \mathcal{B}u \in \mathbf{L}^{2,s}_\sigma.$$

Lemma 3.2. *Let* (C) *be fulfilled with $s_0 > \frac{1}{2}$. Then, for $t \geqslant 0$,*

$$\left\|(\mathcal{L}^\sigma_n \mathcal{H} - \mathcal{H}^0_n)u\right\|_{\sigma,t} \leqslant c\, n^{t-s_0} \|u\|_{\sigma,0} \quad \forall\, u \in \mathbf{L}^2_\sigma,$$

where $c \neq c(n,t,u)$.

Proof. Since $(\mathcal{L}^\sigma_n \mathcal{H} - \mathcal{H}^0_n)u \in \operatorname{im}\mathcal{P}_n$ we have

$$\begin{aligned}
&\left\|(\mathcal{L}^\sigma_n \mathcal{H} - \mathcal{H}^0_n)u\right\|^2_{\sigma,t} \\
&\qquad \leqslant n^{2t} \left\|(\mathcal{L}^\sigma_n \mathcal{H} - \mathcal{H}^0_n)u\right\|^2_{\sigma,0} \\
&\qquad = n^{2t}\, \frac{\pi}{n} \sum_{j=1}^{n} \left| \frac{1}{\pi}\int_{-1}^{1} u(y)\left[h(x^\sigma_{nj},.) - \mathcal{L}^\sigma_{n-1} h(x^\sigma_{nj},.)\right](y)\sigma(y)\,dy \right|^2 \\
&\qquad \leqslant n^{2t} \|u\|^2_{\sigma,0} \frac{1}{\pi n} \sum_{j=1}^{n} \left\|h(x^\sigma_{nj},.) - \mathcal{L}^\sigma_{n-1} h(x^\sigma_{nj},.)\right\|^2_{\sigma,0} \\
&\qquad \overset{(3.7)}{\leqslant} c\, n^{2(t-s_0)} \|u\|^2_{\sigma,0} \frac{1}{n} \sum_{j=1}^{n} \left\|h(x^\sigma_{nj},.)\right\|^2_{\sigma,s_0} \leqslant c\, n^{2(t-s_0)} \|u\|^2_{\sigma,0},
\end{aligned}$$

where we used the algebraic accuracy of the Gaussian rule and (3.7) together with assumption (C). $\square$

Corollary 3.1. *Additionally to the assumptions of Lemma* (3.1), *let condition* (C) *be satisfied. Then, for all sufficiently large n,*

$$\left\|(\mathcal{B}+\mathcal{H}_n^0)u-\zeta_0 p_0^\sigma\right\|_{\sigma,s} \geqslant \frac{\gamma_{ts}}{2}\left\|(u,\zeta_0)\right\|_{\sigma,t,\sim} \quad \forall\,(u,\zeta_0)\in\mathbf{L}_\sigma^2\times\mathbb{C}:\mathcal{B}u\in\mathbf{L}_\sigma^{2,s}.$$

Proof. From Lemma 3.1 and Lemma 3.2 we get

$$\begin{aligned}\gamma_{ts}\left\|(u,\zeta_0)\right\|_{\sigma,t,\sim} &\leqslant \left\|(\mathcal{B}+\mathcal{H}_n^0)u-\zeta_0 p_0^\sigma\right\|_{\sigma,s}+\left\|\left(\mathcal{L}_n^\sigma\mathcal{H}-\mathcal{H}_n^0\right)u\right\|_{\sigma,s}\\ &\leqslant \left\|(\mathcal{B}+\mathcal{H}_n^0)u-\zeta_0 p_0^\sigma\right\|_{\sigma,s}+c\,n^{s-s_0}\left\|(u,\zeta_0)\right\|_{\sigma,s,\sim},\end{aligned}$$

which proves the corollary. □

Proposition 3.2. [**6**, Prop. 4.1] *Let $3\leqslant 3+\tau<s<s_0$, $0\leqslant t\leqslant\tau$, $f\in\mathbf{L}_\sigma^{2,s}$, and assume that conditions* (A), (B), *and* (C) *are fulfilled. Then, for all sufficiently large n, equation* (3.10) *has a unique solution (u_n,ζ_0^n). Moreover, there is a constant $c\neq c(n,f,t)$ such that*

$$|\zeta_0^n-\zeta_0^*|\leqslant c\,n^{-\tau}\|f\|_{\sigma,s} \quad \textit{and} \quad \|u_n-u^*\|_{\sigma,t}\leqslant c\,n^{t-\tau}\|f\|_{\sigma,s}$$

where $(u^*,\zeta_0^*)\in\mathbf{L}_\sigma^{2,\tau}\times\mathbb{C}$ is the unique solution of (3.8).

The proof of the unique solvability of (3.8) in $\mathbf{L}_\sigma^{2,\tau}\times\mathbb{C}$ is part of the proof of [**6**, Prop. 4.1].

Remark 3.1. Since $\chi_i^n=\sum_{k=0}^{n-2}p_k^\sigma(x_{n-1,i}^\sigma)\xi_k^n$ holds for $\xi^n=\left[\xi_k^n\right]_{k=0}^{n-2}$ (cf. (3.5)), a matrix version of equation (3.10) is given by

$$(\mathbf{B}_n+\mathbf{D}_n\mathbf{C}_n^2\mathbf{H}_n\mathbf{C}_{n-1}^3\mathbf{D}_{n-1})\xi^n=\eta^n+\begin{bmatrix}\zeta_0^n & 0 & \cdots & 0\end{bmatrix}^T,$$

where

$$\mathbf{B}_n=\begin{bmatrix} 0 & \beta_0 & 0 & \cdots & 0\\ -\beta_0 & 0 & \beta_1 & \cdots & 0\\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & -\beta_{n-4} & 0 & \beta_{n-3}\\ 0 & \cdots & 0 & -\beta_{n-3} & 0\\ 0 & \cdots & 0 & 0 & -\beta_{n-2}\end{bmatrix}, \tag{3.11}$$

$$\mathbf{H}_n=\left[h(x_{nj}^\sigma,x_{n-1,k}^\sigma)\right]_{j=1,\,k=1}^{n,\,n-1},\quad \mathbf{C}_n^2=\left[\cos\tfrac{j(2k-1)\pi}{2n}\right]_{j=0,k=1}^{n-1,\,n}, \tag{3.12}$$

$$\mathbf{C}_{n-1}^3=\left(\mathbf{C}_{n-1}^2\right)^T,\quad \mathbf{D}_n=\tfrac{1}{n}\,\mathrm{diag}\begin{bmatrix}1 & \sqrt{2} & \cdots & \sqrt{2}\end{bmatrix}, \tag{3.13}$$

and the vector $\eta^n=[\eta_k^n]_{k=0}^{n-1}$ is defined by (3.6).

4. A fast algorithm

Now, our aim is to design a fast algorithm according to the pattern given in [**1**], which was also used, for example, in [**2**]. So, on one hand, we want to reduce the complexity of the method to $O(n\log n)$ and on the other hand, we want to keep the convergence rate stated in Propositon 3.2, at least in a smaller range for the parameter t (see Prop. 4.3). For this, we use the structure properties of that part of the operator, which is given by the $(y-x)\ln|y-x|$-kernel (cf. (3.11)), and the smoothing properties of the remaining part, which are due to the assumptions (A) and (C).

In what follows we assume that the conditions (A), (B), and (C) are fulfilled. Let us remark that the main difference of the present situation to the situations in [**1**, **2**] is that the main part $\mathcal{B} : \mathbf{L}^{2,s}_\sigma \to \mathbf{L}^{2,s+2}$ of the operator $\mathcal{B} + \mathcal{H}$ of the original equation does not have a closed image and, consequently, is not bounded invertible.

We search for an approximate solution $(u_n, \zeta_0^n) \in \operatorname{im} \mathcal{P}_{n-1} \times \mathbb{C}$ of equation (3.8) and write u_n in the form

$$u_n = \sum_{k=0}^{m-2} \xi_k^n p_k^\sigma + \sum_{k=m-1}^{n-2} \xi_k^n p_k^\sigma, \tag{4.1}$$

where $2 < m < n$.

For $k = m-1, \dots, n-2$, we set $\xi_k^n := \langle v_n, p_k^\sigma \rangle_\sigma$, where $(v_n, \widetilde{\zeta}_0^n) \in \operatorname{im} \mathcal{P}_{n-1} \times \mathbb{C}$ is the unique solution of (cf. (3.4) and (3.5))

$$\mathcal{B} v_n = \mathcal{L}_n^\sigma f + \widetilde{\zeta}_0^n p_0^\sigma. \tag{4.2}$$

Equation (4.2) is equivalent to

$$\mathbf{B}_n \xi^n = \eta^n + \begin{bmatrix} \widetilde{\zeta}_0^n & 0 & \dots & 0 \end{bmatrix}^T \text{ with } \xi^n = [\xi_k^n]_{k=0}^{n-2},$$

where the matrix $\mathbf{B}_n$ is defined in (3.11) and where the entries of

$$\eta^n = [\eta_k^n]_{k=0}^{n-1} = \frac{\sqrt{\pi}}{n} \mathbf{D}_n \mathbf{C}_n^2 [f(x_{nj}^\sigma)]_{j=1}^n$$

(cf. (3.12), (3.13)) can be computed with $O(n \log n)$ complexity. Due to the structure of the matrix $\mathbf{B}_n$ we have

$$\xi_{n-2}^n = -\frac{\eta_{n-1}^n}{\beta_{n-2}}, \quad \xi_{n-3}^n = -\frac{\eta_{n-2}^n}{\beta_{n-3}}, \quad \xi_k^n = \frac{\beta_{k+1} \xi_{k+2}^n - \eta_{k+1}^n}{\beta_k}, \ k = n-4, \dots, m-1.$$

which can be realized with $O(n)$ complexity. Moreover, due to Lemma 2.1,

$$\left(v_n, \widetilde{\zeta}_0^n\right) = \mathcal{B}_0 \mathcal{L}_n^\sigma f. \tag{4.3}$$

Define $\mathcal{Q}_n := \mathcal{I} - \mathcal{P}_n : \mathbf{L}^2_\sigma \to \mathbf{L}^2_\sigma$, i.e., $\mathcal{Q}_n u = \sum_{k=n}^\infty \langle u, p_k^\sigma \rangle_\sigma p_k^\sigma$. Note that

$$\|\mathcal{Q}_n u\|_{\sigma,t} \leqslant (n+1)^{t-s} \|u\|_{\sigma,s}, \quad 0 \leqslant t \leqslant s, \ \ u \in \mathbf{L}^{2,s}_\sigma. \tag{4.4}$$

Proposition 4.1. *Let $3 \leqslant \tau + 3 < s < s_0$ and $f \in \mathbf{L}^{2,s}_\sigma$. Furthermore, let $(u^*, \zeta_0^*) \in \mathbf{L}^{2,\tau}_\sigma \times \mathbb{C}$ be the unique solution of (3.8) and $(v_n, \widetilde{\zeta}_0^n) \in \operatorname{im} \mathcal{P}_{n-1} \times \mathbb{C}$ be the unique solution of (4.2). Then, for every $\varepsilon \in (0, s - \tau - 3)$ there is a constant $c \neq c(m,n,f,t)$ such that, for $0 \leqslant t \leqslant \tau + s_0 - s + \varepsilon$,*

$$\|\mathcal{Q}_{m-1} u^* - \mathcal{Q}_{m-1} v_n\|_{\sigma,t} \leqslant c\left(m^{t-\tau-s_0+s-\varepsilon} \|u^*\|_{\sigma,0} + n^{t-\tau-\varepsilon} \|f\|_{\sigma,s}\right).$$

Proof. Set $\theta := s_0 - s$ and define $\mathcal{R} : \widetilde{\mathbf{L}}^{2,t}_\sigma \to \mathbf{L}^{2,t}_\sigma$, $(u, \zeta) \mapsto u$. Since $\mathcal{B} u^* = f + \zeta_0^* p_0^\sigma - \mathcal{H} u^* \in \mathbf{L}^{2,s}_\sigma$ we have, in view of Lemma 2.1,

$$\mathcal{B}_0 f = \mathcal{B}_0(\mathcal{B} u^* + \mathcal{H} u^* - \zeta_0^* p_0^\sigma) = (u^*, \zeta_0^*) + \mathcal{B}_0 \mathcal{H} u^*.$$

Consequently, for $0 \leqslant t \leqslant \tau$,

$$\begin{aligned}
&\|\mathcal{Q}_{m-1}u^* - \mathcal{Q}_{m-1}v_n\|_{\sigma,t} \\
&\quad= \left\|\mathcal{Q}_{m-1}\mathcal{R}(u^*,\zeta_0^*) - \mathcal{Q}_{m-1}\mathcal{R}(v_n,\widetilde{\zeta}_0^n)\right\|_{\sigma,t} \\
&\quad\overset{(4.3)}{=} \|\mathcal{Q}_{m-1}\mathcal{R}(\mathcal{B}_0 f - \mathcal{B}_0\mathcal{H}u^*) - \mathcal{Q}_{m-1}\mathcal{R}\mathcal{B}_0\mathcal{L}_n^\sigma f\|_{\sigma,t} \\
&\quad\leq \|\mathcal{Q}_{m-1}\mathcal{R}\mathcal{B}_0\mathcal{H}u^*\|_{\sigma,t} + \|\mathcal{Q}_{m-1}\mathcal{R}\mathcal{B}_0(f-\mathcal{L}_n^\sigma f)\|_{\sigma,t} \\
&\quad\overset{(4.4)}{\leq} m^{t-(\tau+\theta+\varepsilon)}\|\mathcal{B}_0\mathcal{H}u^*\|_{\sigma,\tau+\theta+\varepsilon,\sim} + \|\mathcal{B}_0(f-\mathcal{L}_n^\sigma f)\|_{\sigma,t,\sim} \\
&\quad\overset{\text{Lemma } 2.1}{\leq} m^{t-\tau-\theta-\varepsilon}\|\mathcal{B}_0\|_{\mathbf{L}_\sigma^{2,s_0}\to\widetilde{\mathbf{L}}_\sigma^{2,\tau+\theta+\varepsilon}}\|\mathcal{H}u^*\|_{\sigma,s_0} \\
&\qquad\qquad + \|\mathcal{B}_0\|_{\mathbf{L}_\sigma^{2,t+3+\varepsilon_1}\to\widetilde{\mathbf{L}}_\sigma^{2,t}}\|f-\mathcal{L}_n^\sigma f\|_{\sigma,t+3+\varepsilon_1} \\
&\quad\overset{(3.7)}{\leqslant} c\big(m^{t-\tau-\theta-\varepsilon}\|u^*\|_{\sigma,0} + n^{t+3+\varepsilon_1-s}\|f\|_{\sigma,s}\big),
\end{aligned}$$

where $\varepsilon_1 := s-\tau-3-\varepsilon > 0$, so that $t+3+\varepsilon_1-s = t-\tau-\varepsilon$. Moreover, we took into account that, due to Remark 2.2, $\|\mathcal{B}_0\|_{\mathbf{L}_\sigma^{2,t+3+\varepsilon_1}\to\widetilde{\mathbf{L}}_\sigma^{2,t}} \leqslant \rho_{3+\varepsilon_1}$ independently of t. □

The numbers ξ_k^n, $k = 0,\ldots,m-2$ in (4.1) we take from the solution $w_m \in \operatorname{im}\mathcal{P}_{m-1}$ of (cf. (3.10))

$$\big(\mathcal{B}+\mathcal{H}_n^0\big)\,w_m = \mathcal{L}_n^\sigma f + \widehat{\zeta}_0^m p_0^\sigma - \mathcal{L}_m^\sigma\mathcal{B}\mathcal{Q}_{m-1}v_n, \tag{4.5}$$

where $v_n \in \operatorname{im}\mathcal{P}_{n-1}$ is the solution of (4.2), i.e., $\xi_k^n := \langle w_m, p_k^\sigma\rangle_\sigma$, $k = 0,\ldots,m-2$. Moreover, we set $\zeta_0^n := \widehat{\zeta}_0^m$.

In view of Remark 3.1 the solving (4.5) is equivalent to solving the system

$$\big(\mathbf{B}_m + \mathbf{D}_m\mathbf{C}_m^2\mathbf{H}_m\mathbf{C}_{m-1}^3\mathbf{D}_{m-1}\big)\widehat{\xi}^m = \widehat{\eta}^m + \begin{bmatrix}\widehat{\zeta}_0^m & 0 & \cdots & 0\end{bmatrix}^T,$$

which can be done with $O(m^3)$ complexity if the numbers

$$\widehat{\eta}_k^m = \langle\mathcal{L}_m^\sigma f - \mathcal{L}_m^\sigma\mathcal{B}\mathcal{Q}_{m-1}v_n, p_k^\sigma\rangle_\sigma\,, \quad k = 0,\ldots,m-1,$$

are computed. In order to do this analogously to (3.6), we need the function values $(\mathcal{B}\mathcal{Q}_{m-1}v_n)(x_{mj}^\sigma)$, $j = 1,\ldots,m$. To get these values effectively we can choose m and n in such a way that $\frac{m}{n}$ is an odd integer, since in this case we have $\{x_{mj}^\sigma : j = 1,\ldots,m\} \subset \{x_{nj}^\sigma : j = 1,\ldots,n\}$. Let us write

$$\mathcal{B}\mathcal{Q}_{m-1}v_n = \sum_{k=0}^{n-1}\chi_k^n p_k^\sigma.$$

In view of Proposition 2.1 we get, setting $\xi_{n-1}^n = \xi_n^n = 0$,

$$\begin{aligned}
\mathcal{B}\mathcal{Q}_{m-1}v_n &= \sum_{k=m-1}^{n-2}\xi_k^n\left(\beta_{k-1}p_{k-1}^\sigma - \beta_k p_{k+1}^\sigma\right) \\
&= \beta_{m-2}\xi_{m-1}^n p_{m-1}^\sigma + \beta_{m-1}\xi_m^n p_m^\sigma + \sum_{k=m}^{n-1}\left(\beta_k\xi_{k+1}^n - \beta_{k-1}\xi_{k-1}^n\right)p_k^\sigma,
\end{aligned}$$

such that, due to (4.2),

$$\chi_k^n := \begin{cases} 0 & 0 \leqslant k \leqslant m-3, \\ \beta_k \xi_{k+1}^n & k = m-2, m-1, \\ \eta_k^n & m \leqslant k \leqslant n-1. \end{cases}$$

Finally,

$$\left(\mathcal{B}\mathcal{Q}_{m-1} v_n\right)\left(x_{nj}^\sigma\right) = \sum_{k=0}^{n-1} \chi_k p_k^\sigma(x_{nj}^\sigma), \quad j = 1, \ldots, n,$$

which can be realized with $O(n \log n)$ complexity using the discrete cosine transform $\mathbf{C}_{n-1}^2$. Consequently, (4.5) can be solved with $O(m^3) + O(n \log n) = O(n \log n)$ complexity if $\frac{m^3}{n} \leqslant \gamma$ for some constant γ.

Proposition 4.2. *Let $3 \leqslant \tau + 3 < s < s_0$ and $f \in \mathbf{L}_\sigma^{2,s}$. Assume that there is an $\varepsilon \in (0, s-\tau-3)$ such that $\tau + s_0 - s - 1 + \varepsilon > 0$. Then, for every $\delta \in (0, \tau + s_0 - s - 1 + \varepsilon]$, there exists a constant $c \neq c(m,n,f)$ such that, for all sufficiently large m and n with $0 < m < n$,*

$$\sqrt{\left\|w_m - \mathcal{P}_{m-1}u^*\right\|_{\sigma,0}^2 + \left|\widehat{\zeta}_0^m - \zeta_0^*\right|^2} \leqslant c\big(m^{1+\delta-\tau-s_0+s-\varepsilon}\left\|u^*\right\|_{\sigma,0} + n^{1+\delta-\tau-\varepsilon}\|f\|_{\sigma,s}\big)$$

where (u^*, ζ_0^*) and $(w_m, \widehat{\zeta}_0^m)$ are the solutions of (3.8) and (4.5), respectively.

Proof. We compute

$$\begin{aligned}
&\big(\mathcal{B} + \mathcal{H}_m^0\big)(w_m - \mathcal{P}_{m-1}u^*) \\
&\quad = \mathcal{L}_m^\sigma f + \widehat{\zeta}_0^m p_0^\sigma - \mathcal{L}_m^\sigma \mathcal{B}\mathcal{Q}_{m-1} v_n - \big(\mathcal{B} + \mathcal{H}_m^0\big)\mathcal{P}_{m-1}u^* \\
&\quad = \mathcal{L}_m^\sigma(\mathcal{B} + \mathcal{H})u^* - \zeta_0^* p_0^\sigma + \widehat{\zeta}_0^m p_0^\sigma - \mathcal{L}_m^\sigma \mathcal{B}\mathcal{Q}_{m-1} v_n - \big(\mathcal{B} + \mathcal{H}_m^0\big)\mathcal{P}_{m-1}u^* \\
&\quad = (\mathcal{L}_m^\sigma - \mathcal{I})\mathcal{B}\mathcal{P}_{m-1}u^* + \big(\widehat{\zeta}_0^m - \zeta_0^*\big)p_0^\sigma \\
&\qquad\qquad + \mathcal{L}_m^\sigma \mathcal{B}\mathcal{Q}_{m-1}(u^* - v_n) + \big(\mathcal{L}_m^\sigma \mathcal{H} - \mathcal{H}_m^0\big)u^* + \mathcal{H}_m^0 \mathcal{Q}_{m-1}u^* \\
&\quad = \mathcal{L}_m^\sigma \mathcal{B}\mathcal{Q}_{m-1}(u^* - v_n) + \big(\widehat{\zeta}_0^m - \zeta_0^*\big)p_0^\sigma + \big(\mathcal{L}_m^\sigma \mathcal{H} - \mathcal{H}_m^0\big)u^*,
\end{aligned}$$

where we took into account that $(\mathcal{L}_m^\sigma - \mathcal{I})\,\mathcal{B}\mathcal{P}_{m-1}u^* = 0$ because of $\mathcal{B}\mathcal{P}_{m-1}u^* \in \operatorname{im}\mathcal{P}_m$ and that $\mathcal{H}_m^0 \mathcal{Q}_{m-1}u^* = \mathcal{L}_m^\sigma g_m$ with (cf. (3.9))

$$\begin{aligned}
g_m(x) &= \frac{1}{\pi}\int_{-1}^1 [\mathcal{L}_{m-1}h(x,.)](y)\,(\mathcal{Q}_{m-1}u^*)(y)\,\sigma(y)\,dy \\
&= \frac{1}{\pi}\int_{-1}^1 \sum_{k=1}^{m-1} h(x, x_{m-1,k}^\sigma)\,\ell_{m-1,k}^\sigma(y)(\mathcal{Q}_{m-1}u^*)(y)\,\sigma(y)\,dy \\
&= \frac{1}{\pi}\sum_{k=1}^{m-1} h(x, x_{m-1,k}^\sigma)\left\langle \mathcal{Q}_{m-1}u^*, \ell_{m-1,k}^\sigma\right\rangle_\sigma = 0,
\end{aligned}$$

where $\ell_{m-1,k}^\sigma$ denote the respective fundamental Lagrange interpolation polynomials of degree $m-2$. With the help of Cor. 3.1, Prop. 2.1, Prop. 4.1, and Lemma 3.2

we get

$$
\begin{aligned}
&\left\|(w_m - \mathcal{P}_{m-1}u^*, \widehat{\zeta}_0^m - \zeta_0^*)\right\|_{\sigma,0,\sim} \\
&\quad \leqslant \frac{2}{\gamma_{0,3+\delta}} \left\| \left(\mathcal{B} + \mathcal{H}_m^0\right)\left(w_m - \mathcal{P}_{m-1}u^*\right) - \left(\widehat{\zeta}_0^m - \zeta_0^*\right)p_0^\sigma \right\|_{\sigma,3+\delta} \\
&\quad \leqslant c\Big(\|\mathcal{L}_m^\sigma \mathcal{B}\mathcal{Q}_{m-1}(u^* - v_n)\|_{\sigma,3+\delta} + \left\|\left(\mathcal{L}_m^\sigma \mathcal{H} - \mathcal{H}_m^0\right)u^*\right\|_{\sigma,3+\delta} \Big) \\
&\quad \leqslant c\Big(\|\mathcal{Q}_{m-1}(u^* - v_n)\|_{\sigma,1+\delta} + \left\|\left(\mathcal{L}_m^\sigma \mathcal{H} - \mathcal{H}_m^0\right)u^*\right\|_{\sigma,3+\delta} \Big) \\
&\quad \leqslant c\Big(m^{1+\delta-\tau-s_0+s-\varepsilon} \|u^*\|_{\sigma,0} + n^{1+\delta-\tau-\varepsilon}\|f\|_{\sigma,s} + m^{3+\delta-s_0} \|u^*\|_{\sigma,0} \Big) \\
&\quad \leqslant c\Big(m^{1+\delta-\tau-s_0+s-\varepsilon} \|u^*\|_{\sigma,0} + n^{1+\delta-\tau-\varepsilon}\|f\|_{\sigma,s} \Big),
\end{aligned}
$$

and the assertion is proved. □

The following proposition shows that, for the $O(n \log n)$-algorithm, we can attain the same convergence rate (in a restricted interval for t) as in Prop. 3.2 under the assumptions that $s_0 > s > 4$.

PROPOSITION 4.3. *Let u_n in (4.1) together with $\zeta_0^n = \widehat{\zeta}_0^m$ be determined by the above described algorithm and let (u^*, ζ_0) be the solution of (3.8), where we assume that there is a constant $\gamma > 0$ such that $\frac{n}{m^3} \leqslant \gamma$ and that $4 \leqslant \tau + 4 < s < s_0$, $f \in \mathbf{L}_\sigma^{2,s}$. Then, there is a constant $c \neq c(m,n,t)$ such that*

$$
\|u_n - u^*\|_{\sigma,t} \leqslant c\, n^{t-\tau}, \quad \max\{0, \tau - (s_0 - s)/2\} \leqslant t \leqslant \tau.
$$

Moreover, if additionally $\tau \leqslant (s_0 - s)/2$, then $|\zeta_0^n - \zeta_0^*| \leqslant c\, n^{-\tau}$.

PROOF. Choose $\varepsilon \in (1, s-\tau-3)$ and $\delta = \varepsilon - 1$. Then $0 < \delta < \tau + s_0 - s - 1 + \varepsilon$, and we can apply Prop. 4.1 and Prop. 4.2 to estimate

$$
\begin{aligned}
\|u^* - u_n\|_{\sigma,t} &\leqslant \|\mathcal{P}_{m-1}u^* - \mathcal{P}_{m-1}u_n\|_{\sigma,t} + \|\mathcal{Q}_{m-1}u^* - \mathcal{Q}_{m-1}u_n\|_{\sigma,t} \\
&= \|\mathcal{P}_{m-1}u^* - w_m\|_{\sigma,t} + \|\mathcal{Q}_{m-1}u^* - \mathcal{Q}_{m-1}v_n\|_{\sigma,t} \\
&\leqslant c\Big(m^t \|\mathcal{P}_{m-1}u^* - w_m\|_{\sigma,0} + m^{t-\tau-s_0+s-\varepsilon} \|u^*\|_{\sigma,0} + n^{t-\tau-\varepsilon}\|f\|_{\sigma,s} \Big) \\
&\leqslant c\Big(m^{t-\tau-s_0+s} \|u^*\|_{\sigma,0} + m^t n^{-\tau}\|f\|_{\sigma,s} + n^{t-\tau-\varepsilon}\|f\|_{\sigma,s} \Big) \leqslant cn^{t-\tau}
\end{aligned}
$$

if $\tau - (s_0 - s)/2 \leqslant t$. Furthermore, due to Prop. 4.2,

$$
|\zeta_0^n - \zeta_0^*| = \left|\widehat{\zeta}_0^m - \zeta_0^*\right| \leqslant c\big(m^{-\tau-s_0+s} \|u^*\|_{\sigma,0} + n^{-\tau}\|f\|_{\sigma,s}\big) \leqslant cn^{-\tau}
$$

if $t \leqslant (s_0 - s)/2$. □

5. Numerical examples

As an example we take the equation

$$
-\frac{1}{\pi} \int_{-1}^{1} [(y-x)\ln|y-x| - \cos(xy + 2y)] \frac{u(y)\,dy}{\sqrt{1-y^2}} = f(x) + \frac{\zeta_0}{\sqrt{\pi}} \tag{5.1}
$$

which was already considered in [**6**, Section 5]. For the right-hand side function $f(x)$ we choose $f(x) = x^3|x|$, $f(x) = x^4|x|$, and

$$f(x) = \frac{1}{\pi}\Big[\frac{2\sin(x+2)}{x+2} - \frac{1}{2}(1-x)^2\Big(\ln(1-x) - \frac{1}{2}\Big) + \frac{1}{2}(1+x)^2\Big(\ln(1+x) - \frac{1}{2}\Big)\Big]. \tag{5.2}$$

In case of (5.2), $u^*(x) = \sqrt{1-x^2}$ with $\zeta_0^* = 0$ is the exact solution. In the following tables one can see the numerical results (we restrict to 9 decimal digits) of the described fast algorithm (FA) for various choices of n and m in comparison with the results of the quadrature method (QM) presented in [**6**]. The computations are performed with double precision.

Equation (5.1), Fourier coefficients, $f(x) = x^3|x|$

	n	m	ξ_0^n	ξ_1^n	ξ_2^n	ξ_3^n	ξ_4^n
FA	256	8	-8.47904757	-1.12502515	-7.02640549	0.14626959	-1.38702022
FA	256	16	-8.47874416	-1.12489088	-7.02624013	0.14632797	-1.38700379
QM	256		-8.47874416	-1.12489088	-7.02624013	0.14632797	-1.38700379
FA	512	8	-8.47904761	-1.12502516	-7.02640553	0.14626959	-1.38702028
FA	512	16	-8.47874420	-1.12489089	-7.02624017	0.14632797	-1.38700386
QM	512		-8.47874420	-1.12489089	-7.02624017	0.14632797	-1.38700386
FA	1024	8	-8.47904761	-1.12502516	-7.02640554	0.14626959	-1.38702029
FA	1024	16	-8.47874420	-1.12489089	-7.02624017	0.14632797	-1.38700386

Equation (5.1), ζ_0^n and function values, $f(x) = x^3|x|$

	n	m	ζ_0^n	$u_n(-0.9)$	$u_n(0)$	$u_n(0.5)$	$u_n(0.95)$
FA	256	8	1.36088509	-7.34653331	-0.64725315	-1.79908541	-10.4375062
FA	256	16	1.36088532	-7.34636880	-0.64720210	-1.79895947	-10.4371124
QM	256		1.36088532	-7.34636881	-0.64720206	-1.79895946	-10.4371124
FA	512	8	1.36088510	-7.34653709	-0.64731563	-1.79908137	-10.4375044
FA	512	16	1.36088533	-7.34637260	-0.64726455	-1.79895542	-10.4371106
QM	512		1.36088533	-7.34637260	-0.64726455	-1.79895542	-10.4371106
FA	1024	8	1.36088510	-7.34653652	-0.64733128	-1.79908188	-10.4375040
FA	1024	16	1.36088533	-7.34637202	-0.64728020	-1.79895593	-10.4371102

In cases $n = 256, 512$, for the fast algorithm we get the same results (with respect to 9 decimal digits) as in the quadrature method of [**6**] already for $m = 16$, which remains true if we increase n. In the following two tables we consider a more smooth right-hand side which results in the fact, that the fast algorithm gives the same results as the quadrature method already for $m = 8$.

Equation (5.1), Fourier coefficients, $f(x) = x^4|x|$

	n	m	ξ_0^n	ξ_1^n	ξ_2^n	ξ_3^n	ξ_4^n
FA	256	8	0.00000000	-4.25538432	0.00000000	-5.47120842	0.00000000
QM	256		0.00000000	-4.25538432	0.00000000	-5.47120842	0.00000000
FA	512	8	0.00000000	-4.25538432	0.00000000	-5.47120842	0.00000000
QM	512		0.00000000	-4.25538432	0.00000000	-5.47120842	0.00000000
FA	1024	8	0.00000000	-4.25538432	0.00000000	-5.47120842	0.00000000

Equation (5.1), ζ_0^n and function values, $f(x) = x^4|x|$

	n	m	ζ_0^n	$u_n(-0.9)$	$u_n(0)$	$u_n(0.5)$	$u_n(0.95)$
FA	256	8	0.32152132	3.57217748	0.00000000	2.34570884	-5.78339205
QM	256		0.32152132	3.57217748	0.00000000	2.34570884	-5.78339205
FA	512	8	0.32152132	3.57217748	0.00000000	2.34570883	-5.78339205
QM	512		0.32152132	3.57217748	0.00000000	2.34570883	-5.78339205
FA	1024	8	0.32152132	3.57217748	0.00000000	2.34570883	-5.78339205

In the last example (see the following two tables), the right-hand side is less smooth. But, beginning with $n = 1024$ the results are exact (w.r.t. 9 or 8 decimal digits) already for $m = 16$.

Equation (5.1), Fourier coefficients, $f(x)$ from (5.2)

	n	m	ξ_0^n	ξ_1^n	ξ_2^n	ξ_3^n	ξ_4^n
FA	256	8	1.12848920	0.00004609	-0.53186634	0.00001504	-0.10638087
FA	256	16	1.12837951	0.00000008	-0.53192266	-0.00000001	-0.10638403
QM	256		1.12837951	0.00000008	-0.53192266	-0.00000001	-0.10638403
FA	512	8	1.12848890	0.00004602	-0.53186667	0.00001505	-0.10638138
FA	512	16	1.12837921	0.00000001	-0.53192299	0.00000000	-0.10638454
QM	512		1.12837921	0.00000001	-0.53192299	0.00000000	-0.10638454
FA	1024	8	1.12848886	0.00004601	-0.53186671	0.00001505	-0.10638144
FA	1024	16	1.12837917	0.00000000	-0.53192303	0.00000000	-0.10638460
u^*			1.12837917	0.00000000	-0.53192304	0.00000000	-0.10638461

Equation (5.1), ζ_0^n and function values, $f(x)$ from (5.2)

	n	m	ζ_0^n	$u_n(-0.9)$	$u_n(0)$	$u_n(0.5)$	$u_n(0.95)$
FA	256	8	0.00000237	0.43594383	1.00001952	0.86607051	0.31238988
FA	256	16	-0.00000007	0.43588980	1.00000006	0.86602553	0.31224956
QM	256		-0.00000007	0.43588980	1.00000006	0.86602553	0.31224956
FA	512	8	0.00000243	0.43594390	1.00001947	0.86607040	0.31239018
FA	512	16	-0.00000001	0.43588988	1.00000001	0.86602542	0.31224986
QM	512		-0.00000001	0.43588988	1.00000001	0.86602542	0.31224986
FA	1024	8	0.00000244	0.43594392	1.00001947	0.86607039	0.31239022
FA	1024	16	0.00000000	0.43588989	1.00000001	0.86602541	0.31224990
$u^*(x)$				0.43588989	1.00000000	0.86602540	0.31224990

In the following two tables, let us show some results with 15 decimal digits for the third example. One can see that, due to rounding errors, the precision of the results decreases when going from $n = 16384$ to $n = 32768$.

Equation (5.1), Fourier coefficients, $f(x)$ from (5.2)

	n	m	ξ_0^n	ξ_4^n
FA	8192	8	1.12848885640610	-0.10638144804001
FA	8192	16	1.12837916711157	-0.10638460808187
FA	8192	32	1.12837916711008	-0.10638460808242
FA	16384	8	1.12848885640011	-0.10638144805023
FA	16384	16	1.12837916710552	-0.10638460809210
FA	16384	32	1.12837916710401	-0.10638460809269
FA	32768	8	1.12848885641034	-0.10638144803278
FA	32768	16	1.12837916711585	-0.10638460807464
FA	32768	32	1.12837916711433	-0.10638460807525
u^*			1.12837916709551	-0.10638460810705

Equation (5.1), ζ_0^n and function values, $f(x)$ from (5.2)

	n	m	$u_n(-0.9)$	$u_n(0.5)$
FA	8192	8	0.43594391928916	0.86607039879272
FA	8192	16	0.43588989107899	0.86602541682333
FA	8192	32	0.43588989107818	0.86602541682336
FA	16384	8	0.43594401611548	0.86607034614225
FA	16384	16	0.43588998790529	0.86602536417281
FA	16384	32	0.43588998790464	0.86602536417291
FA	32768	8	0.43594421947381	0.86607061508862
FA	32768	16	0.43589019126367	0.86602563311924
FA	32768	32	0.43589019126272	0.86602563311909
$u^*(x)$			0.43588989435407	0.86602540378444

The last table presents the comparison of the CPU-time used for establishing and solving the system by the quadrature method and by the fast algorithm (in case of $m = 16$).

CPU-time in seconds for QM and FA ($m = 16$)

n	64	128	256	512	1024	2048	4096	8192	16384	32768
QM	0.02	0.16	1.25	9.52						
FA	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.02	0.03	0.06

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