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AN ALGEBRAIC EXPOSITION OF UMBRAL CALCULUS WITH APPLICATION TO GENERAL LINEAR INTERPOLATION PROBLEM – A SURVEY

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Abstract. A systematic exposition of Sheffer polynomial sequences via determinantal form is given. A general linear interpolation problem related to Sheffer sequences is considered. It generalizes many known cases of linear interpolation. Numerical examples and conclusions close the paper.

1. The modern umbral calculus

In the 1970s Rota and his collaborators [**17**, **19**, **20**] began to construct a completely rigorous foundation for the classical umbral calculus, consisted primarily of a symbolic technique for the manipulation of numerical and polynomial sequences. The theory of Rota et al. was based on the relatively modern ideas of linear functional, linear operator and adjoint. This theory followed that less efficient of generating function methods; in fact, Appell [**1**], Sheffer [**22**] and Steffensen [**23**] based their theories on formal power series. These theories can be criticized both for their formalism not suitable for nonspecialists and for insufficient computational tools. The umbral calculus, because of its numerous applications in many branches of mathematics, physics, chemistry and engineering [**24**], has received many attentions from researchers. Recently, Di Bucchianico and Loeb [**14**] summarized and documented more than five hundred old and new findings related to umbral calculus. In last years attention has centered on finding novel approaches. For example, in [**21**] the connection between Sheffer polynomials and Riordan array is sketched and in [**16**] the isomorphism between the Sheffer group and the Riordan Group is proved. In [**5**, **27**] two different matrix approaches to Appell polynomials are given, in [**9**, **26**], these methods have been extended to Sheffer polynomials, and in [**11**], to binomial polynomial sequences. Recently, the relation between the umbral calculus and the general linear interpolation problem has been highlighted [**6**–**8**, **10**, **11**]. In this survey we give a unitary matrix approach to Sheffer polynomials, including Appell and binomial type polynomial sequences. Our theory of Sheffer sequences assumes binomial type polynomial sequences, therefore the paper is organized as

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follows: in Section 2 we give the preliminaries on binomial type sequences; in Section 3 we provide the new determinantal definition of Sheffer polynomials; in Section 4 we give the relationship with δ-operator and functional, finding the connection with Rota's theory; in Section 5 we recall the determinantal definition of Yang [**26**]; in Section 6 we define the umbral interpolation problem and in Section 7 we provide classical and nonclassical examples; in Section 8 we furnish some numerical examples and finally, in Section 9, conclusions close the paper.

2. Preliminaries

In order to render the work self-contained we provide the necessary preliminary tools on binomial polynomial sequences. Let $\mathcal{P}_n$ be the commutative algebra of all polynomials in a simple variable x, with coefficients in a field $\mathbb{K}$ (typically $\mathbb{R}$ or $\mathbb{C}$).

Definition 2.1. [**19**] A polynomial sequence $\{p_n(x)\}_{n\in\mathbb{N}}$, where n is the degree of $p_n(x)$, is said to be of binomial type (b.t.) if and only if it satisfies the binomial identity

$$p_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}p_k(x)p_{n-k}(y), \tag{2.1}$$

for all $n\in\mathbb{N}$ and $x,y\in\mathbb{R}$.

Well-known examples of polynomial sequences can been found in [**17–19**] and references therein. For a polynomial sequence $p_n(x)$ we could consider its coefficients $a_{n,k}$ in its decomposition over power monomials x^k, that is

$$p_n(x)=\sum_{k=0}^{n}a_{n,k}x^k. \tag{2.2}$$

Relation (2.1), defining polynomial sequences of b.t., can be expressed in terms of coefficients $a_{n,k}$ as follows [**19**, p. 111]

$$\binom{i+j}{i}a_{n,i+j}=\sum_{l=0}^{n}\binom{n}{l}a_{l,i}a_{n-l,j}. \tag{2.3}$$

From (2.1) we can observe that $a_{n,0}=\delta_{n,0}$, and from (2.3) that all the coefficients $a_{n,k}$, $k=1,\dots,n$, $n=0,1,\dots$, are completely determined by a sequence $\{b_n\}_{n\in\mathbb{N}}$, with $b_0=0$, $b_1\neq 0$, as follows

$$a_{n,1}=b_n,\quad a_{n,k}=\frac{1}{k}\sum_{i=1}^{n-k+1}\binom{n}{i}a_{i,1}a_{n-i,k-1},\quad k=2,\dots,n,$$

We also have [**2**]

$$a_{n,k}=\frac{1}{k!}\sum\binom{n}{\nu_1,\dots,\nu_k}b_{\nu_1}\cdots b_{\nu_k},\quad k=2,\dots,n,$$

where the summation is over all k-tuples $(\nu_1, \ldots, \nu_k)$ with $\nu_i \geqslant 1$ and $\sum \nu_i = n$. Finally, for the coefficients $a_{n,k}$ defined in (2.2) we have

$$a_{n,0} = \delta_{n,0}, \quad a_{n,1} = b_n, \tag{2.4}$$

$$a_{n,k} = \frac{1}{k} \sum_{i=1}^{n-k+1} \binom{n}{i} a_{i,1} a_{n-i,k-1} = \frac{1}{k!} \sum \binom{n}{\nu_1, \ldots, \nu_k} b_{\nu_1} \cdots b_{\nu_k}, \quad k = 2, \ldots, n.$$

Therefore, a polynomial sequence of b.t. $p_n(x)$, $n = 0, 1, \ldots$, is associated to a real numerical sequence b_n, $n = 0, 1, \ldots$, with $b_0 = 0$ and $b_1 \neq 0$. Moreover, let us define the formal power series

$$B(t) = \sum_{n=1}^{\infty} b_n \frac{t^n}{n!}, \tag{2.5}$$

then we have

$$\sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!} = e^{xB(t)},$$

that is, the function $e^{xB(t)}$ is the generating function for polynomials $p_n(x)$ [**19**, p. 106]. Moreover, we also have

$$(B(x))^k = \sum_{n=1}^{\infty} a_{n,k} \frac{x^n}{n!},$$

with $a_{n,k}$ defined by (2.4).

Now, let us set $X_n(x) = [1, x, \ldots, x^n]^T$, $P_n(x) = [p_0(x), p_1(x), \ldots, p_n(x)]^T$ ($P_n(x)$ which will be called b.t. vector), and

$$(A)_{i,j} = \begin{cases} a_{i,j} & i \geqslant j \\ 0 & \text{otherwise} \end{cases} \qquad i, j = 0, \ldots, n, \tag{2.6}$$

with $a_{i,j}$, $i, j = 0, \ldots, n$, defined by (2.4). Then, from (2.2), we can write

$$P_n(x) = AX_n(x), \tag{2.7}$$

and, observing that A is invertible,

$$X_n(x) = A^{-1} P_n(x). \tag{2.8}$$

Remark 2.1. [**9**] For the calculation of the inverse matrix

$$\widehat{A} \equiv A^{-1} = \left(A^{-1}\right)_{i,j} \equiv \begin{cases} \widehat{a}_{i,j} & i \geqslant j \\ 0 & \text{otherwise} \end{cases} \qquad i, j = 0, \ldots, n,$$

we can use the following algorithm.

- determine the sequence $\widehat{b}_n$, $n = 0, 1, \ldots$, as follows [**15**]

$$\widehat{b}_0 = 0, \quad \widehat{b}_1 = \frac{1}{a_{1,1}}, \quad \widehat{b}_n = -\frac{1}{a_{n,n}} \sum_{k=1}^{n-1} \widehat{b}_k a_{n,k}, \quad n = 2, \ldots, \tag{2.9}$$

  with $a_{n,k}$ defined by (2.4);

- determine the sequence $\widehat{a}_{n,k}$, $k = 0, \dots, n$, $n = 0, 1, \dots$, as follows [**7**, **9**]

$$\widehat{a}_{n,0} = \delta_{n,0}, \quad \widehat{a}_{n,1} = \widehat{b}_n, \tag{2.10}$$

$$\widehat{a}_{n,k} = \frac{1}{k}\sum_{i=1}^{n-k+1}\binom{n}{i}\widehat{a}_{i,1}\widehat{a}_{n-i,k-1} = \frac{1}{k!}\sum\binom{n}{\nu_1, \dots, \nu_k}\widehat{b}_{\nu_1}\cdots\widehat{b}_{\nu_k}, \quad k = 2, \dots, n.$$

Remark 2.2. We observe explicitly that from (2.7) and (2.8) we have

$$p_n(x) = \sum_{k=0}^{n} a_{n,k}x^k, \qquad x^n = \sum_{k=0}^{n}\widehat{a}_{n,k}p_k(x),$$

that is, these two relations are inverse ones. Therefore, $p_n(x)$ in a basis for $\mathcal{P}_n$.

Now, we define the linear operator on $\mathcal{P}_n$

$$Qy = \sum_{n=0}^{\infty}\frac{\widehat{b}_n}{n!}y^{(n)}, \tag{2.11}$$

with $\widehat{b}_n$ defined by (2.9). Then we have

Theorem 2.1. [**11**] *The relation* $Qp_n(x) = np_{n-1}(x)$, $n = 1, 2, \dots$ *holds.*

Remark 2.3. According to the Rota theory, after Theorem 2.1 Q is the δ-operator associated to the polynomial sequence of b.t. $p_n(x)$.

3. Sheffer polynomial sequences

Let $p_n(x)$, $n = 0, 1, \dots$, be a polynomial sequence of b.t. and β_n, $n = 0, 1, \dots$, a real numerical sequence with $\beta_0 \neq 0$.

Definition 3.1. [**9**] The polynomial sequence $s_n(x)$, $n = 0, 1, \dots$, defined by

$$s_0(x) = \frac{1}{\beta_0}, \tag{3.1}$$

$$s_n(x) = \frac{(-1)^n}{(\beta_0)^{n+1}}\begin{vmatrix} p_0 & p_1(x) & \cdots & p_{n-1}(x) & p_n(x) \\ \beta_0 & \beta_1 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \quad n = 1, 2, \dots$$

is called the Sheffer polynomial sequence for $(\beta_n, p_n(x))$.

Remark 3.1. For $p_n(x) = x^n$ the sequence $s_n(x)$ is the Appell sequence for β_n [**5**]; for $\beta_0 = 1$, and $\beta_n = 0$, $n = 1, 2, \dots$, we get $s_n(x) = p_n(x)$. In general, a Sheffer sequence $s_n(x)$ is the umbral composition [**19**] of an Appell sequence and a binomial type polynomial sequence.

Now, we give several characterizations for Sheffer polynomial sequences.

THEOREM 3.1. [**9**] *The sequence $s_n(x)$ is the Sheffer polynomial sequence for $(\beta_n, p_n(x))$ if and only if*

$$s_n(x) = \alpha_n p_0 + \binom{n}{1}\alpha_{n-1}p_1(x) + \binom{n}{2}\alpha_{n-2}p_2(x) + \cdots + \alpha_0 p_n(x), \quad n = 0, 1, \ldots$$

with

$$\alpha_0 = \frac{1}{\beta_0}, \quad \alpha_i = -\frac{1}{\beta_0}\sum_{k=0}^{i-1}\binom{i}{k}\beta_{i-k}\alpha_k, \quad i = 1, 2, \ldots, n. \tag{3.2}$$

REMARK 3.2. [**9**] Let

$$(M)_{i,j} = \begin{cases} \binom{i}{j}\beta_{i-j} & i \geqslant j \\ 0 & \text{otherwise} \end{cases}, \qquad i, j = 0, \ldots, n. \tag{3.3}$$

The matrix M is invertible and

$$(M^{-1})_{i,j} = \begin{cases} \binom{i}{j}\alpha_{i-j} & i \geqslant j \\ 0 & \text{otherwise} \end{cases}, \qquad i, j = 0, \ldots, n,$$

with α_n defined by (3.2).

Therefore, setting

$$\beta_0 = \frac{1}{\alpha_0}, \quad \beta_i = -\frac{1}{\alpha_0}\sum_{k=0}^{i-1}\binom{i}{k}\alpha_{i-k}\beta_k, \quad i = 1, 2, \ldots, n, \tag{3.4}$$

Relations (3.2) and (3.4) are inverse ones.

THEOREM 3.2 (Recurrence, [**9**]). *The sequence $s_n(x)$, $n = 0, 1, \ldots$, is the Sheffer sequence for $(\beta_n, p_n(x))$ if and only if*

$$s_n(x) = \frac{1}{\beta_0}\left(p_n(x) - \sum_{k=0}^{n-1}\binom{n}{k}\beta_{n-k}s_k(x)\right), \quad n = 1, 2, \ldots$$

Now, we have the following

THEOREM 3.3 (Sheffer identity, [**9**]). *The sequence $s_n(x)$ is the Sheffer polynomial sequence for $(\beta_n, p_n(x))$ if and only if*

$$s_n(x+y) = \sum_{i=0}^{n}\binom{n}{i}s_{n-i}(x)p_i(y) = \sum_{i=0}^{n}\binom{n}{i}s_i(x)p_{n-i}(y), \quad n = 1, 2, \ldots$$

REMARK 3.3. After Theorem 3.3 the polynomial sequence $s_n(x)$ defined by (3.1) is the Sheffer sequence associated to the polynomial sequence of b.t. $p_n(x)$, as defined in [**19**, p. 139].

THEOREM 3.4 (Multiplication theorem, [**9**]). *The sequence $s_n(x)$, $n = 0, 1, \ldots$, is the Sheffer sequence for $(\beta_n, p_n(x))$ if and only if*

$$s_n(mx) = \sum_{i=0}^{n}\binom{n}{i}s_{n-i}(x)p_i((m-1)x), \qquad \begin{matrix} n = 0, 1, \ldots, \\ m = 1, 2, \ldots \end{matrix}$$

Now, we give the relationship with generating function and the equivalence with Sheffer theory [**22**].

THEOREM 3.5 (Generating function, [**9**]). *The sequence* $s_n(x)$ *is the Sheffer polynomial sequence for* $(\beta_n, p_n(x))$ *if and only if there exists a unique numerical sequence* α_n, $n = 0, 1, \ldots$, $\alpha_0 \neq 0$, *such that, setting*

$$a(t) = \alpha_0 + \alpha_1 t + \alpha_2 \frac{t^2}{2} + \cdots + \alpha_n \frac{t^n}{n!} + \ldots, \quad \alpha_0 \neq 0,$$

we have

$$a(t) e^{xB(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!},$$

where $B(t)$ *is defined in* (2.5).

4. Relationship with δ-operator and functional

Let Q be the δ-operator [**19**] associated to the polynomial sequence of b.t. $p_n(x)$, that is given by (2.11). Then we have the following

THEOREM 4.1. [**9**] *If* $s_n(x)$, $n = 0, 1, \ldots$ *is the Sheffer sequence for* $(\beta_n, p_n(x))$, *then*

$$Qs_n(x) = ns_{n-1}(x) \;\; n = 1, 2, \ldots. \tag{4.1}$$

THEOREM 4.2. [**9**] *If* $s_n(x)$, $n = 0, 1, \ldots$ *be the Sheffer sequence for* $(\beta_n, p_n(x))$ *and* Q *the* δ*-operator associated to the polynomial sequence of b.t.* $p_n(x)$, *then* $s_n(x)$ *satisfies the functional equation*

$$\frac{\beta_n}{n!} Q^n y(x) + \frac{\beta_{n-1}}{(n-1)!} Q^{n-1} y(x) + \cdots + \frac{\beta_2}{2!} Q^2 y(x) + \beta_1 Q y(x) + \beta_0 y(x) = p_n(x)$$

Let, now, L be the linear functional on $\mathcal{P}_n$ defined by $\beta_n = L(p_n(x))$, $n = 0, 1, \ldots$. Then we have the following

THEOREM 4.3. *The sequence* $s_n(x)$, $n = 0, 1, \ldots$ *is the Sheffer sequence for* $(\beta_n, p_n(x))$ *if and only if*

$$L(Q^k s_n(x)) = n! \delta_{n,k}, \quad k = 0, \ldots, n. \tag{4.2}$$

PROOF. It follows from (3.1), (4.1) and from the linearity of L. □

REMARK 4.1. After Theorem 4.3 we have the equivalence of our matrix approach with Roman theory [**18**].

REMARK 4.2. Putting $L_i = LQ^i$, $i = 0, \ldots, n$, relations (4.2) can be interpreted as a linear interpolation problem on $\mathcal{P}_n$, the solution of which is the polynomial sequence $s_n(x)$ given by (3.1).

Let us, now, introduce the Sheffer vector.

DEFINITION 4.1. If $s_n(x)$ is the Sheffer polynomial sequence for $(\beta_n, p_n(x))$, then the vector of functions $S_n(x) = [s_0(x), \ldots, s_n(x)]^T$ is called the Sheffer vector for $(\beta_n, p_n(x))$.

Then we have

THEOREM 4.4 (Matrix form, [**9**]). *Let $S_n(x)$ be a vector of polynomial function. It is the Sheffer vector for $(\beta_n, p_n(x))$ if and only if*

$$S_n(x) = SX_n(x), \quad X_n(x) = \widehat{S}S_n(x),$$

where $S = M^{-1}A$, $\widehat{S} \equiv S^{-1} = A^{-1}M$, with M and A defined by (3.3) *and* (2.6) *respectively.*

REMARK 4.3. Putting

$$S \equiv (S)_{i,j} \equiv \begin{cases} s_{i,j} & i \geqslant j \\ 0 & \text{otherwise} \end{cases}, \qquad i,j = 0,\dots,n,$$

$$\widehat{S} \equiv (\widehat{S})_{i,j} \equiv \begin{cases} \widehat{s}_{i,j} & i \geqslant j \\ 0 & \text{otherwise} \end{cases}, \qquad i,j = 0,\dots,n, \tag{4.3}$$

we get

$$s_{i,j} = \sum_{k=j}^{i} \binom{i}{k} \alpha_{i-k} a_{k,j}, \quad i = 0,\dots,n, \ j = 0,\dots,i,$$

$$\widehat{s}_{i,j} = \sum_{k=j}^{i} \binom{i}{k} \beta_{i-k} \widehat{a}_{k,j} = \frac{L(Q^j x^i)}{j!}, \quad i = 0,\dots,n, \ j = 0,\dots,i. \tag{4.4}$$

THEOREM 4.5 (Representation Theorem, [**9**]). *If $s_n(x)$ is the Sheffer sequence for $(\beta_n, p_n(x))$ and $q_n(x)$ is a polynomial of degree $\leqslant n$ such that $q_n(x) = \sum_{k=0}^{n} c_k x^k$, $n = 0, 1, \dots,$ then*

$$q_n(x) = \sum_{k=0}^{n} \frac{L(Q^k q_n(x))}{k!} s_k(x) = \sum_{k=0}^{n} \left(\sum_{j=k}^{n} c_k \widehat{s}_{j,k} \right) s_k(x), \quad n = 0, 1, \dots.$$

THEOREM 4.6. [**9**] *The sequence $s_n(x)$ is the Sheffer sequence for $(\beta_n, p_n(x))$ if and only if*

$$s_n(x) = \frac{(-1)^n}{\prod_{k=0}^{n} \widehat{s}_{k,k}} \begin{vmatrix} 1 & x & \cdots & x^{n-1} & x^n \\ \widehat{s}_{0,0} & \widehat{s}_{1,0} & \cdots & \widehat{s}_{n-1,0} & \widehat{s}_{n,0} \\ 0 & \widehat{s}_{1,1} & \cdots & \widehat{s}_{n-1,1} & \widehat{s}_{n,1} \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & \widehat{s}_{n-1,n-1} & \widehat{s}_{n,n-1} \end{vmatrix}, \quad n = 0, 1, \dots,$$

with $\widehat{s}_{i,j}$, $i, j = 0, \dots, n$, defined by (4.4).

REMARK 4.4. For $\widehat{s}_{i,j} = \binom{i}{j}\beta_{i-j}$, $s_n(x)$ is the Appell polynomial sequence as defined in [**5**].

THEOREM 4.7 (Binomial sequences, [**9**]). *If $p_n(x)$, $n = 0, 1, \dots,$ is a binomial type polynomial sequence associated to b_n then*

$$p_0(x) = 1, \quad p_n(x) = \frac{(-1)^{n+1}}{\prod_{k=0}^{n} \widehat{a}_{k,k}} \begin{vmatrix} x & x^2 & \cdots & x^{n-1} & x^n \\ \widehat{a}_{1,1} & \widehat{a}_{2,1} & \cdots & \widehat{a}_{n-1,1} & \widehat{a}_{n,1} \\ 0 & \widehat{a}_{2,2} & \cdots & \widehat{a}_{n-1,2} & \widehat{a}_{n,2} \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & \widehat{a}_{n-1,n-1} & \widehat{a}_{n,n-1} \end{vmatrix},$$

$n = 1, 2, \dots,$ with $\widehat{a}_{i,j}$, $i, j = 0, \dots, n$, defined by (2.10).

REMARK 4.5. In [**11**] the inverse of Theorem 4.7 is proved.

THEOREM 4.8 (Connection constants, [**9**]). *Let $S_n(x)$ and $T_n(x)$ be the Sheffer vectors for $(\beta_n,\ p_n(x))$ and $(\gamma_n,\ q_n(x))$, respectively, with $p_n(x) = \sum_{k=0}^{n} a_{n,k}x^k$ and $q_n(x) = \sum_{k=0}^{n} d_{n,k}x^k$. Then $T_n(x) = CS_n(x)$, where $C = TS^{-1}$, with S^{-1} defined by* (4.3) *and* (4.4)*, and*

$$(T)_{i,j} = \begin{cases} \sum_{k=j}^{i} \binom{i}{k} \delta_{i-k} d_{k,j} & i \geqslant j \\ 0 & \textit{otherwise} \end{cases}, \qquad i, j = 0, \dots, n,$$

where δ_n are defined by $\sum_{k=0}^{n} \binom{n}{k} \delta_k \gamma_{n-k} = \left\{ \begin{smallmatrix} 1, & n=0, \\ 0, & n=1,\dots \end{smallmatrix} \right.$

5. The algebraic approach of Yang

Yang in [**26**] gives a new determinantal form and a recurrence relation for Sheffer sequences; his approach is based on the production matrix [**13**] of an invertible lower triangular matrix. There is the following

THEOREM 5.1. [**26**] *Let $[g(t), f(t)]$ be an exponential Riordan array with the c-sequences $(c_i)_{i\geqslant 0}$ and r-sequences $(r_i)_{i\geqslant 0}$. Let $(a_n(x))_{x\geqslant 0}$ be the Sheffer polynomial sequence for $(g(t), f(t))$. Then $(a_n(x))_{x\geqslant 0}$ satisfies the recurrence relation*

$$\begin{aligned} a_{n+1}(x) &= (x - c_0 - nr_1)a_n(x) - \frac{n!}{(n-1)!}(c_1 + (n-1)r_2)a_{n-1}(x) - \dots \\ &\quad - \frac{n!}{2!}(c_{n-2} + 2r_{n-1})a_2(x) - \frac{n!}{1!}(c_{n-1} + r_n)a_1(x) - n!c_n a_0(x), \end{aligned}$$

with initial conditions $a_0(x) = 1$ and $a_1(x) = x - c_0$. For $n \geqslant 0$, $a_{n+1}(x)$ is given by

$$\begin{aligned} a_{n+1}(x) &= (-1)^{n+1} \\ &\times \begin{vmatrix} c_0 - x & 1 & 0 & \cdots & 0 \\ 1!c_1 & \frac{1!}{1!}(c_0 + r_1) - x & 1 & \cdots & 0 \\ 2!c_2 & \frac{2!}{1!}(c_1 + r_2) & \frac{2!}{2!}(c_0 + 2r_1) - x & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n!c_n & \frac{n!}{1!}(c_{n-1} + r_n) & \frac{n!}{2!}(c_{n-2} + 2r_{n-1}) & \cdots & \frac{n!}{n!}(c_0 + nr_1) - x \end{vmatrix} \\ &= (-1)^{n+1}|P_{n+1} - xI_{n+1}| = |xI_{n+1} - P_{n+1}|. \end{aligned}$$

Thus, the Sheffer sequences are characteristic polynomials of production matrices.

6. The umbral interpolation

The umbral interpolation [**10**] is the linear interpolation problem which is expressed through a basis of Sheffer polynomials (Appell or binomial polynomials in particular). Let X be the linear space of real functions defined in the interval $[a, b]$ continuous and with continuous derivatives of all necessary orders. Let Q be a δ-operator on $\mathcal{P}_n$ and $p_n(x)$, $n = 0, 1, \ldots$, the polynomial sequence of binomial type associated to it. Let L be a linear functional on X such that $L(1) \neq 0$. Let us we set $\beta_n = L(p_n(x))$, $n = 0, 1, \ldots$, and let $s_n(x)$ be the Sheffer polynomial sequence for $(\beta_n, p_n(x))$.

DEFINITION 6.1. [**10**] The polynomial sequence $s_n(x)$, $n = 0, 1, \ldots$, is a basis for $\mathcal{P}_n$ and we call it *Umbral basis* for (L, Q).

Let be $\omega_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, and consider the problem

$$L(Q^i P_n) = i!\omega_i, \quad i = 0, \ldots, n, \quad P_n(x) \in \mathcal{P}_n.$$

We call it *Umbral interpolation problem.*

THEOREM 6.1 (Main theorem, [**10**]). *Let $s_i(x)$ be the Umbral basis for (L, Q) and $\omega_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$; the polynomial $P_n(x) = \sum_{i=0}^{n} \omega_i s_i(x)$ is the unique polynomial of degree less than or equal to n such that $L(Q^i P_n) = i!\omega_i$, $i = 0, \ldots, n$.*

REMARK 6.1. [**10**] For each $P_n(x) \in \mathcal{P}_n$ we have $P_n(x) = \sum_{i=0}^{n} \frac{L(Q^i P_n)}{i!} s_i(x)$.

Let us consider a function $f \in X$, such that $Q^i f$ is defined in X. Then we have

THEOREM 6.2. [**10**] *Let $f \in X$ such that $Q^i f \in X$. The polynomial*

$$P_n[f](x) = \sum_{i=0}^{n} \frac{L(Q^i f)}{i!} s_i(x) \tag{6.1}$$

is the unique polynomial of degree less than or equal to n such that

$$L(Q^i P_n[f]) = L(Q^i f), \quad i = 0, \ldots, n.$$

DEFINITION 6.2. The polynomial $P_n[f](x)$, if not identically zero, is called *Umbral interpolation polynomial* of the function f for (L, Q); it satisfies

$$L(Q^i P_n) = L(Q^i f), \ i = 0, 1, \ldots, n.$$

Therefore, it is interesting to consider the estimation of the remainder. Let us set $R_n[f](x) = f(x) - P_n[f](x)$, for all $x \in [a, b]$.

THEOREM 6.3 (Exactness, [**10**]). *For any $f \in \mathcal{P}_n$ and each $x \in [a, b]$,*

$$R_n[f](x) = 0 \text{ and } R_n[p_{n+1}(x)] \neq 0.$$

For a fixed x, we may consider $R_n[f](x)$ as a linear functional which acts on f and annihilates all elements of $\mathcal{P}_n$. By Peano's theorem, if a linear functional has this property, then it must have a simple representation in terms of $f^{(n+1)}$.

Theorem 6.4. [**10**] *For $f \in C^{n+1}[a,b]$, the following relation holds*

$$\forall x \in [a,b], \quad R_n[f](x) = \frac{1}{n!}\int_a^b K_n(x,t) f^{(n+1)}(t)\,dt$$

where

$$K_n(x,t) = R_n\big[(x-t)_+^n\big] = (x-t)_+^n - \sum_{i=0}^{n} \frac{L_x\big[Q^i(x-t)_+^n\big]}{i!} s_i(x).$$

Now, let us fix $z \in [a,b]$ and consider the polynomial

$$\overline{P}_n[f,z](x) \equiv f(z) + P_n[f](x) - P_n[f](z) = f(z) + \sum_{i=1}^{n} \frac{L(Q^i f)}{i!}(s_i(x) - s_i(z)). \tag{6.2}$$

Definition 6.3. The polynomial $\overline{P}_n[f,z](x)$ is called *Umbral interpolation polynomial of f centered at z.*

Theorem 6.5. [**10**] *The polynomial $\overline{P}_n[f,z](x)$ is an approximating polynomial of degree n for $f(x)$, i.e.,*

$$\forall\, x \in [a,b], \; f(x) = \overline{P}_n[f,z](x) + \overline{R}_n[f](x),$$

with $\overline{R}_n[p_i(x)] = 0$, $i = 0, \ldots, n$ and $\overline{R}_n[p_{n+1}(x)] \neq 0$.

Theorem 6.6. [**10**] *The polynomial $\overline{P}_n[f,z](x)$ satisfies the interpolation conditions $\overline{P}_n[f,z](z) = f(z)$, $L(Q^i \overline{P}_n[f,z]) = L(Q^i f)$, $i = 1, \ldots, n$.*

7. Examples

7.1. Appell interpolation [6]. With the previous notation, let $Qf = Df = f'(x)$. Then the associated sequence is the canonical basis x^n, $n = 0, 1, \ldots$.

Now, let L be a linear functional verifying $L(1) \neq 0$. Then the umbral interpolation polynomials (6.1) and (6.2) become

$$P_n[f](x) = L(f) + \sum_{i=1}^{n} \frac{L(f^{(i)}(x))}{i!} A_i(x), \tag{7.1}$$

$$\overline{P}_n[f,0](x) = f(0) + \sum_{i=1}^{n} \frac{L(f^{(i)}(x))}{i!}(A_i(x) - A_i(0)),$$

where $A_i(x)$ is the umbral basis for (L, D), and in particular, it is an Appell sequence.

7.1.1. *Taylor interpolation* [**6, 12**]. Let $L(f) = f(x_0)$, $x_0 \in [a,b]$. Then the umbral basis for (L, D) is the sequence $A_0(x) = 1$, $A_n(x,a) = (x - x_0)^n$, $n = 1, 2, \ldots$. The interpolation polynomial (6.1) becomes

$$T_n[f](x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i$$

i.e., the Umbral interpolation is the well known Taylor interpolation [**12**]. Therefore, (7.1) can be seen as a generalization of Taylor interpolation.

7.1.2. *Bernoulli interpolation* [**3**, **6**]. Let $L(f) = \int_0^1 f(x)\,dx$. Then the umbral basis for (L, D) is the Bernoulli sequence $B_n(x)$ [**4**, **5**]. Interpolation polynomials (6.1) and (6.2) become

$$B_n[f](x) = \int_0^1 f(x)dx + \sum_{i=1}^{n} \frac{f^{(i-1)}(1) - f^{(i-1)}(0)}{i!} B_i(x),$$

$$\overline{B}_n[f,0](x) = f(0) + \sum_{i=1}^{n} \frac{f^{(i-1)}(1) - f^{(i-1)}(0)}{i!} (B_i(x) - B_i(0)). \tag{7.2}$$

7.1.3. *Euler interpolation* [**6**]. We consider $L(f) = \frac{1}{2}(f(0) + f(1))$. Then the umbral basis for (L, D) is the Euler sequence $E_n(x)$ [**5**]. Interpolation polynomials (6.1) and (6.2) become

$$E_n[f](x) = \frac{f(0) + f(1)}{2} + \sum_{i=1}^{n} \frac{f^{(i)}(0) + f^{(i)}(1)}{2i!} E_i(x), \tag{7.3}$$

$$\overline{E}_n[f,0](x) = f(0) + \sum_{i=1}^{n} \frac{f^{(i)}(0) + f^{(i)}(1)}{2i!} (E_i(x) - E_i(0)).$$

7.2. Abel–Sheffer interpolation [10]. With the previous notation, let $Qf = D_a f = f'(x+a)$, $a \in \mathbb{R}$, $a \neq 0$. Then the associated sequence is the Abel sequence $A_0(x,a) = 1$, $A_n(x,a) = x(x - an)^{n-1}$, $n = 1, 2, \dots$ [**19**].

Now, let L be a linear functional verifying $L(1) \neq 0$. Then umbral interpolation polynomials (6.1) and (6.2) become

$$P_n[f](x) = L(f) + \sum_{i=1}^{n} \frac{L(f^{(i)}(x + ai))}{i!} s_i(x), \tag{7.4}$$

$$\overline{P}_n[f,0](x) = f(0) + \sum_{i=1}^{n} \frac{L(f^{(i)}(x + ai))}{i!} (s_i(x) - s_i(0)),$$

where $s_i(x)$ is the umbral basis for (L, D_a).

7.2.1. *Abel–Gontscharoff interpolation* [**10**, **12**]. Let $L(f) = f(x_0)$, $x_0 \in [a, b]$. Then the umbral basis for (L, D_a) is the sequence

$$\widetilde{G}_0(x) = 1,\ \widetilde{G}_n(x,a) = (x - x_0)(x - x_0 - an)^{n-1},\ n = 1, 2, \dots,$$

that is the classical Abel–Gontscharoff sequence [**12**] on the equidistant points $x_i = x_0 + ai$, $i = 0, \dots, n$. Interpolation polynomial (6.1) becomes

$$\widetilde{G}_n[f](x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0 + ai)}{i!} \widetilde{G}_i(x,a)$$

i.e., the Umbral interpolation is the well known Abel–Gontscharoff interpolation [**12**, **25**] on the equidistant points $x_i = x_0 + ai$, $i = 0, \dots, n$. Therefore (7.4) can be seen as a generalization of Abel–Gontscharoff interpolation on the equidistant points.

7.2.2. *Abel–Bernoulli interpolation* [**10**]. Let $L(f) = \int_0^1 f(x)\,dx$. Then the umbral basis for (L, D_a) is the Bernoulli–Abel sequence $\widetilde{B}_n(x,a)$ [**9**]. Interpolation polynomials (6.1) and (6.2) become

$$\widetilde{B}_n[f](x) = \int_0^1 f(x)\,dx + \sum_{i=1}^{n} \frac{f^{(i-1)}(1+ai) - f^{(i-1)}(ai)}{i!} \widetilde{B}_i(x,a),$$

$$\overline{\widetilde{B}}_n[f,0](x) = f(0) + \sum_{i=1}^{n} \frac{f^{(i-1)}(1+ai) - f^{(i-1)}(ai)}{i!} (\widetilde{B}_i(x,a) - \widetilde{B}_i(0,a)). \tag{7.5}$$

7.2.3. *Abel–Euler interpolation* [**10**]. We consider $L(f) = \frac{1}{2}(f(0)+f(1))$. Then the umbral basis for (L, D_a) is the Euler–Abel sequence $\widetilde{E}_n(x,a)$ [**9**]. Interpolation polynomials (6.1) and (6.2) become

$$\widetilde{E}_n[f](x) = \frac{f(0)+f(1)}{2} + \sum_{i=1}^{n} \frac{f^{(i)}(ai) + f^{(i)}(1+ai)}{2i!} \widetilde{E}_i(x,a), \tag{7.6}$$

$$\overline{\widetilde{E}}_n[f,0](x) = f(0) + \sum_{i=1}^{n} \frac{f^{(i)}(ai) + f^{(i)}(1+ai)}{2i!} (\widetilde{E}_i(x,a) - \widetilde{E}_i(0,a)).$$

7.3. Δ_h-Appell interpolation [8]. Let $Qf = \Delta_h f(x) = f(x+h) - f(x)$. Moreover, let Δ_h^{-1} be the inverse operator of Δ_h, such that

$$\Delta_h^{-1}\varphi(x) = f(x) \Leftrightarrow \Delta_h f(x) = \varphi(x).$$

Then the associated sequence to Δ_h is the falling factorial sequence [**19**]

$$(x)_0 = 1, \ \ (x)_n = x(x-h)(x-2h)\cdots(x-(n-1)h), \ \ n = 1, 2, \dots.$$

Now, let L be a linear functional verifying $L(1) \neq 0$. Then umbral interpolation polynomials (6.1) and (6.2) become

$$P_n[f](x) = \sum_{i=0}^{n} \frac{L(\Delta_h^i f)}{h^i i!} \mathcal{A}_i(x), \tag{7.7}$$

$$\overline{P}_n[f,0](x) = f(0) + \sum_{i=1}^{n} \frac{L(\Delta_h^i f)}{h^i i!} (\mathcal{A}_i(x) - \mathcal{A}_i(0)),$$

where $\mathcal{A}_i(x)$ is the umbral basis for (L, Δ_h) and, in particular, it is a Δ_h-Appell sequence [**8**].

As in the previous example we can consider the following cases.

7.3.1. *Newton interpolation on equidistant points* [**8, 12**]. Let $L(f) = f(x_0)$. The umbral basis for (L, Δ_h) is $(x - x_0)_n$. Interpolation polynomial (6.1) becomes

$$P_n[f](x) = f(x_0) + \sum_{i=1}^{n} \frac{\Delta_h^i f(x_0)}{h^i i!} (x - x_0)_i.$$

It is known as Newton interpolation on equidistant points, therefore (7.7) can be seen as a generalization of Newton interpolation on equidistant points.

7.3.2. *Bernoulli interpolation of second kind* [**8**]. Let $L(f) = (D\Delta_h^{-1}f)_{x=0}$. We call the umbral basis for (L, Δ_h) Bernoulli polynomial sequence of second kind $B_n^{II}(x)$ [**8**]. Interpolation polynomials (6.1) and (6.2) become

$$B_n^{II}[f](x) = (D\Delta_h^{-1}f)_{x=0} + \sum_{i=1}^{n} \frac{\Delta_h^{i-1}f'(0)}{h^i i!} B_i^{II}(x),$$

$$\overline{B}_n^{II}[f,0](x) = f(0) + \sum_{i=1}^{n} \frac{\Delta_h^{i-1}f'(0)}{h^i i!} (B_i^{II}(x) - B_i^{II}(0)). \tag{7.8}$$

7.3.3. *Boole interpolation* [**8**]. Let $L(f) = (M_h f)_{x=0} = \frac{1}{2}(f(0)+f(h))$. We call the umbral basis for (L, Δ_h) Boole polynomial sequence $E_n^{II}(x)$ [**8**]. Interpolation polynomials (6.1) and (6.2) become

$$E_n^{II}[f](x) = \frac{f(0)+f(h)}{2} + \sum_{i=1}^{n} \frac{\Delta_h^i(f(0)+f(h))}{2h^i i!} E_i^{II}(x), \tag{7.9}$$

$$\overline{E}_n^{II}[f,0](x) = f(0) + \sum_{i=1}^{n} \frac{\Delta_h^i(f(0)+f(h))}{2h^i i!} (E_i^{II}(x) - E_i^{II}(0)).$$

7.4. δ_h-Sheffer interpolation [10]. Let

$$Qf = \delta_h f(x) = \frac{1}{h}\Big(f\Big(x+\frac{1}{2}h\Big) - f\Big(x-\frac{1}{2}h\Big)\Big)$$

and let δ_h^{-1} be the inverse operator of δ_h, such that

$$\delta_h^{-1}\varphi(x) = f(x) \Leftrightarrow \delta_h f(x) = \varphi(x).$$

Then the associated sequence to δ_h is the sequence [**19**]

$$x^{[0]} = 1, \ \ x^{[n]} \equiv x\Big(x+\Big(\frac{n}{2}-1\Big)h\Big)_{n-1} = x\Big(x+\Big(\frac{n}{2}-1\Big)h\Big)\cdots\Big(x+\Big(-\frac{n}{2}+1\Big)h\Big),$$

$n = 1, 2, \ldots$. Now, let L be a linear functional verifying $L(1) \neq 0$. Then umbral interpolation polynomials (6.1) and (6.2) become

$$P_n[f](x) = \sum_{i=0}^{n} \frac{L(\delta_h^i f)}{i!} s_i(x), \tag{7.10}$$

$$\overline{P}_n[f,0](x) = f(0) + \sum_{i=1}^{n} \frac{L(\delta_h^i f)}{i!} (s_i(x) - s_i(0)),$$

where $s_i(x)$ is the umbral basis for (L, δ_h).

As in the previous example we can consider the following cases.

7.4.1. *δ_h-central interpolation* [**10**,**12**]. Let $L(f) = f(0)$. The umbral basis for (L, δ_h) is $s_n(x) = x^{[n]}$. Interpolation polynomial (6.1) becomes

$$P_n[f](x) = f(0) + \sum_{i=1}^{n} \frac{\delta_h^i f(0)}{i!} x^{[i]}.$$

It is known as interpolation formula with central differences [**23**, p. 32], therefore (7.10) can be seen as a generalization of central interpolation.

7.4.2. *δ_h-Bernoulli interpolation* [**10**]. Let $L(f) = \left(D\delta_h^{-1} f\right)_{x=0}$. We call the umbral basis for (L, δ_h) δ_h-Bernoulli polynomial sequence $\widehat{B}_n(x)$. Interpolation polynomials (6.1) and (6.2) become

$$\widehat{B}_n[f](x) = \left(D\delta_h^{-1} f\right)_{x=0} + \sum_{i=1}^{n} \frac{\delta_h^{i-1} f'(0)}{i!} \widehat{B}_i(x),$$

$$\overline{\widehat{B}}_n[f](x) = f(0) + \sum_{i=1}^{n} \frac{\delta_h^{i-1} f'(0)}{i!} \left(\widehat{B}_i(x) - \widehat{B}_i(0)\right). \tag{7.11}$$

7.4.3. *δ_h-Euler interpolation* [**10**]. Let

$$L(f) = (M_h f)_{x=0} = \frac{1}{2}\left(f(h/2) + f(-h/2)\right).$$

We call the umbral basis for (L, δ_h) δ_h-Euler polynomial sequence $\widehat{E}_n(x)$. Interpolation polynomials (6.1) and (6.2) become

$$\widehat{E}_n[f](x) = \frac{f(h/2) + f(-h/2)}{2} + \sum_{i=1}^{n} \frac{\delta_h^i\left(f(h/2) + f(-h/2)\right)}{2i!} \widehat{E}_i(x), \tag{7.12}$$

$$\overline{\widehat{E}}_n[f,0](x) = f(0) + \sum_{i=1}^{n} \frac{\delta_h^i\left((h/2) + f(-h/2)\right)}{2i!} \left(\widehat{E}_i(x) - \widehat{E}_i(0)\right).$$

8. Numerical examples

Now we consider some interpolation test problem and report the numerical results obtained by using an ad hoc “Mathematica" code. We compare the error in approximating a given function with Appell, Abel–Sheffer, Δ_h-Appell and δ_h-Sheffer interpolation polynomials. In particular we compare numerical results obtained by applying:

- Abel-Bernoulli interpolation polynomial $\overline{\widetilde{B}}_n[f,0](x)$ (7.5)
- Bernoulli interpolation polynomial $\overline{B}_n[f,0](x)$ (7.2)
- δ_h-Bernoulli interpolation polynomial $\overline{\widehat{B}}_n[f,0](x)$ (7.11)
- Bernoulli interpolation polynomial of second kind $\overline{B}_n^{II}[f,0](x)$ (7.8)
- Abel-Euler interpolation polynomial $\widetilde{E}_n[f](x)$ (7.6)
- Euler interpolation polynomial $E_n[f](x)$ (7.3)
- δ_h-Euler interpolation polynomial $\widehat{E}_n[f](x)$ (7.12)
- Boole interpolation polynomial $E_n^{II}[f](x)$ (7.9)

We emphasize that the compared polynomials of the same degree have the same degree of exactness.

Example 8.1. For the function $f(x) = e^{(x+1)/2}$, $x \in [0,1]$, the interpolation error is reported in the following tables.

	$\widetilde{\overline{B}}_n[f,0](x)$	$\overline{B}_n[f,0](x)$	$\widehat{\overline{B}}_n[f,0](x)$	$\overline{B}_n^{II}[f,0](x)$
$n=5$	$2.774*10^{-6}$	$1.102*10^{-6}$	$1.628*10^{-5}$	$6.949*10^{-7}$
$n=6$	$1.460*10^{-7}$	$8.619*10^{-8}$	$8.814*10^{-7}$	$1.733*10^{-8}$
$n=7$	$1.463*10^{-8}$	$6.885*10^{-9}$	$4.160*10^{-8}$	$4.354*10^{-10}$
$n=8$	$7.589*10^{-10}$	$5.458*10^{-10}$	$1.742*10^{-9}$	$8.609*10^{-12}$

	$\widetilde{E}_n[f](x)$	$E_n[f](x)$	$\widehat{E}_n[f](x)$	$E_n^{II}[f](x)$
$n=5$	$5.978*10^{-5}$	$4.460*10^{-5}$	$2.606*10^{-6}$	$1.318*10^{-7}$
$n=6$	$9.432*10^{-6}$	$7.103*10^{-6}$	$1.171*10^{-7}$	$2.925*10^{-9}$
$n=7$	$1.483*10^{-6}$	$1.130*10^{-6}$	$4.715*10^{-9}$	$5.817*10^{-11}$
$n=8$	$2.354*10^{-7}$	$1.799*10^{-7}$	$1.726*10^{-10}$	$1.041*10^{-12}$

EXAMPLE 8.2. For the function $f(x) = \ln(x^2+10)$, $x \in [0,1]$, the interpolation error is reported in the following tables.

	$\widetilde{\overline{B}}_n[f,0](x)$	$\overline{B}_n[f,0](x)$	$\widehat{\overline{B}}_n[f,0](x)$	$\overline{B}_n^{II}[f,0](x)$
$n=5$	$1.994*10^{-6}$	$4.526*10^{-6}$	$1.311*10^{-4}$	$2.823*10^{-6}$
$n=6$	$2.482*10^{-6}$	$1.760*10^{-6}$	$1.737*10^{-6}$	$3.744*10^{-7}$
$n=7$	$5.442*10^{-7}$	$3.457*10^{-7}$	$5.477*10^{-6}$	$1.579*10^{-8}$
$n=8$	$1.267*10^{-7}$	$2.559*10^{-7}$	$6.832*10^{-8}$	$4.487*10^{-9}$

	$\widetilde{E}_n[f](x)$	$E_n[f](x)$	$\widehat{E}_n[f](x)$	$E_n^{II}[f](x)$
$n=5$	$1.183*10^{-4}$	$2.138*10^{-4}$	$1.974*10^{-5}$	$4.435*10^{-7}$
$n=6$	$1.257*10^{-4}$	$1.410*10^{-4}$	$1.559*10^{-6}$	$6.225*10^{-8}$
$n=7$	$6.482*10^{-5}$	$8.666*10^{-5}$	$5.843*10^{-7}$	$1.624*10^{-9}$
$n=8$	$5.736*10^{-5}$	$7.829*10^{-5}$	$4.823*10^{-8}$	$5.279*10^{-10}$

EXAMPLE 8.3. For the function $f(x) = 10\cos(x) + \frac{1}{10}\sin^2(x)$, $x \in [0,1]$, the interpolation error is reported in the following tables.

	$\widetilde{\overline{B}}_n[f,0](x)$	$\overline{B}_n[f,0](x)$	$\widehat{\overline{B}}_n[f,0](x)$	$\overline{B}_n^{II}[f,0](x)$
$n=5$	$4.652*10^{-4}$	$2.319*10^{-4}$	$4.227*10^{-3}$	$1.483*10^{-4}$
$n=6$	$1.882*10^{-5}$	$2.451*10^{-6}$	$3.613*10^{-6}$	$5.090*10^{-7}$
$n=7$	$1.3378*10^{-5}$	$2.695*10^{-6}$	$1.224*10^{-5}$	$9.767*10^{-8}$
$n=8$	$1.981*10^{-6}$	$2.058*10^{-6}$	$4.386*10^{-7}$	$3.399*10^{-8}$

	$\widetilde{E}_n[f](x)$	$E_n[f](x)$	$\widehat{E}_n[f](x)$	$E_n^{II}[f](x)$
$n=5$	$9.016*10^{-3}$	$9.265*10^{-3}$	$6.613*10^{-4}$	$2.737*10^{-5}$
$n=6$	$1.880*10^{-3}$	$3.195*10^{-4}$	$2.770*10^{-6}$	$6.299*10^{-8}$
$n=7$	$1.490*10^{-3}$	$7.061*10^{-4}$	$1.201*10^{-6}$	$1.661*10^{-8}$
$n=8$	$6.494*10^{-4}$	$6.746*10^{-4}$	$3.200*10^{-7}$	$4.111*10^{-9}$

9. Conclusions

We have given a construction of Sheffer sequences via determinantal form, proving the equivalence with previous theories (Rota et al., Sheffer). Moreover, all the main known and some new properties have been shown. Another recent determinantal form has been mentioned. Afterwards, a general linear interpolation problem has been proposed and solved, giving many examples. Further developments both in the multivariate case and in computational applications (stability, conditioning, etc.) are possible.

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