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CONVERGENCE IN CAPACITY OF RATIONAL APPROXIMANTS OF MEROMORPHIC FUNCTIONS

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Abstract. Let f be meromorphic on the compact set $E \subset \mathbb{C}$ with maximal Green domain of meromorphy $E_{\rho(f)}$, $\rho(f) < \infty$. We investigate rational approximants with numerator degree $\leqslant n$ and denominator degree $\leqslant m_n$ for f. We show that the geometric convergence rate on E implies convergence in capacity outside E if $m_n = o(n)$ as $n \to \infty$. Further, we show that the condition is sharp and that the convergence in capacity is uniform for a subsequence $\Lambda \subset \mathbb{N}$.

1. Introduction

Let E be compact in $\mathbb{C}$ with connected complement $\Omega = \overline{\mathbb{C}} \setminus E$. The set Ω is called *regular* if there exists a Green function $G(z) = G(z, \infty)$ on Ω with pole at ∞ satisfying $G(z) \to 0$ as $z \to \partial\Omega$. Note that $\lim_{z\to\infty}(G(z) - \log|z|) = -\log \operatorname{cap} E$. Here, $\operatorname{cap} E$ is the *logarithmic capacity* and $\operatorname{cap} E > 0$ if Ω is regular (cf. Tsuji [**5**]). Moreover, we define the Green domains E_ρ by

$$E_\rho := \{z \in \Omega : G(z) < \log \rho\} \cup E, \quad \rho > 1$$

and $E_1 := E^\circ$, where E° is the set of interior points of E.

For $B \subset \mathbb{C}$, we denote by $C(B)$ the class of *continuous functions* on B, and $\mathcal{M}(B)$ represents the class of functions f that are meromorphic in some open neighborhood of B.

If $f \in \mathcal{M}(E)$, then there exists a maximal $\rho(f) > 1$ such that $f \in \mathcal{M}(E_{\rho(f)})$. $\rho(f) = \infty$ if and only if f is meromorphic on $\mathbb{C}$.

Given $n, m \in \mathbb{N}_0$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, let $\mathcal{R}_{n,m}$ be the collection of all rational functions,

$$\mathcal{R}_{n,m} := \{r = p/q : p \in \mathcal{P}_n,\ q \in \mathcal{P}_m,\ q \not\equiv 0\},$$

where $\mathcal{P}_n$ (resp. $\mathcal{P}_m$) denotes the collection of all algebraic polynomials with degree at most n (resp. m).

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Let $r^*_{n,m} := r^*_{n,m}(f) \in \mathcal{R}_{n,m}$ denote a rational function of best uniform approximation to f on E, i.e.,

$$e_{n,m}(f) := \inf_{r \in \mathcal{R}_{n,m}} \|f - r\|_E = \|f - r^*_{n,m}\|_E,$$

where we use $\|\cdot\|_B$ for the supremum norm on $B \subset \mathbb{C}$.

By Walsh's theorem (cf. Walsh [**6**]), we know that

$$\limsup_{n\to\infty} \|f - r^*_{n,m_n}\|_E^{1/n} \leqslant \frac{1}{\rho(f)},$$

if $\lim_{n\to\infty} m_n = \infty$. Now, the starting point in [**1**] was a sequence of rational approximants $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|_{\partial E}^{1/n} \leqslant \frac{1}{\tau} < 1. \tag{1.1}$$

In [**1**] the problem was considered whether the convergence (1.1) can be transferred to domains E_σ, $\sigma > 1$. Concerning the convergence of $\{r_{n,m_n}\}_{n\in\mathbb{N}}$, the m_1-*measure* was used: Let e be subset of $\mathbb{C}$, and set $m_1(e) := \inf\{\sum |U_\nu|\}$, where the infimum is taken over all coverings $\{U_\nu\}$ of e by disks U_ν, and $|U_\nu|$ is the radius of the disk U_ν.

Let D be domain in $\mathbb{C}$ and φ a function defined in D with values in $\overline{\mathbb{C}}$. A sequence of functions $\{\varphi_n\}$, meromorphic in D, is said to converge to a function φ m_1-*almost uniformly inside* D if for any compact set $K \subset D$ and any $\varepsilon > 0$ there exists a set K_ε such $m_1(K_\varepsilon) < \varepsilon$ and $\{\varphi_n\}$ converges uniformly to φ on $K \setminus K_\varepsilon$.

THEOREM 1.1. [**1**, Theorem 1] *Let E be compact in $\mathbb{C}$ with regular, connected complement $\Omega = \overline{\mathbb{C}} \setminus E$, $\{m_n\}_{n=1}^\infty$ a sequence in $\mathbb{N}_0$ with $m_n = o(n/\log n)$ as $n\to\infty$, $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ a sequence of rational functions, $r_{n,m_n} \in \mathcal{R}_{n,m_n}$, such that for $f \in \mathcal{M}(E)$*

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|_{\partial E}^{1/n} \leqslant \frac{1}{\tau} < 1.$$

Then there exists an extension $\tilde f$ of f to E_τ with the following property:

For any $\varepsilon > 0$ there exists a subset $\Omega(\varepsilon) \subset \mathbb{C}$ with $m_1(\Omega(E)) < \varepsilon$ such that $\tilde f$ is a continuous function on $E_\tau \setminus \Omega(\varepsilon)$ with

$$\limsup_{n\to\infty} \|\tilde f - r_{n,m_n}\|_{\overline{E}_\sigma \setminus \Omega(\varepsilon)}^{1/n} \leqslant \frac{\sigma}{\tau}$$

for any σ with $1 < \sigma < \tau$ and $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ converges m_1-almost uniformly to $\tilde f$ inside E_τ.

In [**1**] it was noted that it is not known for $f \in \mathcal{M}(E_\rho)$, $\rho > 1$, whether the continuous extension $\tilde f$ of Theorem 1.1 is m_1-equivalent to f on $E_\tau \cap E_\rho$ if $\lim_{n\to\infty} m_n = \infty$, $m_n = o(n/\log n)$ as $n \to \infty$.

The main result of this paper is to show that this is true even if $m_n = o(n)$ as $n\to\infty$. Moreover, we can show not only convergence in m_1-measure, but more stronger, convergence in capacity and even uniform convergence in capacity at least for a subsequence of $\{r_{n,m_n}\}_{n\in\mathbb{N}}$.

2. Convergence in capacity

Let D be a domain in $\mathbb{C}$, φ a function in $\mathbb{C}$ with values in $\overline{\mathbb{C}}$. A sequence $\varphi_n : D \to \overline{\mathbb{C}}$, $n \in \mathbb{N}$, *converges in capacity inside* D if for any compact set $K \subset D$ and any $\varepsilon > 0$ one has $\operatorname{cap}(\{z \in K : |(\varphi - \varphi_n)(z)| \geqslant \varepsilon\} \to 0$ as $n \to \infty$. Moreover, $\{\varphi_n\}_{n\in\mathbb{N}}$ converges *uniformly in capacity inside* D to φ if for any compact set $K \subset D$ and any $\varepsilon > 0$ there exists a set $K_\varepsilon \subset K$ such that $\operatorname{cap} K_\varepsilon < \varepsilon$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ converges uniformly to φ on $K \smallsetminus K_\varepsilon$ (cf. Gonchar [**2**]).

Our main theorems for convergence in capacity can be formulated as follows.

THEOREM 2.1. *Let E be compact in $\mathbb{C}$ with regular connected complement, $\{m_n\}_{n\in\mathbb{N}}$ a sequence in $\mathbb{N}$ with*

$$m_n = o(n) \text{ as } n \to \infty \text{ and } \lim_{n\to\infty} m_n = \infty. \tag{2.1}$$

Let $f \in \mathcal{M}(E)$ and let $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ be a sequence of rational functions, $r_{n,m_n} \in \mathcal{R}_{n,m_n}$, such that

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|_{\partial E}^{1/n} \leqslant \frac{1}{\tau} < 1.$$

Then the sequence $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ converges in capacity to f inside $E_{\min(\tau,\rho(f))}$.

Theorem 2.1 can be proved by using the methods of the proof of the following Theorem 2.2. For the statement of Theorem 2.2 we choose a parameter $d > 1$ such that $\operatorname{diameter}(E_{\rho(f)}) < d$, if $\rho(f) < \infty$. The parameter d results from the subadditivity theorem for the capacity, due to Nevanlinna [**3**, p. 217] (cf. Pommerenke [**4**]).

THEOREM 2.2. *Let $f \in \mathcal{M}(E)$ with $\rho(f) < \infty$, $\{m_n\}_{n\in\mathbb{N}}$ with* (2.1), *$\{r_{n,m_n}\}_{n\in\mathbb{N}}$ a sequence of rational functions, $r_{n,m_n} \in \mathcal{R}_{n,m_n}$, such that*

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|_{\partial E}^{1/n} \leqslant \frac{1}{\rho(f)}. \tag{2.2}$$

Let $\sigma, 1 < \sigma < \rho(f)$, and $1 < \theta < \rho(f)/\sigma$. Then there exists $n_0 = n_0(\sigma,\theta)$ and compact sets $\Omega_n(\sigma,\theta) \subset \overline{E}_\sigma$ such that for all $n \geqslant n_0(\sigma,\theta)$

$$\operatorname{cap} \Omega_n(\sigma,\theta) \leqslant d^{1/2}\Big(1 - \frac{\theta - 1}{1 + 3\theta}\Big)^{n/2m_n}, \tag{2.3}$$

$$\|f - r_{n,m_n}\|_{\overline{E}_\sigma \smallsetminus \Omega_n(\sigma,\theta)} \leqslant \Big(\frac{\theta\sigma}{\rho(f)}\Big)^n. \tag{2.4}$$

Concerning uniform convergence inside $E_{\rho(f)}$ the following theorem holds.

THEOREM 2.3. *Let f, $\{m_n\}_{n\in\mathbb{N}}$ and $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ be as in Theorem* 2.2 *and let* (2.2) *hold. Then there exists a subset $\{n_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that the subsequence $\{r_{n_k,m_{n_k}}\}_{k\in\mathbb{N}}$ converges uniformly in capacity to f inside $E_{\rho(f)}$.*

Such a type of geometric convergence in capacity was proved by Gonchar [**2**] for the Padé approximation. In [**1**] geometric uniform convergence in m_1-measure of real rational approximants to real functions was proved only for Chebyshev approximation on an interval. So far Theorem 2.3 seems to be the first result for uniform convergence in capacity.

3. Proofs

As already mentioned, we may restrict ourselves to the proof of Theorem 2.2.

PROOF OF THEOREM 2.2. For abbreviation, we write $\rho = \rho(f)$. Let $\varepsilon := (\theta-1)/4$; then we get

$$\varepsilon = \frac{\theta-1}{4} < \frac{\rho/\sigma - 1}{4} = \frac{1}{4}\,\frac{\rho-\sigma}{\sigma} < \rho - \sigma.$$

We choose τ such that $\rho - \varepsilon < \tau < \rho$, and we denote by h^τ the monic polynomial whose zeros are the poles of f in $\overline{E}_\tau$, counted with their multiplicities. Then

$$(fh^\tau)(z) = f(z)h^\tau(z)$$

is holomorphic in $\overline{E}_\tau$. Let us denote by $p_n^\tau \in \mathcal{P}_n$ the best uniform approximation of fh^τ on E. Then there exists $n_1 = n_1(\sigma,\varepsilon)$ such that for $n \geqslant n_1(\sigma,\varepsilon)$

$$\|fh^\tau - r_{n,m_n}h^\tau\|_{\partial E} \leqslant \frac{1}{2}\Big(\frac{1}{\rho-\varepsilon}\Big)^n, \tag{3.1}$$

$$\|fh^\tau - p_n^\tau\|_{E} \leqslant \frac{1}{2}\Big(\frac{1}{\rho-\varepsilon}\Big)^n, \tag{3.2}$$

$$\|fh^\tau - p_n^\tau\|_{\overline{E}_\sigma} \leqslant \frac{1}{2}\Big(\frac{\sigma}{\rho-\varepsilon}\Big)^n, \tag{3.3}$$

$$\operatorname{degree}(h^\tau) \leqslant m_n. \tag{3.4}$$

For (3.1) we have used (2.2), the theorem of Bernstein–Walsh for (3.2) and (3.3), (3.4) follows from (2.1).

Combining (3.1) and (3.2),

$$\|r_{n,m_n}h^\tau - p_n^\tau\|_{\partial E} \leqslant \Big(\frac{1}{\rho-\varepsilon}\Big)^n, \quad n \geqslant n_1(\sigma,\varepsilon). \tag{3.5}$$

Let $r_{n,m_n}(z) = p_n(z)/q_{m_n}(z)$, normalized by

$$q_{m_n}(z) := q^*_{m_n}(z)\prod_{\xi_{n,i}\notin E_\rho}\Big(1-\frac{z}{\xi_{n,i}}\Big) \quad\text{and}\quad q^*_{m_n}(z) := \prod_{\xi_{n,i}\in E_\rho}(z-\xi_{n,i})$$

where $\xi_{n,i}$ denote the poles of r_{n,m_n}. Then for any compact set $K \subset \mathbb{C}$

$$\limsup_{n\to\infty}\|q_{m_n}\|_K^{1/n} \leqslant 1.$$

Because of (3.5) and the normalization of $q_{m,n}$, there exists a constant $c > 0$ such that for $z \in E$

$$|p_n(z)h^\tau(z) - p_n^\tau(z)q_{m_n}(z)| \leqslant c^{m_n}\Big(\frac{1}{\rho-\varepsilon}\Big)^n.$$

We apply the lemma of Bernstein–Walsh to the polynomial

$$w(z) = p_n(z)h^\tau(z) - p_n^\tau(z)q_{m_n}(z) \in \mathcal{P}_{n+m_n}$$

and obtain $|w(z)| \leqslant (c\sigma)^{m_n}\big(\frac{\sigma}{\rho-\varepsilon}\big)^n$ for $z \in \overline{E}_\sigma$. Consequently, for $z \in \overline{E}_\sigma$, where $q_{m_n}(z) \neq 0$, we get

$$|r_{n,m_n}(z)h^\tau(z) - p_n^\tau(z)| = \Big|\frac{w(z)}{q_{m_n}(z)}\Big| \leqslant (c\sigma)^{m_n}\Big(\frac{\sigma}{\rho-\varepsilon}\Big)^n\frac{1}{|q_{m_n}(z)|}.$$

Hence, there exists $n_2 = n_2(\sigma, \varepsilon)$, $n_2 \geqslant n_1$, such that

$$|r_{n,m_n}(z)h^\tau(z) - p_n^\tau(z)| \leqslant \frac{1}{2}\Big(\frac{(1+\varepsilon)\sigma}{\rho-\varepsilon}\Big)^n \frac{1}{|q_{m_n}^*(z)|}$$

for all $z \in \overline{E}_\sigma$ with $q_{m_n}^*(z) \neq 0$ and all $n \geqslant n_2$. Let us consider the set

$$S_n(\sigma,\varepsilon) := \Big\{z \in \overline{E}_\sigma : |r_{n,m_n}(z)h^\tau(z) - p_n^\tau(z)| \geqslant \frac{1}{2}\Big(\frac{(1+2\varepsilon)\sigma}{\rho-\varepsilon}\Big)^n\Big\};$$

then

$$S_n(\sigma,\varepsilon) \subset e_n = e_n(\sigma,\varepsilon) := \Big\{z \in \overline{E}_\sigma : |q_{m_n}^*(z)| \leqslant \Big(\frac{1+\varepsilon}{1+2\varepsilon}\Big)^n\Big\}.$$

Since $q_{m_n}^*$ is monic and degree $(q_{m_n}^*) \leqslant m_n$, we obtain

$$\operatorname{cap} e_n \leqslant \Big(\frac{1+\varepsilon}{1+2\varepsilon}\Big)^{\frac{n}{\operatorname{degree}(q_{m_n}^*)}} \leqslant \Big(\frac{1+\varepsilon}{1+2\varepsilon}\Big)^{\frac{n}{m_n}}. \tag{3.6}$$

Therefore, we have shown that for $z \in \overline{E}_\sigma \setminus e_n$ and $n \geqslant n_2 = n_2(\sigma,\varepsilon)$

$$|r_{n,m_n}(z)h^\tau(z) - p_n^\tau(z)| \leqslant \frac{1}{2}\Big(\frac{(1+2\varepsilon)\sigma}{\rho-\varepsilon}\Big)^n. \tag{3.7}$$

By (3.3) and (3.7), we have for $z \in \overline{E}_\sigma \setminus e_n$ and $n \geqslant n_2$

$$|f(z)h^\tau(z) - r_{n,m_n}(z)h^\tau(z)| \leqslant \Big(\frac{(1+2\varepsilon)\sigma}{\rho-\varepsilon}\Big)^n$$

or

$$|f(z) - r_{n,m_n}(z)| \leqslant \Big(\frac{(1+2\varepsilon)\sigma}{\rho-\varepsilon}\Big)^n \frac{1}{|h^\tau(z)|},$$

when $h^\tau(z) \neq 0$. Let us consider

$$\tilde{S}_n(\sigma,\varepsilon) := \Big\{z \in \overline{E}_\sigma : |f(z) - r_{n,m_n}(z)| \geqslant \Big(\frac{(1+3\varepsilon)\sigma}{\rho-\varepsilon}\Big)^n\Big\};$$

then

$$\tilde{S}_n(\sigma,\varepsilon) \subset \tilde{e}_n = \tilde{e}_n(\sigma,\varepsilon) := \Big\{z \in \overline{E}_\sigma : |h^\tau(z)| \leqslant \Big(\frac{1+2\varepsilon}{1+3\varepsilon}\Big)^n\Big\}$$

and by (3.4)

$$\operatorname{cap} \tilde{e}_n \leqslant \Big(\frac{1+2\varepsilon}{1+3\varepsilon}\Big)^{n/m_n}. \tag{3.8}$$

Summarizing, we have obtained for $z \in \overline{E}_\sigma \setminus (e_n \cup \tilde{e}_n)$ and $n \geqslant n_2$

$$|f(z) - r_{n,m_n}(z)| \leqslant \Big(\frac{(1+3\varepsilon)\sigma}{\rho-\varepsilon}\Big)^n. \tag{3.9}$$

Because of the subadditivity of the capacity (Nevanlinna [**3**], Pommerenke [**4**])

$$1\big/\log\frac{d}{\operatorname{cap}(e_n \cup \tilde{e}_n)} \leqslant 1\big/\log\frac{d}{\operatorname{cap} e_n} + 1\big/\log\frac{d}{\operatorname{cap} \tilde{e}_n},$$

where d is greater than the diameter of $e_n \cup \tilde{e}_n$ and $d > 1$. The parameter d of Theorem 2.2 fulfills these conditions. Using (3.7) and (3.9), we get

$$\begin{aligned} 1/\log \frac{d}{\operatorname{cap}(e_n \cup \tilde{e}_n)} &\leqslant 1/\log \left[d\Big(\frac{1+2\varepsilon}{1+\varepsilon}\Big)^{n/m_n} \right] + 1/\log \left[d\Big(\frac{1+3\varepsilon}{1+2\varepsilon}\Big)^{n/m_n} \right] \\ &\leqslant 2/\log \left[d\Big(\frac{1+3\varepsilon}{1+2\varepsilon}\Big)^{n/m_n} \right] \end{aligned}$$

or

$$\log \frac{d}{\operatorname{cap}(e_n \cup \tilde{e}_n)} \geqslant \frac{1}{2} \log \left[d\Big(\frac{1+3\varepsilon}{1+2\varepsilon}\Big)^{n/m_n} \right]$$

and finally

$$\operatorname{cap}(e_n \cup \tilde{e}_n) \leqslant d^{1/2} \Big(\frac{1+2\varepsilon}{1+3\varepsilon}\Big)^{n/2m_n}. \tag{3.10}$$

Since $\varepsilon = (\theta - 1)/4$, we obtain

$$\frac{1+2\varepsilon}{1+3\varepsilon} = \frac{2+2\theta}{1+3\theta} = 1 - \frac{\theta - 1}{1+3\theta} < 1$$

and some calculations show that

$$\frac{1+3\varepsilon}{\rho - \varepsilon} < \frac{\theta}{\rho}.$$

Inserting these inequalities into (3.9) and (3.10), and define the compact sets $\Omega_n(\sigma, \theta) := e_n(\sigma, \theta) \cup \tilde{e}_n(\sigma, \theta)$. Then we have proved the inequalities (2.3) and (2.4) of Theorem 2.2. □

We remark that Theorem 2.1 follows directly from Theorem 2.2 if we choose θ so small that $\theta < \rho(f)/\sigma$ and keeping in mind that

$$\lim_{n\to\infty} \operatorname{cap}(e_n(\sigma, \varepsilon) \cup \tilde{e}_n(\sigma, \varepsilon)) = 0$$

with $\varepsilon = (\theta - 1)/4$. Moreover, for $\tau < \rho(f)$ the same method of proof leads to the result of Theorem 2.1 under the condition

$$\limsup_{n\to\infty} \|f - r_{n,m_n}\|_{\partial E}^{1/n} \leqslant \frac{1}{\tau} < 1. \tag{3.11}$$

Then the technique of the proof leads immediately to the following Corollary of Theorem 2.2.

Corollary 3.1. *Let $f \in \mathcal{M}(E)$ with $\rho(f) < \infty$, $\{m_n\}_{n\in\mathbb{N}}$ a sequence with* (2.1)*, $\{r_n, m_n\}_{n\in\mathbb{N}}$, $r_{n,m_n} \in \mathcal{R}_{n,m_n}$, a sequence such that* (3.11) *holds. Then there exists for any σ, $1 < \sigma < \min(\tau, \rho(f))$ and arbitrary θ, $1 < \theta < \min(\tau, \rho(f))/\sigma$, a natural number $n_0 = n_0(\sigma, \theta)$ and set $\Omega_n(\sigma, \theta) \subset \overline{E}_\sigma$ such that* (2.3) *and* (2.4) *hold for $n \geqslant n_0(\sigma, \theta)$.*

Proof of Theorem 2.3. We consider a monotonically increasing sequence $\{\sigma_i\}$ such that $\lim_{i\to\infty} \sigma_i = \rho(f)$ and a monotonically decreasing sequence $\{\theta_i\}$,

$1 < \theta_i < \rho(f)/\sigma_i$, such that $\lim_{i\to\infty} \theta_i = 1$. Let $\Omega_n(\sigma,\theta)$ and $n_0(\sigma,\theta)$ be defined as in Theorem 2.2, i.e., for $n \geqslant n_0(\sigma,\theta)$

$$|f(z) - r_{n,m_n}(z)| \leqslant \Big(\frac{\theta\sigma}{\rho(f)}\Big)^n \text{ for } z \in \overline{E}_\sigma \setminus \Omega_n(\sigma,\theta)$$

and $\operatorname{cap}\Omega_n(\sigma,\theta) \leqslant d^{1/2}\gamma^{n/2m_n}$, where $\gamma = 1 - \frac{\theta-1}{1+3\theta}$. Replacing (σ,θ) by (σ_1,θ_1), we can find, by using $m_n = o(n)$ as $n \to \infty$, a subsequence $\Lambda_1 = \{n_j^{(1)}\}_{j=1}^{\infty}$ of $\mathbb{N}$ such that $n_j^{(1)} \geqslant n_0(\sigma_1,\theta_1)$ and

$$\frac{m_{n_j^{(1)}}}{n_j^{(1)}} \Big/ \log\frac{1}{\gamma_1} \leqslant \frac{2}{(j+1)^2}$$

for $j = 1, 2, \ldots$, where

$$\gamma_1 = 1 - \frac{\theta_1 - 1}{1 + 3\theta_1}.$$

Recursively, we can define subsequences $\Lambda_k = \{n_j^{(k)}\}_{j=1}^{\infty} \subset \Lambda_{k-1}$ $(k = 2, 3, \ldots)$ such that $n_j^{(k)} \geqslant n_0(\sigma_k,\theta_k)$ and

$$\frac{m_{n_j^{(k)}}}{n_j^{(k)}} \Big/ \log\frac{1}{\gamma_k} \leqslant \frac{2}{(k+j)^2}$$

for $j = 1, 2, \ldots$. We define $\Lambda := \{n_1^{(k)}\}_{k=1}^{\infty}$ and we have to show that Λ fulfills the assertions of our theorem.

Let K be compact in $E_{\rho(f)}$ and $\varepsilon > 0$. For ε we can find an index $i^* > 1$ such that

$$2\sum_{j=1}^{\infty} \frac{1}{(i^*+j)^2} < 1 \Big/ \log\frac{d}{\varepsilon}. \tag{3.12}$$

Then we define

$$k^* := \max(i^*, \min\{i : K \subset E_{\sigma_i}\}) \text{ and } K_\varepsilon := \overline{\bigcup_{j=1}^{\infty} \Omega_{n_j^{(k^*)}}(\sigma_{k^*},\theta_{k^*})}.$$

We know that

$$|f(z) - r_{n,m_n}(z)| \leqslant \Big(\frac{\theta_{k^*}\sigma_{k^*}}{\rho(f)}\Big)^n \tag{3.13}$$

for $z \in \overline{E}_{\sigma_{k^*}} \setminus \Omega_{n_j^{(k^*)}}(\sigma_{k^*},\theta_{k^*})$ and

$$\operatorname{cap}\Omega_{n_j^{(k^*)}}(\sigma_{k^*},\theta_{k^*}) \leqslant d^{1/2}(\gamma_k^*)^{n_j^{(k^*)}/2m_{n_j^{(k^*)}}} \tag{3.14}$$

for $j = 1, 2, \ldots$. The subadditivity of the capacity (Nevanlinna) yields with (3.12), (3.14)

$$1\Big/\log\frac{d}{\operatorname{cap}K_\varepsilon}\leqslant\sum_{j=1}^\infty 1\Big/\log\frac{d}{\operatorname{cap}\Omega_{n_j^{(k^*)}}(\sigma_{k^*},\theta_{k^*})}$$
$$\leqslant\sum_{j=1}^\infty\left(\log d-\frac12\log d-\frac{n_j^{(k^*)}}{2m_{n_j^{(k^*)}}}\log\gamma_k^*\right)^{-1}$$
$$\leqslant 2\sum_{j=1}^\infty\frac{m_{n_j^{(k^*)}}}{n_j^{(k^*)}}\Big/\log\frac{1}{\gamma_k^*}\leqslant 2\sum_{j=1}^\infty\frac{1}{(k^*+j)^2}<1\Big/\log\frac{d}{\varepsilon},$$

and consequently $\operatorname{cap}K_\varepsilon<\varepsilon$. Since $\{n\in\Lambda: n\geqslant n_1^{(k^*)}\}\subset\Lambda_{k^*}$ we obtain by (3.13)

$$|f(z)-r_{n,m_n}(z)|\leqslant\left(\frac{\theta_{k^*}\sigma_{k^*}}{\rho(f)}\right)^n,\ z\in K\setminus K_\varepsilon.$$

for all $n\in\Lambda$, $n\geqslant n_1^{(k^*)}$. Hence the uniform convergence in capacity of $\{r_{n,m_n}\}_{n\in\Lambda}$ to f inside $E_{\rho(f)}$ is proven. □

4. Sharpness of the theorems

The result in Theorem 2.1 is sharp in the sense that in (2.1) the condition $m_n=o(n)$ as $n\to\infty$ is essential. To verify this we consider the following example: Let E be compact with regular connected complement, $f\in\mathcal{M}(E)$ with $\rho(f)<\infty$. If $\{m_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathbb{N}$ with (2.1), then Walsh's theorem implies that there exist best uniform rational approximants $r^*_{n,m_n}\in\mathcal{R}_{n,m_n}$ to f on E such that

$$\limsup_{n\to\infty}\|f-r^*_{n,m_n}\|_E^{1/n}\leqslant\frac{1}{\rho(f)}.$$

According to Theorem 2.1 the sequence $\{r^*_{n,m_n}\}_{n\in\mathbb{N}}$ converges in capacity to f inside $E_{\rho(f)}$.

Furthermore, let $\{\tilde m_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ with

$$\tilde m_n\geqslant m_n \text{ and } \lim_{n\to\infty}\frac{\tilde m_n}{n}>0. \tag{4.1}$$

We choose a point $\xi\in E_{\rho(f)}\setminus E$, hence $\alpha:=\operatorname{dist}(\xi,E)>0$. Then we define the sequence

$$r_{n,\tilde m_n}:=r^*_{n,m_n}+R_n(z)\in\mathcal{R}_{n,\tilde m_n},\ \text{where } R_n(z)=\frac{\alpha^{\tilde m_n-m_n}}{\rho(f)^n}\,\frac{1}{(z-\xi)^{\tilde m_n-m_n}}.$$

Then

$$\|f-r_{n,\tilde m_n}\|_E\leqslant\|f-r^*_{n,m_n}\|_E+\frac{1}{\rho(f)^n},$$
$$\limsup_{n\to\infty}\|f-r_{n,\tilde m_n}\|_E^{1/n}\leqslant\frac{1}{\rho(f)}.$$

Consider the disks

$$D_n:=\left\{z\in\mathbb{C}:|z-\xi|\leqslant\frac{\alpha}{\rho(f)^{n/(\tilde m_n-m_n)}}\right\}.$$

Using (2.1) and (4.1), we conclude that there exists a number $\kappa > 0$ and $n_0 \in \mathbb{N}$ such that

$$\frac{n}{\tilde{m}_n - m_n} \leqslant \kappa \text{ for all } n \geqslant n_0.$$

Hence, with $r := \alpha/\rho(f)^\kappa$ we get $D_n \supset K := K_r(\xi) = \{z \in \mathbb{C} : |z - \xi| \leqslant r\}$ for all $n \geqslant n_0$. Moreover, we can choose κ big enough such that $K \subset E_{\rho(f)}$.

Now, fix $\varepsilon > 0$, $0 < \varepsilon < 1$, and consider the sets

$$S_n(\varepsilon) := \{z \in K : |(f - r^*_{n,m_n})(z)| \geqslant \varepsilon\}.$$

By Theorem 2.1, we know that $\{r^*_{n,m_n}\}_{n\in\mathbb{N}}$ converges in capacity to f inside $E_{\rho(f)}$. Therefore

$$\lim_{n\to\infty} \operatorname{cap} S_n(\varepsilon) = 0. \tag{4.2}$$

By definition of $r_{n,\tilde{m}_n}$, we have

$$|(f - r_{n,\tilde{m}_n})(z)| \geqslant |R_n(z)| - |(f - r^*_{n,m_n})(z)|.$$

Consequently,

$$|(f - r_{n,\tilde{m}_n})(z)| \geqslant 1 - \varepsilon \text{ for all } z \in K \setminus S_n(\varepsilon).$$

Since $\operatorname{cap} K = r > 0$, we obtain by Nevanlinna's inequality, together with (4.2), that

$$\liminf_{n\to\infty} \operatorname{cap}\{z \in K : |(f - r_{n,\tilde{m}_n})(z)| \geqslant 1 - \varepsilon\} > 0.$$

Hence, $\{r_{n,\tilde{m}_n}\}_{n\in\mathbb{N}}$ does not converge in capacity to f inside $E_{\rho(f)}$, and the condition "$m_n = o(n)$ as $n \to \infty$" is essential in Theorem 2.1.

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