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COMPLETELY PSEUDO-VALUATION RINGS AND THEIR EXTENSIONS

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ABSTRACT. Recall that a commutative ring R is said to be a pseudo-valuation ring if every prime ideal of R is strongly prime. We define a completely pseudo-valuation ring. Let R be a ring (not necessarily commutative). We say that R is a completely pseudo-valuation ring if every prime ideal of R is completely prime. With this we prove that if R is a commutative Noetherian ring, which is also an algebra over $\mathbb{Q}$ (the field of rational numbers) and δ a derivation of R, then R is a completely pseudo-valuation ring implies that $R[x;\delta]$ is a completely pseudo-valuation ring. We prove a similar result when prime is replaced by minimal prime.

1. Introduction

A ring R means an associative ring with identity $1 \neq 0$, and any R-module unitary. $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Z}$ denotes the ring of integers and $\mathbb{N}$ denotes the set of positive integers unless otherwise stated. Let R be a ring. The set of prime ideals of R is denoted by $\mathrm{Spec}(R)$, the set of associated prime ideals of R (viewed as a right R-module) is denoted by $\mathrm{Ass}(R_R)$, the set of minimal prime ideals of R is denoted by $\mathrm{MinSpec}(R)$ and the set of completely prime ideals of R is denoted by $\mathrm{C.Spec}(R)$. For any two ideals I, J of R, $I \subset J$ means that I is strictly contained in J.

Let R be a ring, σ an automorphisms of R and δ a σ-derivation of R; i.e., $\delta : R \to R$ is an additive mapping satisfying $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$.

For example let σ be an automorphism of a ring R and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ be a map defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \quad \text{for all } r \in R.$$

Then ϕ is a ring homomorphism if and only if δ is a σ-derivation of R.

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We recall that the Ore extension $R[x;\sigma,\delta] = \{f = \sum x^i a_i,\ a_i \in R\}$, $0 \leqslant i \leqslant n$ with usual addition and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We denote $R[x;\sigma,\delta]$ by $O(R)$. If I is an ideal of R such that I is σ-stable (i.e. $\sigma(I) = I$) and is also δ-invariant (i.e., $\delta(I) \subseteq I$), then clearly $I[x;\sigma,\delta]$ is an ideal of $O(R)$, and we denote it as usual by $O(I)$.

In the case σ is the identity map, we denote the ring of differential operators $R[x;\delta]$ by $D(R)$. If J is an ideal of R such that J is δ-invariant (i.e., $\delta(J) \subseteq J$), then clearly $J[x;\delta]$ is an ideal of $D(R)$, and we denote it as usual by $D(J)$.

In the case δ is the zero map, we denote $R[x;\sigma]$ by $S(R)$. If K is an ideal of R such that K is σ-stable (i.e., $\sigma(K) = K$), then clearly $K[x;\sigma]$ is an ideal of $S(R)$, and we denote it as usually by $S(K)$.

We recall that the skew Laurent polynomial ring

$$R[x, x^{-1};\sigma] = \{f = \textstyle\sum_{i=-n}^{m} x^i a_i,\ \ a_i \in R\};\ \ m, n \in \mathbb{N}$$

where multiplication is subject to the relation $ax = x\sigma(a)$ for all $a \in R$. We denote $R[x, x^{-1};\sigma]$ by $L(R)$. If U is an ideal of R such that U is σ-stable (i.e., $\sigma(U) = U$), then clearly $U[x, x^{-1};\sigma]$ is an ideal of $L(R)$, and we denote it as usually by $L(U)$.

We now have the following famous result concerning Ore extensions.

THEOREM 1.1 (Hilbert Basis Theorem). *Let R be a right/left Noetherian ring. Let σ be an automorphisms of R and δ a σ-derivation of R. Then the Ore extension $O(R) = R[x;\sigma,\delta]$ is right/left Noetherian. Also $R[x, x^{-1};\sigma]$ is right/left Noetherian.*

PROOF. See Theorems (1.12) and (1.17) of Goodearl and Warfield [**11**]. □

2. Preliminaries

In this section we mention some known results that lead to the investigation of completely prime ideals of Ore extensions.

LEMMA 2.1 (Lemma (10.6.4) of McConnell and Robson [**13**]). *Let R be a ring and σ its automorphism. Then*

(1) *If P is a prime ideal of $S(R)$ such that $x \notin P$, then $P \cap R$ is a prime ideal of R and $\sigma(P \cap R) = P \cap R$.*
(2) *If U is a prime ideal of R such that $\sigma(U) = U$, then $S(U)$ is a prime ideal of $S(R)$ and $S(U) \cap R = U$.*

THEOREM 2.1 (Theorem (2.22) of Goodearl and Warfield [**11**]). *Let R be a commutative Noetherian $\mathbb{Q}$-algebra. Let δ be a derivation of R. Then*

(1) *If P is a prime ideal of $D(R)$, then $P \cap R$ is a prime ideal of R and $\delta(P \cap R) \subseteq P \cap R$.*
(2) *If U is a prime ideal of R such that $\delta(U) \subseteq U$, then $D(U)$ is a prime ideal of $D(R)$ and $D(U) \cap R = U$.*

Goodearl and Warfield proved in (2ZA) of [**11**] that if R is a commutative Noetherian ring, and if σ is an automorphism of R, then an ideal I of R is of the form $P \cap R$ for some prime ideal P of $R[x, x^{-1};\sigma]$ if and only if there is a prime ideal S of R and a positive integer m with $\sigma^m(S) = S$, such that $I = \bigcap \sigma^i(S)$, $i = 1, 2, \ldots, m$. They proved in Theorem (2.22) of [**11**] that if δ is a derivation of a commutative Noetherian ring R which is also an algebra over $\mathbb{Q}$ and P is a prime ideal of $R[x;\delta]$, then $P \cap R$ is a prime ideal of R and if S is a prime ideal of R with $\delta(S) \subseteq S$, then $S[x;\delta]$ is a prime ideal of $R[x;\delta]$. Gabriel proved in [**10**] that if R is a right Noetherian ring which is also an algebra over $\mathbb{Q}$ and P is a prime ideal of $R[x;\delta]$, then $P \cap R$ is a prime ideal of R.

Let R be a right Noetherian ring. Then we know that $\mathrm{MinSpec}(R)$ is finite by Theorem (2.4) of Goodearl and Warfield [**11**] and for any automorphism σ of R, $P \in \mathrm{MinSpec}(R)$ implies that $\sigma^j(P) \in \mathrm{MinSpec}(R)$ for all positive integers j. Therefore, there exists some $m \in \mathbb{N}$ such that $\sigma^m(P) = P$ for all $P \in \mathrm{MinSpec}(R)$. We denote $\bigcap_{i=1}^m \sigma^i(P)$ by P^0.

The author of this paper has proved in Theorem (2.4) of [**6**] that if R is a Noetherian ring and σ an automorphism of R, then $P \in \mathrm{MinSpec}(S(R))$ if and only if there exists $U \in \mathrm{MinSpec}(R)$ Such that $S(P \cap R) = P$ and $P \cap R = U^0$.

It has also been proved in Theorem (3.7) of Bhat [**6**], that if R is a Noetherian $\mathbb{Q}$-algebra and δ a derivation of R, then $P \in \mathrm{MinSpec}(D(R))$ if and only if $P = D(P \cap R)$ and $P \cap R \in \mathrm{MinSpec}(R)$.

For more details and some basic results for the rings $R[x;\sigma,\delta]$, $R[x;\sigma]$, and $R[x;\delta]$, the reader is referred to chapters (1) and (2) of Goodearl and Warfield [**11**].

2.1. Completely prime ideals. Recall that an ideal P of a ring R is completely prime if R/P is a domain, i.e., $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$ (McCoy [**14**]).

In the commutative case completely prime and prime have the same meaning. We also note that every completely prime ideal of a ring R is a prime ideal, but the converse need not be true. The following example shows that a prime ideal need not be a completely prime ideal.

Example 2.1. Let $R = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right) = M_2(\mathbb{Z})$. If p is a prime number, then the ideal $P = M_2(p\mathbb{Z})$ is a prime ideal of R, but is not completely prime, since for $a = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we have $ab \in P$, even though $a \notin P$ and $b \notin P$.

Regarding the relation between the completely prime ideals of a ring R and those of $O(R)$ the following result has been proved by Bhat [**8**].

Theorem 2.2 (Theorem 2.4 of Bhat [**8**]). *Let R be a ring, σ its automorphism and δ a σ-derivation of R. Then*

(1) *For any completely prime ideal P of R with $\delta(P) \subseteq P$ and $\sigma(P) = P$, $O(P)$ is a completely prime ideal of $O(R)$.*
(2) *For any completely prime ideal U of $O(R)$, $U \cap R$ is a completely prime ideal of R.*

2.2. An analogue of pseudo-valuation rings in noncommutative set up. We recall that as in Hedstrom [**12**], an integral domain R with quotient field F, is called a pseudo-valuation domain (PVD) if each prime ideal P of R is strongly prime ($ab \in P$, $a \in F$, $b \in F$ implies that either $a \in P$ or $b \in P$).

For example let $F = \mathbb{Q}(\sqrt{2})$. Set $V{=}F + xF[x] = F[x]$. Then V is a pseudo-valuation domain. We also note that $S = \mathbb{Q} + \mathbb{Q}x + x^2V$ is not a pseudo-valuation domain (Badawi [**5**]). In Badawi, Anderson and Dobbs [**3**], the study of pseudo-valuation domains was generalized to arbitrary rings in the following way.

A prime ideal P of R is said to be strongly prime if aP and bR are comparable (under inclusion) for all a, $b \in R$, or equivalently if R/P has no nonzero nil ideals. A commutative ring R is said to be a pseudo-valuation ring (PVR) if each prime ideal P of R is strongly prime. We note that a commutative PVR is quasilocal by Lemma 1(b) of Badawi, Anderson and Dobbs [**3**].

An integral domain is a PVR if and only if it is a PVD by Proposition (3.1) of Anderson [**1**], Proposition (4.2) of Anderson [**2**] and Proposition (3) of Badawi [**4**].

Motivated by above, we define a completely pseudo-valuation ring R in the following way.

Definition 2.1. Let R be a ring (not necessarily commutative). Then R is said to be a completely pseudo-valuation ring if every prime ideal of R is completely prime.

The following is an example of a completely pseudo-valuation ring.

Example 2.2. Let $R = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$. Then R is a completely pseudo-valuation ring.

Now much is not known about the relation between the prime ideals of R and those of $O(R)$ (even in the case when R is Noetherian), but in certain cases the minimal prime ideals of $O(R)$ have been characterized, therefore, this motivates us to define a near completely pseudo-valuation ring R in the following way.

Definition 2.2. Let R be a ring (not necessarily commutative). Then R is said to be a near completely pseudo-valuation ring if every minimal prime ideal of R is completely prime.

For example a reduced ring is a near completely pseudo-valuation ring. The ring R as in example (2.2) being a completely pseudo-valuation ring is also a near completely pseudo-valuation ring.

We would like to mention that the word *valuation* is not completely appropriate in the above definitions, but the notion introduced here stems from the commutative case, so we have decided to keep this word. It would be interesting to see whether the notion of valuation retains some of its attributes in the non-commutative case as we have here.

3. Minimal prime ideals of polynomial rings

The set of power series $\{f = \sum_{i=0}^{\infty} x^i a_i,\ a_i \in R\}$ endowed with usual addition and a multiplication subject to the relation $ax = x\sigma(a)$ for all $a \in R$ is the skew power series ring, denoted by $R[x; \sigma]$.

Proposition 3.1. *Let R be a Noetherian $\mathbb{Q}$-algebra. Let σ and δ be as usually such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then $e^{t\delta}$ is an automorphism of $R[t; \sigma]$, where $e^{t\delta} = 1 + t\delta + \frac{t^2\delta^2}{2!} + \cdots$.*

Proof. The proof is on the same lines as in a sketch provided by Blair and Small in [**9**]. □

Lemma 3.1. *Let R be a Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then $P \in \mathrm{MinSpec}(R)$ with $\sigma(P) = P$ implies that $\delta(P) \subseteq P$.*

Proof. On the basis of Lemma (2.6) of [**7**]. □

We now have the following.

Theorem 3.1. *Let R be a Noetherian $\mathbb{Q}$-algebra. Consider $O(R)$ as usual such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then $P_1 \in \mathrm{MinSpec}(R)$ with $\sigma(P_1) = P_1$ implies that $O(P_1) \in \mathrm{MinSpec}(O(R)$. Conversely $P \in \mathrm{MinSpec}(O(R))$ with $P \cap R \in \mathrm{MinSpec}(R)$, $\sigma(P \cap R) = P \cap R$ and $\delta(P \cap R) \subseteq P \cap R$ implies that $P \cap R \in \mathrm{MinSpec}(R)$.*

Proof. Let $P_1 \in \mathrm{MinSpec}(R)$. Then $\delta(P_1) \subseteq P_1$ by Lemma (3.1). Now it can be seen that that $D(P_1) \in \mathrm{Spec}(O(R))$. Suppose $O(P_1) \notin \mathrm{MinSpec}(O(R))$ and $P_2 \subset O(P_1)$ be a minimal prime ideal of $O(R)$. Then $P_2{=}O(P_2 \cap R) \subset O(P_1) \subseteq \mathrm{MinSpec}(O(R))$. Therefore $P_2 \cap R \subset P_1$ which is a contradiction, as $P_2 \cap R \in \mathrm{Spec}(R)$. Hence $O(P_1) \in \mathrm{MinSpec}(O(R))$.

Conversely suppose that $P \in \mathrm{MinSpec}(O(R))$. Now $P \cap R \in \mathrm{Spec}(R)$, $\sigma(P \cap R) = P \cap R$ and $\delta(P \cap R) \subseteq P \cap R$ implies that $O(P \cap R) \in \mathrm{Spec}(O(R))$. Now $O(P \cap R) \subseteq P$ and P being minimal implies that $O(P \cap R) = P$. We now show that $P \cap R \in \mathrm{MinSpec}(R)$. Suppose $P_1 \subset P \cap R$ is a minimal prime ideal of R. Then $O(P_1) \subset O(P \cap R)$ and as in the first paragraph $O(P_1) \in \mathrm{Spec}(O(R))$ which is a contradiction. Hence $P \cap R \in \mathrm{MinSpec}(R)$. □

4. Completely pseudo-valuation rings

In this section we state and prove the results concerning completely pseudo-valuation rings and their extensions.

Theorem 4.1. *Let R be a commutative Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let δ be a derivation of R. Then R is a completely pseudo-valuation ring implies that $D(R)$ is a completely pseudo-valuation ring.*

Proof. Let R be a completely pseudo-valuation ring which is also an algebra over $\mathbb{Q}$. Now $D(R)$ is Noetherian by Theorem (1.1). Let $J \in \mathrm{Spec}(D(R))$. Then by Theorem (2.1) $J{\cap}R \in \mathrm{Spec}(R)$. Now R is a completely pseudo-valuation $\mathbb{Q}$-algebra, therefore $J{\cap}R \in \mathrm{C.Spec}(R)$. Also $\delta(J{\cap}R) \subseteq J{\cap}R$ and $D(J{\cap}R) = J$ by Theorem (2.1). Now Theorem (2.2) implies that $D(J \cap R) \in \mathrm{C.Spec}(D(R))$. Therefore $J \in \mathrm{C.Spec}(D(R))$. Hence $D(R)$ is a completely pseudo-valuation ring. □

Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Consider $O(R)$ as usual. It is not known whether a contraction of a prime ideal of $O(R)$ is a prime ideal of R. Therefore, for the time being we shall not attack the following problem.

QUESTION 4.1. *Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let R be a completely pseudo-valuation ring. Is $O(R)$ a completely pseudo-valuation ring?*

Recall that the nature of minimal prime ideals of a Noetherian $\mathbb{Q}$-algebra R is known (with respect to automorphisms and derivations of R). Moreover the relation between the minimal prime ideals of R and those of polynomial rings is also known now. For example [**6, 9, 10, 11**]. We, therefore, have the following result concerning near completely pseudo-valuation rings.

THEOREM 4.2. *Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that $\sigma(P) = P$ for all $P \in \mathrm{MinSpec}(R)$. Consider $O(R)$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$, $\sigma(P_1 \cap R) = P_1 \cap R$ and $\delta(P_1 \cap R) \subseteq P_1 \cap R$ for all $P_1 \in \mathrm{MinSpec}(O(R))$. Then R is a near completely pseudo-valuation ring implies that $O(R)$ is a near completely pseudo-valuation ring.*

PROOF. The proof is on the same lines as in Theorem (4.1). □

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