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ON WEAK α-SKEW MCCOY RINGS

Mohammad Javad Nikmehr,
Ali Nejati and Mansoureh Deldar

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Abstract. Let α be an endomorphism of a ring R. We introduce the notion of weak α-skew McCoy rings which are a generalization of the α-skew McCoy rings and the weak McCo rings. Some properties of this generalization are established, and connections of properties of a weak α-skew McCoy ring R with $n \times n$ upper triangular $T_n(R)$ are investigated. We study relationship between the weak skew McCoy property of a ring R and its polynomial ring, $R[x]$. Among applications, we show a number of interesting properties of a weak α-skew McCoy ring R such as weak skew McCoy property in a ring R.

1. Introduction

Throughout this note, R denotes an associative ring with unity and α is a ring endomorphism. We denote $R[x;\alpha]$ the *Ore extension* whose elements are the polynomials $\sum_{i=0}^{n} a_i x^i$, $a_i \in R$, where the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x$ for any $a \in R$. $nil(R)$ denotes the set of all the nilpotent elements of R. Rege and Chhawchharia [**7**] introduced the notion of an Armendariz ring. They defined a ring R to be an *Armendariz ring* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. The name "Armendariz ring" was chosen because Armendariz had showed that a reduced ring (i.e., a ring without nonzero nilpotent elements) satisfies this condition. Hong, Kim, and Kwak [**3**] called R an *α-skew Armendariz* ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x;\alpha]$ satisfy $f(x)g(x) = 0$, then $a_i\alpha^i(b_j) = 0$ for each i, j, which is a generalization of the Armendariz rings. Liu and Zhao [**4**] called a ring R *weak Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then a_ib_j is nilpotent element of R for each i and j. Motivated by the above results, Zhang and Chen [**8**] called a ring R *weak α-skew Armendariz* if whenever polynomials

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$f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x;\alpha]$ satisfy $f(x)g(x) = 0$, then $a_i\alpha^i(b_j) \in \mathrm{nil}(R)$ for each i and j. It is obvious that a weak α-skew Armendariz ring is a generalization of the α-skew Armendariz rings and the weak Armendariz rings. Recall that a ring R is called *reversible* if $ab = 0$ implies $ba = 0$, for all $a, b \in R$. R is called *semicommutative* if for all $a, b \in R$, $ab = 0$ implies $aRb = \{0\}$. Reduced rings are clearly reversible and reversible rings are semicommutative, but the converse is not true in general [**6**]. According to Nielson [**6**], a ring R is called *right McCoy* (resp., *left McCoy*) if, for any polynomials $f(x), g(x) \in R[x] \setminus \{0\}$, $f(x)g(x) = 0$ implies $f(x)r = 0$ (resp., $sg(x) = 0$) for some $0 \neq r \in R$ (resp., for some $0 \neq s \in R$). A ring is called *McCoy* if it is both left and right McCoy. By McCoy [**5**], commutative rings are McCoy rings. Reduced rings are Armendariz and Armendariz rings are McCoy. A ring R is *right weak McCoy* whenever, $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x] \setminus \{0\}$ satisfy $f(x)g(x) = 0$, then $a_is \in \mathrm{nil}(R)$ for some $0 \neq s \in R$, and every i. *Left weak McCoy rings* are defined similarly. If a ring is both left and right weak McCoy we say that the ring is *weak McCoy ring.* Also in [**2**] investigated this generalization of McCoy rings and their properties.

A ring R is called *α-skew McCoy ring* with respect to α if for any nonzero polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x;\alpha]$ satisfy $f(x)g(x) = 0$, implies $f(x)s = 0$ for some nonzero $s \in R$. It is clear that a ring R is right McCoy if R is id_R-skew McCoy, where id_R is the identity endomorphism of R. In [**1**], Basser, Kwak, Lee showed that every domain with an endomorphism α is α-skew McCoy, and R is α-skew McCoy if and only if the factor ring $R[x]/(x^n)$ is ŕ$\bar{\alpha}$-skew McCoy, where $\bar{\alpha} : R[x] \to R[x]$ defined by $\bar{\alpha}(f(x)) = \sum_{i=0}^{m} \alpha(a_i)x^i$ for any $f(x) = a_0 + a_1x + \cdots + a_nx^n$ is an endomorphism of $R[x]$. Also they proved that for a ring isomorphism $\sigma : R \to S$, R is a α-skew McCoy ring if and only if S is an $\sigma\alpha\sigma^{-1}$-skew McCoy ring.

Motivated by the above results, for an endomorphism α of a ring R, we investigate a generalization of the α-skew McCoy rings and the weak McCoy rings which we call a *weak α-skew McCoy ring* and study several results.

2. Weak α-Skew McCoy rings

We begin this section by the following definition and also we study properties of weak α-skew McCoy rings.

DEFINITION 2.1. Let α be an endomorphism of a ring R. The ring R is called *weak α-skew McCoy* with respect to α if for any nonzero polynomials $p(x) = \sum_{i=0}^{n} a_ix^i$ and $q(x) = \sum_{j=0}^{m} b_jx^j$ in $R[x;\alpha]$ with $p(x)q(x) = 0$, there exists $s \in R - \{0\}$ such that $a_i\alpha^i(s) \in nil(R)$ for $0 \leqslant i \leqslant n$.

It can be easily checked that if R is a weak McCoy ring then it is a weak id_R-skew McCoy ring, where id_R is an identity endomorphism of R. Also every weak Armendariz ring is weak McCoy and therefore is weak id_R-skew McCoy. If $\mathrm{nil}(R) \trianglelefteq R$, then R is weak Armendariz and so R will be weak McCoy ring and so R is weak id_R-skew McCoy.

PROPOSITION 2.1. *Let α be an endomorphism of a ring R. Then every weak α-skew Armendariz ring is a weak α-skew McCoy ring.*

PROOF. Let $f(x)=\sum_{i=0}^{n}a_ix^i$, $g(x)=\sum_{j=0}^{m}b_jx^j\in R[x;\alpha]\setminus\{0\}$ and assume that $f(x)g(x)=0$. Since R is weak α-skew Armendariz, $a_i\alpha^i(b_j)\in \mathrm{nil}(R)$ for all i, j. Let $r=b_t$ for $0\leqslant t\leqslant m$, and hence $a_i\alpha^i(r)\in\mathrm{nil}(R)$ for all i. Therefore R is weak α-skew McCoy. $\square$

Let I be an ideal of R. If $\alpha(I)\subseteq I$, then defined $\bar{\alpha}:R/I\to R/I$ by $\bar{\alpha}(a+I)=\alpha(a)+I$ for $a\in R$, is an endomorphism of the factor ring R/I. Now we have the following proposition.

PROPOSITION 2.2. *Let α be an endomorphism of a ring R and I be an ideal of R with $\alpha(I)\subseteq I$. If $I\subseteq \mathrm{nil}(R)$ and R/I is weak $\bar{\alpha}$-skew McCoy, then R is weak α-skew McCoy.*

PROOF. Let $f(x)=a_0+a_1x+\cdots+a_mx^m$ and $g(x)=b_0+b_1x+\cdots+b_nx^n\in R[x;\alpha]\setminus\{0\}$ such that $f(x)g(x)=0$. Then $\big(\sum_{i=0}^{m}\bar{a}_ix^i\big)\big(\sum_{j=0}^{n}\bar{b}_jx^j\big)=0$ in R/I. Thus there exists n_i such that $(\bar{a}_i\bar{\alpha}^i(\bar{s}))^{n_i}=0$ for some $s\in R\setminus I$. Hence $a_i\alpha^i(s)\in\mathrm{nil}(R)$ and so R is weak α-skew McCoy. $\square$

Let R be a ring, α an automorphism of R and Δ a multiplicatively closed subset of R consisting of central regular elements. The ring $\Delta^{-1}R$ is called the *ring of fractions* of R with respect to Δ. We define $\Delta^{-1}\alpha:\Delta^{-1}R\to\Delta^{-1}R$ by $\Delta^{-1}\alpha(b^{-1}a)=(\alpha(b))^{-1}\alpha(a)$ for any $b^{-1}a\in\Delta^{-1}R$. Then $\Delta^{-1}\alpha$ is an automorphism of $\Delta^{-1}R$.

PROPOSITION 2.3. *Let R be weak α-skew McCoy. Then $\Delta^{-1}R$ is weak $\Delta^{-1}\alpha$-skew McCoy.*

PROOF. Let $f(x)=\sum_{i=0}^{m}c_ix^i$ and $g(x)=\sum_{j=0}^{n}d_jx^j$ be nonzero polynomials in $\Delta^{-1}R[x;\Delta^{-1}\alpha]$ such that c_i, d_j are in $\Delta^{-1}R$ for all i,j. Then we can assume that $c_i=a_iu^{-1}$ and $d_j=b_jv^{-1}$ for some a_i, $b_j\in R$ and $u,v\in\Delta$. Let $f_1(x)=\sum_{i=0}^{m}a_ix^i$, $g_1(x)=\sum_{j=0}^{n}b_jx^j$. Thus $f_1(x)g_1(x)=0$ in$R[x;\alpha]$. Thus $a_i\alpha^i(s)\in\mathrm{nil}(R)$ for some $0\neq s\in R$ for $0\leqslant i\leqslant m$. So $c_i(\Delta^{-1}\alpha)^i(s)\in\mathrm{nil}(\Delta^{-1}R)$ for $0\leqslant i\leqslant m$. Thus $\Delta^{-1}R$ is a weak $\Delta^{-1}\alpha$-skew McCoy ring. $\square$

Let $R[x;x^{-1}]$ be the ring of *Laurent polynomials*, i.e., the formal sums $\sum_{i=k}^{n}a_ix^i$, where k,n are (possibly negative) integers. For an automorphism α of R, $\bar{\alpha}:R[x;x^{-1}]\to R[x;x^{-1}]$ defined by $\bar{\alpha}\big(\sum_{i=k}^{n}a_ix^i\big)=\sum_{i=k}^{n}\alpha(a_i)x^i$ is an automorphism of $R[x;x^{-1}]$. The restriction of $\bar{\alpha}$ to $R[x]$, we also denote by $\bar{\alpha}$.

COROLLARY 2.1. *Let $R[x]$ be weak $\bar{\alpha}$-skew McCoy ring. Then $R[x;x^{-1}]$ is a weak $\bar{\alpha}$-skew McCoy ring.*

PROOF. It is clear that $\Delta=\{1,x,x^2,\ldots\}$ is multiplicatively closed subset of $R[x]$. Since $R[x;x^{-1}]=\Delta^{-1}R[x]$, it follows that $R[x;x^{-1}]$ is a weak $\bar{\alpha}$-skew McCoy ring. $\square$

Let α be an endomorphism of a ring R and $M_n(R)$ be the $n \times n$ matrix over R, and $\bar{\alpha} : M_n(R) \to M_n(R)$ defined by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$. Then $\bar{\alpha}$ is an endomorphism of $M_n(R)$. It is obvious that the restriction of $\bar{\alpha}$ to $T_n(R)$ is an endomorphism of $T_n(R)$, where $T_n(R)$ is the $n \times n$ upper triangular matrix ring over R. We also denote $\bar{\alpha}|_{T_n(R)}$ by $\bar{\alpha}$.

For a ring R, $T_n(R)$ $(n \geqslant 2)$ is a weak McCoy ring. Now we have the following proposition.

Proposition 2.4. *Let α be an endomorphism of a ring R. Then, for any n, $T_n(R)$ is a weak $\bar{\alpha}$-skew McCoy ring if R is a weak α-skew McCoy ring.*

Proof. Let $f(x) = A_0 + A_1x + \cdots + A_px^p$ and $g(x) = B_0 + B_1x + \cdots + B_qx^q$ be elements of $T_n(R)[x; \bar{\alpha}]$ satisfying $f(x)g(x) = 0$, where

$$A_i = \begin{pmatrix} a_{11}^{(i)} & a_{12}^{(i)} & \cdots & a_{1n}^{(i)} \\ 0 & a_{22}^{(i)} & \cdots & a_{2n}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^{(i)} \end{pmatrix}, \quad B_j = \begin{pmatrix} b_{11}^{(j)} & b_{12}^{(j)} & \cdots & b_{1n}^{(j)} \\ 0 & b_{22}^{(j)} & \cdots & b_{2n}^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn}^{(j)} \end{pmatrix}.$$

Then from $f(x)g(x) = 0$, it follows that $\left(\sum_{i=0}^{p} a_{ss}^{(i)} x^i\right)\left(\sum_{j=0}^{q} b_{ss}^{(j)} x^j\right) = 0$ in $R[x; \alpha]$ for each s with $1 \leqslant s \leqslant n$. Since R is a weak α-skew McCoy ring, there exists $s_k \neq 0$ such that $a_{ss}^{(i)}\alpha^i(s_k) \in \mathrm{nil}(R)$ for $1 \leqslant k \leqslant n$. Therefore $\left(a_{ss}^{(i)}\alpha^i(s_k)\right)^{m_k} = 0$ for some $m_k \in \mathbb{Z}$. Let $m = \max\{m_1, m_2, \ldots, m_n\}$. We define

$$S = \begin{pmatrix} s_1 & * & \cdots & * \\ 0 & s_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_n \end{pmatrix},$$

where $*$ stands for any element of R. Then

$$\left(A_i\bar{\alpha}^i(S)\right)^m = \begin{pmatrix} a_{11}^{(i)}\alpha^i(s_1) & * & \cdots & * \\ 0 & a_{22}^{(i)}\alpha^i(s_2) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^{(i)}\alpha^i(s_n) \end{pmatrix}^m = \begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It implies that $T_n(R)$ is a weak $\bar{\alpha}$-skew McCoy ring. $\square$

Example 2.1. [**1**] Let α be an endomorphism on the 2×2 matrices ring $R = M_2(\mathbb{Z}_3)$ over $\mathbb{Z}_3$ defined by $\alpha\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)$. For $p(x) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)x$, $q(x) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)x \in R[x; \alpha]$, one has $p(x)q(x) = 0$. It can be easily checked that $p(x)c \neq 0$ for any nonzero $c \in R$. Therefore R is not α-skew McCoy. This also shows that the 2×2 upper triangular matrix ring $\left\{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a, b, c \in \mathbb{Z}_3\right\}$ over $\mathbb{Z}_3$ is not α-skew McCoy.

We note that the α-skew McCoy ring is weak α-skew McCoy, but the converse is not always true by the following example.

EXAMPLE 2.2. Since $R = \mathbb{Z}_3$ is a domain, it is α-skew Armndariz ring for any endomorphism α of R by [**3**, Proposition 10]. Hence R is α-skew McCoy. Thus R is weak α-skew McCoy, therefore $T_2(\mathbb{Z}_3)$ is weak $\bar{\alpha}$-skew McCoy ring by Propositin 2.4. But $T_2(\mathbb{Z}_3)$ is not α-skew McCoy ring the Example 2.1.

In the following, we provide a connection between abelian and weak α-skew McCoy rings.

PROPOSITION 2.5. *Let R be an abelian ring and α be an endomorphism with $\alpha(e) = e$ for every $e^2 = e \in R$. Then R is a weak α-skew McCoy ring if eR and $(1-e)R$ are weak α-skew McCoy for some $e^2 = e \in R$.*

PROOF. Let $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n$ in $R[x;\alpha]$ with $f(x)g(x) = 0$. Let $f_1(x) = ef(x)$, $f_2(x) = (1-e)f(x)$, $g_1(x) = eg(x)$, $g_2(x) = (1-e)g(x)$. Then $f_1g_1(x) = 0$, $f_2g_2(x) = 0$. Since eR and $(1-e)R$ are weak α-skew McCoy, there exist m_i, n_i such that $e(a_i\alpha^i(s))^{m_i} = ((ea_i)\alpha^i(es))^{m_i} = 0$ and $(1-e)(a_i\alpha^i(t))^{n_i} = \big(((1-e)a_i)\,\alpha^i((1-e)t)\big)^{n_i} = 0$ for some $s \in eR$, $t \in (1-e)R$. Let $k_i = \max\{m_i, n_i\}$. Then $(a_i\alpha^i(st))^{k_i} = 0$. This means that R is weak α-skew McCoy. $\square$

Let R_i be a ring and α_i an endomorphism of R_i for each $i \in I$. Then, for the product $\prod_{i\in I} R_i$ of R_i and the endomorphism $\bar{\alpha} : \prod_{i\in I} R_i \to \prod_{i\in I} R_i$ defined by $\bar{\alpha}((a_i)) = (\alpha_i(a_i))$, $\prod_{i\in I} R_i$ is weak $\bar{\alpha}$-skew McCoy if and only if each R_i is weak α_i-skew McCoy.

Every homomorphism σ of rings R and S can be extended to the homomorphism of rings $R[x]$ and $S[x]$ defined by $\sum_{i=0}^m a_ix^i \mapsto \sum_{i=0}^m \sigma(a_i)x^i$, which we also denote by σ.

PROPOSITION 2.6. *Let $\sigma : R \to S$ be a ring isomorphism. If R is weak α-skew McCoy, then S is weak $\sigma\alpha\sigma^{-1}$-skew McCoy.*

PROOF. Assume that $f(x) = \sum_{i=0}^m a_ix^i$ and $g(x) = \sum_{j=0}^m b_jx^j$ are polynomials in $S[x,\sigma\alpha\sigma^{-1}]$. Since σ is an isomorphism, there exist $f_1(x) = \sum_{i=0}^m a'_ix^i$ and $g(x) = \sum_{j=0}^m b'_jx^j$ in $R[x,\alpha]$ such that $f(x) = \sigma(f_1(x)) = \sum_{i=0}^m \sigma(a'_i)x^i$ and $g(x) = \sigma(g_1(x)) = \sum_{j=0}^m \sigma(b'_j)x^j$. First we show that $f(x)g(x) = 0$ implies $f_1(x)g_1(x) = 0$. We have

$$a_0b_k + a_1(\sigma\alpha\sigma^{-1})(b_{k-1}) + \cdots + a_k(\sigma\alpha\sigma^{-1})^k(b_0) = 0 \text{ for any } 0 \leqslant k \leqslant m.$$

From the definition of $f_1(x)$ and $g_1(x)$, we have,

$$\sigma(a'_0)\sigma(b'_k) + \sigma(a'_1)(\sigma\alpha\sigma^{-1})\sigma(b'_{k-1}) + \cdots + \sigma(a'_k)(\sigma\alpha\sigma^{-1})^k\sigma(b'_0) = 0,$$

so that $(\sigma\alpha\sigma^{-1})^t = \sigma\alpha^t\sigma^{-1}$ we obtain $a'_0b'_k + a'_1\alpha(b'_{k-1}) + \cdots + a'_k\alpha^k(b'_0) = 0$, which means that $f_1(x)g_1(x)$ in $R[x;\alpha]$. From the fact that R is weak α-skew McCoy, we have $a'_i\alpha^i(r) \in \mathrm{nil}(R)$ for some $r \in R$. Since $a'_i = \sigma^{-1}(a_i)$, $r = \sigma^{-1}(s)$ for some $s \in S$, we have $\sigma^{-1}(a_i)\alpha^i(\sigma^{-1}s) \in \mathrm{nil}(R)$. Therefore we obtain $a_i(\sigma\alpha\sigma^{-1})^i(s) \in \mathrm{nil}(R)$, $0 \leqslant i,j \leqslant m$. Hence S is weak $\sigma\alpha\sigma^{-1}$-skew McCoy. $\square$

Let $E_{ij} = (e_{st})$, $1 \leqslant s,t \leqslant n$, denotes $n \times n$ unit matrices over ring R, in which $e_{ij} = 1$ and $e_{st} = 0$ when $s \neq i$ or $t \neq j$, $0 \leqslant i,j \leqslant n$ for all $n \geqslant 2$. If $V = \sum_{i=1}^{n-1} E_{i,i+1}$, then $V_n(R) = RI_n + RV + \cdots + RV^{n-1}$ is the subring of upper triangular skew matrices.

Corollary 2.2. *Suppose that α is an endomorphism of a ring R. If the factor ring $\frac{R[x]}{(x^n)}$ is weak $\bar{\alpha}$-skew McCoy, then $V_n(R)$ is weak $\bar{\alpha}$-skew McCoy.*

Proof. Assume that $R[x]/(x^n)$ is weak $\bar{\alpha}$-skew McCoy and define the ring isomorphism $\theta : V_n(R) \to R[x]/(x^n)$ defined by

$$\theta(r_0 I_n + r_1 V + \cdots + r_{n-1} V^{n-1}) = r_0 + r_1 x + \cdots + r_{n-1} x^{n-1} + (x^n).$$

Now we have that $V_n(R)$ is weak $\theta^{-1}\bar{\alpha}\theta$-skew McCoy and that

$$\theta^{-1}\bar{\alpha}\theta(r_0 I_n + r_1 V + \cdots + r_{n-1} V^{n-1}) = \bar{\alpha}(r_0 I_n + r_1 V + \cdots + r_{n-1} V^{n-1}),$$

which means that $V_n(R)$ is a weak $\bar{\alpha}$-skew McCoy ring. □

Before stating Theorem 2.1, we need the following proposition.

Proposition 2.7. [**8**] *Let R be a reversible ring and α be an endomorphism of R such that $a\alpha(b) = 0$ whenever $ab = 0$ for any $a, b \in R$. Then R is weak α-skew Armendariz.*

In [**4**] it was shown that if a ring R is semicommutative, then $R[x]$ is weak Armendariz. For the case of weak α-skew McCoy, we have the following theorem.

Theorem 2.1. *Let R be a reversible ring and α be an endomorphism of R such that $a\alpha(b) = 0$ whenever $ab = 0$ for any $a, b \in R$. If for some positive integer t, $\alpha^t = 1_R$, then $R[x]$ is weak α-skew McCoy.*

Proof. Let $p(y) = f_0(x) + f_1(x)y + \cdots + f_m(x)y^m$ and $q(y) = g_0(x) + g_1(x)y + \cdots + g_n(x)y^n$ be in $(R[x])[y;\alpha]$ with $p(y)q(y) = 0$. We also let $f_i(x) = a_{i0} + a_{i1}x + \cdots + a_{iw_i}x^{w_i}$ and $g_j(x) = b_{j0} + b_{j1}x + \cdots + b_{jv_j}x^{v_j}$ for any $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$, where $a_{i0}, a_{i1}, \ldots, a_{iw_i}, b_{j0}, b_{j1}, \ldots, b_{jv_j} \in R$. Take a positive integer k such that $k > \deg(f_0(x)) + \deg(f_1(x)) + \cdots + \deg(f_m(x)) + \deg(g_0(x)) + \deg(g_1(x)) + \cdots + \deg(g_n(x))$, where the degrees of $f_i(x)$ and $g_j(x)$ are as the polynomials in $R[x]$ and the degree of zero polynomial is taken to be 0 for all $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$. Let $f(x) = f_0(x^t) + f_1(x^t)x^{tk+1} + f_2(x^t)x^{2tk+2} + \cdots + f_m(x^t)x^{mtk+m}$ and $g(x) = g_0(x^t) + g_1(x^t)x^{tk+1} + g_2(x^t)x^{2tk+2} + \cdots + g_n(x^t)x^{ntk+n} \in R[x]$. Then the set of coefficients of the $f_i(x)$ (respectively, $g_j(x)$) equals the set of coefficients of $f(x)$ (respectively, $g(x)$). Since $p(y)q(y) = 0$, x commutes with elements of R in the polynomial ring $R[x]$, and $\alpha^t = 1_R$, we have $f(x)g(x) = 0$ in $R[x;\alpha]$. By Proposition 2.7, R is weak α-skew Armendariz, and so R weak α-skew McCoy by Proposition 2.1. Thus there exists $b \neq 0$ in R such that $a_{il}\alpha^i(b) \in \mathrm{nil}(R)$ for any $0 \leqslant i \leqslant m$, $l \in \{0, 1, \ldots, w_0, \ldots, w_m\}$. Since R is reversible, $\sum_l a_{il}\alpha^i(b) \in \mathrm{nil}(R)$, by [**4**, Lemma 3.1]. Therefore $f_i(x)\alpha^i(b) \in \mathrm{nil}(R[x])$ by [**4**, Lemma 3.7] for all i, and hence $R[x]$ is weak $\bar{\alpha}$-skew McCoy. □

Also, for the weak α-skew McCoy, the following result holds.

THEOREM 2.2. *Let R be a reversible ring and α be an endomorphism of R such that $a\alpha(b) = 0$ whenever $ab = 0$ for any $a, b \in R$. If, for some positive integer t, $\alpha^t = 1_R$, then $R[x; \alpha]$ is weak α-skew McCoy.*

PROOF. Let $p(y), q(y)$ and k be the same as in the proof of Theorem 2.1. We claim that $f_i(x)g_j(x) \in \mathrm{nil}(R[x; \alpha])$ for all $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$. Let $p(x^{tk}) = f_0(x) + f_1(x)x^{tk} + \cdots + f_m(x)x^{mtk}$ and $q(x^{tk}) = g_0(x) + g_1(x)x^{tk} + \cdots + g_n(x)x^{ntk} \in R[x; \alpha]$. Then the set of coefficients of $f_i(x)$ (respectively, $g_j(x)$) equals the set of coefficients of $p(x^{tk})$ (respectively, $q(x^{tk})$). Since $p(y)q(y) = 0$ and $\alpha^t = 1_R$, we have $p(x^{tk})q(x^{tk}) = 0$ in $R[x; \alpha]$. Since R is weak α-skew McCoy, by Propositions 2.1 and 2.7, there exists $b \neq 0$ such that $a_{il}\alpha^i(b) \in \mathrm{nil}(R)$ for any $0 \leqslant i \leqslant m$, $0 \leqslant l \leqslant w_i$. Thus $f_i(x)b \in \mathrm{nil}(R[x; \alpha])$. Hence $R[x; \alpha]$ is weak McCoy. □

Let α be an automorphism of a ring R. Suppose that there exists the classical left quotient Q of R. Then for any $b^{-1}a \in Q$, where $a, b \in R$ with b regular, the induced map $\bar{\alpha} : Q(R) \to Q(R)$ defined by $\bar{\alpha}(b^{-1}a) = (\alpha(b))^{-1}\alpha(a)$ is also an automorphism.

PROPOSITION 2.8. *Assume that there exists the classical left quotient Q of a ring R. If R is reversible, then Q is weak α-skew McCoy if R is weak α-skew McCoy.*

PROOF. Let $f(x) = s_0^{-1}a_0 + s_1^{-1}a_1x + \cdots + s_m^{-1}a_mx^m$ and $g(x) = t_0^{-1}b_0 + t_1^{-1}b_1x + \cdots + t_n^{-1}b_nx^n \in Q[x; \bar{\alpha}]$ such that $f(x)g(x) = 0$. Let C be a left denominator set. There exist $s, t \in C$ and $a'_i, b'_j \in R$ such that $s_i^{-1}a_i = s^{-1}a'_i$ and $t_j^{-1}b_j = t^{-1}b'_j$ for $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$. Then $s^{-1}(a'_0 + a'_1x + \cdots + a'_mx^m)t^{-1}(b'_0 + b'_1x + \cdots + b'_nx^n) = 0$. It follows that $(a'_0 + a'_1x + \cdots + a'_mx^m)t^{-1}(b'_0 + b'_1x + \cdots + b'_nx^n) = 0$. Thus $(a'_0t^{-1} + a'_1(\alpha(t))^{-1}x + \cdots + a'_m(\alpha^m(t))^{-1}x^m)(b'_0 + b'_1x + \cdots + b'_nx^n) = 0$. For $(a'_i\alpha^i(t))^{-1}$, there exist $t' \in C$, $a''_i \in R$ such that $(a'_i\alpha^i(t))^{-1} = t'a''_i$. Hence $t'^{-1}(a''_0 + a''_1x + \cdots + a''_mx^m)(b'_0 + b'_1x + \cdots + b'_nx^n) = 0$. We have that $(a''_0 + a''_1x + \cdots + a''_mx^m)(b'_0 + b'_1x + \cdots + b'_nx^n) = 0$. Since R is weak α-skew McCoy, there exists $b' \neq 0$ such that $a''_i\alpha^i(b') \in \mathrm{nil}(R)$. Suppose that $(a''_i\alpha^i(b'))^{n_i} = 0$. Since R is reversible, Q is semicommutative. Then $(t'^{-1}(a''_i\alpha^i(b')))^{n_i} = 0$. So $(a'_i\bar{\alpha}^i(t^{-1}b'))^{n_i} = ((t'^{-1}a''_i)\alpha^i(b'))^{n_i} = 0$. Similarly $(s^{-1}a'_i)(\bar{\alpha}^i(t^{-1}b'_j))^{n_i} = 0$. Therefore Q is weak α-skew McCoy. □

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Department of Mathematics
K. N. Toosi University of Technology
P.O. Box 16315 − 1618
Tehran, Iran
nikmehr@kntu.ac.ir

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Department of Mathematics
Karaj Branch, Islamic Azad university
Karaj, Iran
algebra56.tau56@yahoo.com

Department of Mathematics
Islamic Azad university, Central Tehran Branch
P.O. Box 14168-94351, Tehran, Iran
man.deldar@iauctb.ac.ir