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ON RICCI TYPE IDENTITIES IN MANIFOLDS WITH NON-SYMMETRIC AFFINE CONNECTION

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Abstract. In [**18**], using polylinear mappings, we obtained several curvature tensors in the space L_N with non-symmetric affine connection ∇. By the same method, we here examine Ricci type identities.

1. Introduction

Consider N-dimensional differentiable manifold $\mathcal{M}_N$ on which a non-symmetric affine connection $\overset{1}{\nabla}$ is defined. If $\mathfrak{X}(\mathcal{M}_N)$ is a Lie algebra of smooth vector fields and $X, Y \in \mathfrak{X}(\mathcal{M}_N)$, then the mapping $\overset{2}{\nabla}: \mathfrak{X}(\mathcal{M}_N) \times \mathfrak{X}(\mathcal{M}_N) \to \mathfrak{X}(\mathcal{M}_N)$ given by

$$\overset{2}{\nabla}_X Y = \overset{1}{\nabla}_Y X + [X, Y] \tag{1.1}$$

defines an other non-symmetric connection $\overset{2}{\nabla}$ on $\mathcal{M}_N$ [**14**]. That means that we have

$$\overset{\theta}{\nabla}_{Y_1+Y_2} X = \overset{\theta}{\nabla}_{Y_1} X + \overset{\theta}{\nabla}_{Y_2} X, \qquad \overset{\theta}{\nabla}_{fY} X = f \overset{\theta}{\nabla}_Y X,$$
$$\overset{\theta}{\nabla}_Y (X_1 + X_2) = \overset{\theta}{\nabla}_Y X_1 + \overset{\theta}{\nabla}_Y X_2, \qquad \overset{\theta}{\nabla}_Y (fX) = Yf \cdot X + f \overset{\theta}{\nabla}_Y X,$$

for $\theta = 1, 2$ and $X, Y, X_1, X_2, Y_1, Y_2 \in \mathfrak{X}(\mathcal{M}_N)$, $f \in \mathcal{F}(\mathcal{M}_N)$, where $\mathcal{F}(\mathcal{M}_N)$ is an algebra of smooth real functions on $\mathcal{M}_N$. In that case we write $L_N = (\mathcal{M}_N, \overset{1}{\nabla}, \overset{2}{\nabla})$ and L_N call a space on non-symmetric connections $\overset{1}{\nabla}, \overset{2}{\nabla}$.

If we introduce local coordinates $x^1, \dots, x^N$ and put $\partial/\partial x^i = \partial_i$, in view of (1.1) it will be

$$\overset{2}{\nabla}_{\partial_j} \partial_k = \overset{1}{\nabla}_{\partial_k} \partial_j. \tag{1.2}$$

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Denoting coefficients of the connection $\overset{1}{\nabla}$ in the base $\partial_1,\dots,\partial_N$ with L^i_{jk}, we have $\overset{1}{\nabla}_{\partial_k}\partial_j = L^i_{jk}\partial_i$, $\overset{2}{\nabla}_{\partial_k}\partial_j \underset{(1.2)}{=} \overset{1}{\nabla}_{\partial_j}\partial_k = L^i_{kj}\partial_i$, where $\underset{(1.2)}{=}$ denotes "equal with respect to (1.2)".

Further, if we take by definition

$$\overset{\theta}{T}(X,Y) = \overset{\theta}{\nabla}_Y X - \overset{\theta}{\nabla}_X Y + [X,Y],\ \ \theta\in\{1,2\},$$

it follows

$$\overset{2}{T}(X,Y) = -\overset{1}{T}(X,Y) \equiv -T(X,Y),$$
$$(\overset{2}{T}(X,Y) = \overset{1}{T}(X,Y)) \Leftrightarrow (\overset{1}{\nabla} = \overset{2}{\nabla} = \nabla).$$

We proved in [**18**] how it is possible to obtain several curvature tensors in L_N by polylinear mappings. It is proved that among these tensors there are 5 independent ones:

$$\overset{1}{R}(X;Y,Z) = \overset{1}{\nabla}_Z\overset{1}{\nabla}_Y X - \overset{1}{\nabla}_Y\overset{1}{\nabla}_Z X + \overset{1}{\nabla}_{[Y,Z]}X, \tag{1.3}$$

$$\overset{2}{R}(X;Y,Z) = \overset{2}{\nabla}_Z\overset{2}{\nabla}_Y X - \overset{2}{\nabla}_Y\overset{2}{\nabla}_Z X + \overset{2}{\nabla}_{[Y,Z]}X, \tag{1.4}$$

$$\overset{3}{R}(X;Y,Z) = \overset{2}{\nabla}_Z\overset{1}{\nabla}_Y X - \overset{1}{\nabla}_Y\overset{2}{\nabla}_Z X + \overset{2}{\nabla}_{\overset{1}{\nabla}_Y Z}X - \overset{1}{\nabla}_{\overset{2}{\nabla}_Z Y}X, \tag{1.5}$$

$$\overset{4}{R}(X;Y,Z) = \overset{2}{\nabla}_Z\overset{1}{\nabla}_Y X - \overset{1}{\nabla}_Y\overset{2}{\nabla}_Z X + \overset{2}{\nabla}_{\overset{2}{\nabla}_Y Z}X - \overset{1}{\nabla}_{\overset{1}{\nabla}_Z Y}X, \tag{1.6}$$

(1.7)
$$\overset{5}{R}(X;Y,Z) = \frac{1}{2}(\overset{1}{\nabla}_Z\overset{1}{\nabla}_Y X - \overset{2}{\nabla}_Y\overset{1}{\nabla}_Z X + \overset{2}{\nabla}_Z\overset{2}{\nabla}_Y X - \overset{1}{\nabla}_Y\overset{2}{\nabla}_Z X + \overset{1}{\nabla}_{[Y,Z]}X + \overset{2}{\nabla}_{[Y,Z]}X),$$

while the rest can be expressed as linear combinations of these five tensors. For $X = \partial/\partial x^j \equiv \partial_j$, $Y=\partial_k$, $Z=\partial_l$, one obtains

$$\overset{1}{R}{}^i_{jkl} = L^i_{jk,l} - L^i_{jl,k} + L^p_{jk}L^i_{pl} - L^p_{jl}L^i_{pk}, \tag{1.8}$$

$$\overset{2}{R}{}^i_{jkl} = L^i_{kj,l} - L^i_{lj,k} + L^p_{kj}L^i_{lp} - L^p_{lj}L^i_{kp}, \tag{1.9}$$

$$\overset{3}{R}{}^i_{jkl} = L^i_{jk,l} - L^i_{lj,k} + L^p_{jk}L^i_{lp} - L^p_{lj}L^i_{pk} + L^p_{lk}(L^i_{pj} - L^i_{jp}), \tag{1.10}$$

$$\overset{4}{R}{}^i_{jkl} = L^i_{jk,l} - L^i_{lj,k} + L^p_{jk}L^i_{lp} - L^p_{lj}L^i_{pk} + L^p_{kl}(L^i_{pj} - L^i_{jp}), \tag{1.11}$$

$$\overset{5}{R}{}^i_{jkl} = \frac{1}{2}(L^i_{jk,l} + L^i_{kj,l} - L^i_{jl,k} - L^i_{lj,k} + L^p_{jk}L^i_{pl} + L^p_{kj}L^i_{lk} - L^p_{jl}L^i_{kp} - L^p_{lj}L^i_{pk}) \tag{1.12}$$

2. Identities for a vector and for a covector by both connections

2.1. Consider an expression

$$(\overset{\nu}{\nabla}_Z\overset{\mu}{\nabla}_Y X - \overset{\mu}{\nabla}_Y\overset{\nu}{\nabla}_Z X)(\omega),\ \ \mu,\nu\in\{1,2\}. \tag{2.1}$$

Let us quote the relations (for $\mu = 1,2$)

$$(\overset{\mu}{\nabla}_Y X)(\omega) = Y[X(\omega)] - X(\overset{\mu}{\nabla}_Y \omega), \qquad (\overset{\mu}{\nabla}_Y \omega)(X) = Y[\omega(X)] - \omega(\overset{\mu}{\nabla}_Y X).$$

and denote

$$a)\ \overset{\mu}{\nabla}_Y X = \overline{X} \in \mathfrak{X}(\mathcal{M}_{\mathcal{N}}), \quad b)\ \overset{\nu}{\nabla}_Z \omega = \overline{\omega} \in \mathfrak{X}^*(\mathcal{M}_{\mathcal{N}}). \tag{2.2}$$

Then we have

$$\begin{aligned}(\overset{\nu}{\nabla}_Z \overset{\mu}{\nabla}_Y X)(\omega) &\underset{(2a)}{=} (\overset{\nu}{\nabla}_Z \overline{X})(\omega) \underset{(2)}{=} Z[\overline{X}(\omega)] - \overline{X}(\overset{\nu}{\nabla}_Z \omega)\\ &\underset{(2)}{=} Z[(\overset{\mu}{\nabla}_Y X)(\omega)] - (\overset{\mu}{\nabla}_Y X)(\overline{\omega})\\ &\underset{(2b)}{=} Z\{Y[X(\omega)] - X(\overset{\mu}{\nabla}_Y X)\} - \{Y[X(\overset{\nu}{\nabla}_Z \omega)] - X(\overset{\mu}{\nabla}_Y \overset{\nu}{\nabla}_Z \omega)\}\\ &= ZY[X(\omega)] - Z[X(\overset{\mu}{\nabla}_Y \omega)] - Y[X(\overset{\nu}{\nabla}_Z \omega)] + X(\overset{\mu}{\nabla}_Y \overset{\nu}{\nabla}_Z \omega),\end{aligned} \tag{2.3}$$

and one gets

$$(\overset{\nu}{\nabla}_Z \overset{\mu}{\nabla}_Y X - \overset{\mu}{\nabla}_Y \overset{\nu}{\nabla}_Z X)(\omega) = [Z,Y][X(\omega)] - X(\overset{\nu}{\nabla}_Z \overset{\mu}{\nabla}_Y \omega - \overset{\mu}{\nabla}_Y \overset{\nu}{\nabla}_Z \omega), \tag{2.4}$$

i.e.,

$$(\overset{\nu}{\nabla}_Z \overset{\mu}{\nabla}_Y X - \overset{\mu}{\nabla}_Y \overset{\nu}{\nabla}_Z X)(\omega) = [Z,Y][X(\omega)] - (\overset{\nu}{\nabla}_Z \overset{\mu}{\nabla}_Y \omega - \overset{\mu}{\nabla}_Y \overset{\nu}{\nabla}_Z \omega)(X),\ \mu,\nu \in \{1,2\}.$$

From

$$(\overset{\nu}{\nabla}_{[Z,Y]} X)(\omega) = \overset{\nu}{\nabla}_{[Z,Y]}[X(\omega)] - X(\overset{\nu}{\nabla}_{[Z,Y]} \omega) = [Z,Y][X(\omega)] + (\overset{\nu}{\nabla}_{[Y,Z]} \omega)(X), \tag{2.5}$$

we find the first addend on the right side and substitute in (2.7). We obtain

$$\begin{aligned}&(\overset{\nu}{\nabla}_Z \overset{\mu}{\nabla}_Y X - \overset{\nu}{\nabla}_Y \overset{\mu}{\nabla}_Z X + \overset{\nu}{\nabla}_{[Y,Z]} X)(\omega)\\ &\qquad = -(\overset{\nu}{\nabla}_Z \overset{\mu}{\nabla}_Y \omega - \overset{\nu}{\nabla}_Y \overset{\mu}{\nabla}_Z \omega + \overset{\nu}{\nabla}_{[Y,Z]} \omega)(X), \quad \mu,\nu \in \{1,2\}\end{aligned} \tag{2.6}$$

Definition 2.1. The equations (2.4) for $\mu,\nu \in \{1,2\}$ are *Ricci type identities for a vector* in L_N.

2.2. Taking $\mu = \nu = 1$, we obtain the corresponding identity for $\overset{1}{\nabla}$:

$$\overset{1}{R}(X;Y,Z)(\omega) = -(\overset{1}{\nabla}_Z \overset{1}{\nabla}_Y \omega - \overset{1}{\nabla}_Y \overset{1}{\nabla}_Z \omega + \overset{1}{\nabla}_{[Y,Z]} \omega)(X). \tag{2.7}$$

Denoting

$$\overset{1}{\overline{R}}(\omega;Y,Z) = \overset{1}{\nabla}_Z \overset{1}{\nabla}_Y \omega - \overset{1}{\nabla}_Y \overset{1}{\nabla}_Z \omega + \overset{1}{\nabla}_{[Y,Z]} \omega, \tag{2.8}$$

the equation (2.7) gives a relation

$$\overset{1}{R}(X;Y,Z)(\omega) = -\overset{1}{\overline{R}}(\omega;Y,Z)(X). \tag{2.9}$$

In order to write the equation (2.4) in local coordinates for $\mu = \nu = 1$, we take $X = X^j\partial_j$, $Y = \partial_k$, $Z = \partial_l$, $\omega = dx^i$. For the left side in (2.4) we obtain

$$\begin{aligned}
\mathcal{L} &= (\overset{1}{\nabla}_{\partial_l}\overset{1}{\nabla}_{\partial_k}X - \overset{1}{\nabla}_{\partial_k}\overset{1}{\nabla}_{\partial_l}X)(dx^i)\\
&= [\overset{1}{\nabla}_{\partial_l}(X^j_{\underset{1}{|}k}\partial_j) - \overset{1}{\nabla}_{\partial_k}(X^j_{\underset{1}{|}l}\partial_j)](dx^i)\\
&= [(X^j_{\underset{1}{|}k})_{,l}\partial_j + X^j_{\underset{1}{|}k}L^p_{jl}\partial_p - (X^j_{\underset{1}{|}l})_{,k}\partial_j - X^j_{\underset{1}{|}l}L^p_{jk}\partial_p](dx^i)\\
&= (X^j_{\underset{1}{|}k})_{,l}\delta^i_j + X^j_{\underset{1}{|}k}L^p_{jl}\delta^i_p - (X^j_{\underset{1}{|}l})_{,k}\delta^i_j - X^j_{\underset{1}{|}l}L^p_{jk}\delta^i_p\\
&= (X^i_{\underset{1}{|}k})_{,l} + X^j_{\underset{1}{|}k}L^i_{jl} - (X^i_{\underset{1}{|}l})_{,k} - X^j_{\underset{1}{|}l}L^i_{jk}\\
&= X^i_{\underset{1}{|}k\underset{1}{|}l} + L^p_{kl}X^i_{\underset{1}{|}p} - X^i_{\underset{1}{|}l\underset{1}{|}k} - L^p_{lk}X^i_{\underset{1}{|}p}\\
&= X^i_{\underset{1}{|}k\underset{1}{|}l} - X^i_{\underset{1}{|}k\underset{1}{|}l} + T^p_{kl}X^i_{\underset{1}{|}p}.
\end{aligned}$$

For the right-hand side in (2.4) we obtain

$$\begin{aligned}
\mathcal{R} &= [\partial_l, \partial_k](X(dx^i)) + X(\overset{1}{\nabla}_{\partial_l}\overset{1}{\nabla}_{\partial_k}dx^i - \overset{1}{\nabla}_{\partial_k}\overset{1}{\nabla}_{\partial_l}dx^i)\\
&= 0 + X[\overset{1}{\nabla}_{\partial_l}(-L^i_{pl}dx^p) - \overset{1}{\nabla}_{\partial_l}(-L^i_{pk}dx^p)]\\
&= X(\underset{1}{R}^i_{pkl}dx^p) = \underset{1}{R}^i_{pkl}dx^p(X) = \underset{1}{R}^i_{pkl}X^p,
\end{aligned}$$

and from $\mathcal{L} = \mathcal{R}$, we have

$$X^i_{\underset{1}{|}kl} - X^i_{\underset{1}{|}lk} = \underset{1}{R}^i_{pkl}X^p - T^p_{kl}X^i_{\underset{1}{|}p}, \tag{2.10}$$

i.e., the known identity in local coordinates. So, we have proved the following theorem.

Theorem 2.1. *In the space L_N, with non-symmetric affine connection $\overset{1}{\nabla}$ by equation* (2.4) *for $\mu = \nu = 1$* the first Ricci type identity for a vector *is given. That identity can be written in forms* (2.5), (2.6), (2.9), *here $\overset{1}{R}$ is given by* (1.3) *and $\overset{1}{\overline{R}}$ by* (2.8). *The corresponding identity in local coordinates is* (2.10).

2.3. By using equation (2.4) and the condition $X(\omega) = \omega(X) \in \mathcal{F}(\mathcal{M}_\mathcal{N})$, we obtain the equation analogous to (2.4) (X and ω have changed the roles):

$$(\overset{\nu}{\nabla}_Z\overset{\mu}{\nabla}_Y\omega - \overset{\mu}{\nabla}_Y\overset{\nu}{\nabla}_Z\omega)(X) = [Z, Y][\omega(X)] - \omega(\overset{\nu}{\nabla}_Z\overset{\mu}{\nabla}_Y X - \overset{\mu}{\nabla}_Y\overset{\nu}{\nabla}_Z X). \tag{2.11}$$

Definition 2.2. Equations (2.11) for $\mu, \nu \in \{1, 2\}$ are *Ricci type identities for a covector* in L_N.

Equation (2.11) can be obtained also by consideration of the expression on the left-hand side in (2.11). The known Ricci identity for a covariant vector in local coordinates can be obtained from (2.11) by substituting $\omega = \omega_j x^j$, $X = \partial_i$, $Y = \partial_k$, $Z = \partial_l$:

$$\omega_{j\underset{1}{|}kl} - \omega_{j\underset{1}{|}lk} = -\underset{1}{R}{}^p_{jkl}\omega_p - T^p_{kl}\omega_{j\underset{1}{|}p}. \tag{2.12}$$

So, the following theorem is valid.

Theorem 2.2. *In the space L_N, with non-symmetric affine connection $\overset{1}{\nabla}$, by equation* (2.11) *for $\mu = \nu = 1$,* the first Ricci type identity for a covector *is given. The corresponding identity in local coordinates is* (2.12).

2.4. For $\mu = \nu = 2$ from (2.4) is obtained

$$(\overset{2}{\nabla}_Z\overset{2}{\nabla}_Y X - \overset{2}{\nabla}_Y\overset{2}{\nabla}_Z X)(\omega) = [Z,Y][X(\omega)] - X(\overset{2}{\nabla}_Z\overset{2}{\nabla}_Y\omega - \overset{2}{\nabla}_Y\overset{2}{\nabla}_Z\omega). \tag{2.13}$$

From here

$$\overset{2}{R}(X;Y,Z)(\omega) = -\overline{\overset{2}{R}}(\omega;Y,Z)(X), \tag{2.14}$$

where $\overline{\overset{2}{R}}$ is expressed by $\overset{2}{\nabla}$ analogously to (2.8) and $\overset{2}{R}$ is given in (1.3). Surpassing to local coordinates, from (2.13) one obtains

$$X^i_{\underset{2}{|}kl} - X^i_{\underset{2}{|}lk} = \underset{2}{R}{}^i_{pkl}X^p + T^p_{kl}X^i_{\underset{2}{|}p}, \tag{2.15}$$

and also equations similar to (2.11), (2.12) (for a covector).

Thus, we state

Theorem 2.3. *In the space L_N with non-symmetric affine connection $\overset{2}{\nabla}$, defined by* (1.1), the second Ricci type identity for a vector *is given by equation* (2.13). *The corresponding identity in local coordinates is* (2.15).

3. Identities for a vector and covector obtained by combinations of both connections

3.1. Putting $\mu = 1$, $\nu = 2$ into (2.4), we get the identity

$$(\overset{2}{\nabla}_Z\overset{1}{\nabla}_Y X - \overset{1}{\nabla}_Y\overset{2}{\nabla}_Z X)(\omega) = [Z,Y][X(\omega)] - (\overset{2}{\nabla}_Z\overset{1}{\nabla}_Y\omega - \overset{1}{\nabla}_Y\overset{2}{\nabla}_Z\omega)(X). \tag{3.1}$$

and from (2.6)

$$(\overset{2}{\nabla}_Z\overset{1}{\nabla}_Y X - \overset{1}{\nabla}_Y\overset{2}{\nabla}_Z X + \overset{2}{\nabla}_{[Y,Z]}X)(\omega) = -(\overset{2}{\nabla}_Z\overset{1}{\nabla}_Y\omega - \overset{1}{\nabla}_Y\overset{2}{\nabla}_Z\omega + \overset{2}{\nabla}_{[Y,Z]}\omega)(X) \tag{3.2}$$

Analogously to (1.5), let us put

$$\overline{\overset{3}{R}}(\omega;Y,Z) = \overset{2}{\nabla}_Z\overset{1}{\nabla}_Y\omega - \overset{1}{\nabla}_Y\overset{2}{\nabla}_Z\omega + \overset{2}{\nabla}_{\overset{1}{\nabla}_Y Z}\omega - \overset{1}{\nabla}_{\overset{2}{\nabla}_Y Z}\omega \in \mathfrak{X}^*(\mathcal{M}_N) \tag{3.3}$$

and (3.2) becomes

$$(3.4)\quad (\overset{3}{R}(X;Y,Z)+\overset{1}{\nabla}_{\overset{2}{\nabla}_Y Z}X-\overset{2}{\nabla}_{\overset{1}{\nabla}_Y Z}X+\overset{2}{\nabla}_{[Y,Z]}X)(\omega)$$
$$=-(\overset{3}{\overline{R}}(\omega;Y,Z)+\overset{1}{\nabla}_{\overset{2}{\nabla}_Z Y}\omega-\overset{2}{\nabla}_{\overset{1}{\nabla}_Y Z}\omega+\overset{2}{\nabla}_{[Y,Z]}\omega)(X).$$

Because of

$$(\overset{1}{\nabla}_{\overset{2}{\nabla}_Z Y}X-\overset{2}{\nabla}_{\overset{1}{\nabla}_Y Z}X+\overset{2}{\nabla}_{[Y,Z]}X)(\omega)=(\overset{1}{\nabla}_{\overset{2}{\nabla}_Z Y}X-\overset{2}{\nabla}_{\overset{1}{\nabla}_Y Z+[Z,Y]}X)(\omega)$$
$$=(\overset{1}{\nabla}_{\overset{2}{\nabla}_Z Y}X-\overset{2}{\nabla}_{\overset{2}{\nabla}_Z Y}X)(\omega)$$

and

$$-(\overset{1}{\nabla}_{\overset{2}{\nabla}_Z Y}\omega-\overset{2}{\nabla}_{\overset{1}{\nabla}_Y Z}\omega+\overset{2}{\nabla}_{[Y,Z]}\omega)(X)=-(\overset{1}{\nabla}_{\overset{2}{\nabla}_Z Y}\omega-\overset{2}{\nabla}_{\overset{1}{\nabla}_Y Z+[Z,Y]}\omega)(X)$$
$$=(\overset{2}{\nabla}_{\overset{2}{\nabla}_Z Y}\omega-\overset{1}{\nabla}_{\overset{2}{\nabla}_Z Y}\omega)(X)$$
$$=\overset{2}{\nabla}_{\overset{2}{\nabla}_Z Y}[\omega(X)]-\omega(\overset{2}{\nabla}_{\overset{2}{\nabla}_Z Y}X)-\overset{1}{\nabla}_{\overset{2}{\nabla}_Z Y}[\omega(X)]+\omega(\overset{1}{\nabla}_{\overset{2}{\nabla}_Z Y}X)$$
$$=(\overset{1}{\nabla}_{\overset{2}{\nabla}_Z Y}X-\overset{2}{\nabla}_{\overset{2}{\nabla}_Z Y}X)(\omega),$$

we see that the right-hand sides of these equations are identical and from (3.4):

$$(3.5)\qquad \overset{3}{R}(X;Y,Z)(\omega)=-\overset{3}{\overline{R}}(\omega;Y,Z)(X).$$

3.2. If we put $X=X^j\partial_j$, $Y=\partial_k$, $Z=\partial_l$, $\omega=dx^i$, equation (3.1) will be written in local coordinates as follows. For the left-hand side $\mathcal{L}$ we have

$$\mathcal{L}=(\overset{2}{\nabla}_{\partial_l}\overset{1}{\nabla}_{\partial_k}X-\overset{1}{\nabla}_{\partial_k}\overset{2}{\nabla}_{\partial_l}X)(dx^i)$$
$$=[\overset{2}{\nabla}_{\partial_l}(X^j_{\underset{1}{|k}}\partial_j)-\overset{1}{\nabla}_{\partial_k}(X^j_{\underset{2}{|l}}\partial_j)](dx^i)$$
$$(3.6)\qquad =[(X^j_{\underset{1}{|k}})_{,l}\partial_j+X^j_{\underset{1}{|k}}L^p_{lj}\partial_p-(X^j_{\underset{2}{|l}})_{,k}\partial_j-X^j_{\underset{2}{|l}}L^p_{jk}\partial_p](dx^i)$$
$$=(X^i_{\underset{1}{|k}})_{,l}+X^j_{\underset{1}{|k}}L^i_{lj}-(X^i_{\underset{2}{|l}})_{,k}-X^j_{\underset{2}{|l}}L^i_{jk}$$
$$=X^i_{\underset{1}{|k}\underset{2}{|l}}-X^i_{\underset{2}{|l}\underset{1}{|k}}-L^p_{lk}(X^i_{\underset{1}{|p}}-X^i_{\underset{2}{|p}})=X^i_{\underset{1}{|k}\underset{2}{|l}}-X^i_{\underset{2}{|l}\underset{1}{|k}}-L^p_{lk}T^i_{sp}X^s,$$

where

$$X^i_{\underset{1}{|k}\underset{2}{|l}}=(X^i_{\underset{1}{|k}})_{,l}+X^p_{\underset{1}{|k}}L^i_{lp}-X^i_{\underset{1}{|p}}L^p_{lk},$$
$$X^i_{\underset{2}{|l}\underset{1}{|k}}=(X^i_{\underset{2}{|l}})_{,k}+X^p_{\underset{2}{|l}}L^i_{pk}-X^i_{\underset{2}{|p}}L^p_{lk}.$$

For the right-hand side one obtains

$$\mathcal{R} = -(\overset{2}{\nabla}_{\partial_l}\overset{1}{\nabla}_{\partial_k}dx^i - \overset{1}{\nabla}_{\partial_k}\overset{2}{\nabla}_{\partial_l}dx^i)(X) \\ = (L^i_{pk,l}dx^p - L^i_{pk}L^p_{ls}dx^s + L^i_{lp,k}dx^p + L^i_{lp}L^p_{sk}dx^s)(X) \\ = (L^i_{pk,l} - L^i_{sk}L^s_{lp} - L^i_{lp,k} + L^i_{ls}L^s_{pk})X^p. \tag{3.7}$$

By virtue of (3.6) and (3.7), from $\mathcal{L} = \mathcal{R}$ it is

$$X^i_{\underset{1}{|}k\underset{2}{|}l} - X^i_{\underset{2}{|}l\underset{1}{|}k} = \underset{3}{R}{}^i_{pkl}X^p, \tag{3.8}$$

and, analogously to that exposed above, for a covariant vector ω it is obtained

$$\omega_{j\underset{1}{|}k\underset{2}{|}l} - \omega_{j\underset{2}{|}l\underset{1}{|}k} = -\underset{3}{R}{}^p_{jkl}\omega_p. \tag{3.9}$$

3.3. Introducing $\overset{4}{R}(X;Y,Z)$ into (3.2) by virtue of (1.6) and defining $\overset{4}{\overline{R}}(\omega;Y,Z)$ according to

$$\overset{4}{\overline{R}}(\omega;Y,Z) = \overset{2}{\nabla}_Z\overset{1}{\nabla}_Y\omega - \overset{1}{\nabla}_Y\overset{2}{\nabla}_Z\omega + \overset{2}{\nabla}_{\overset{2}{\nabla}_Y Z}\omega - \overset{1}{\nabla}_{\overset{1}{\nabla}_Y Z}\omega \in \mathfrak{X}^*(\mathcal{M}_\mathcal{N}), \tag{3.10}$$

equation (3.2) gives

$$(\overset{4}{R}(X;Y,Z) + \overset{1}{\nabla}_{\overset{1}{\nabla}_Z Y}X - \overset{2}{\nabla}_{\overset{2}{\nabla}_Y Z}X + \overset{2}{\nabla}_{[Y,Z]}X)(\omega) \\ = -(\overset{4}{\overline{R}}(\omega;Y,Z) + \overset{1}{\nabla}_{\overset{1}{\nabla}_Z Y}\omega - \overset{2}{\nabla}_{\overset{2}{\nabla}_Y Z}\omega + \overset{2}{\nabla}_{[Y,Z]}\omega)(X).$$

As in the case of $\overset{3}{\overline{R}}$, we obtain

$$\overset{4}{R}(X;Y,Z)(\omega) = -\overset{4}{\overline{R}}(\omega;Y,Z)(X). \tag{3.11}$$

Putting $X = \partial_j$, $Y = \partial_k$, $Z = \partial_l$, $\omega = dx^i$ and taking into consideration (3.10), we get

$$\overset{4}{R}{}^p_{jkl}\partial_p(dx^i) = -[\overset{2}{\nabla}_{\partial_l}(-L^i_{pk}dx^p) - \overset{2}{\nabla}_{\partial_k}(-L^i_{lp}dx^p) + \overset{2}{\nabla}_{L^p_{kl}\partial_p}dx^i - \overset{1}{\nabla}_{L^p_{kl}\partial_p}dx^i](\partial_j),$$

from where for $\overset{4}{R}$ the value (1.11) is obtained. In view of (1.10), (1.11) it is

$$\overset{4}{R}{}^p_{jkl} - \overset{3}{R}{}^p_{jkl} = T^i_{pj}T^p_{kl}.$$

and, using (3.8) and (3.9), we obtain

$$X^i_{\underset{1}{|}k\underset{2}{|}l} - X^i_{\underset{2}{|}l\underset{1}{|}k} = \underset{4}{R}{}^i_{pkl}X^p + T^i_{ps}T^s_{kl}X^p, \tag{3.12}$$

$$\omega_{j\underset{1}{|}k\underset{2}{|}l} - \omega_{j\underset{2}{|}l\underset{1}{|}k} = -\underset{4}{R}{}^p_{jkl}\omega_p + T^p_{sj}T^s_{kl}\omega_p. \tag{3.13}$$

Now, we can state the following theorem

Theorem 3.1. *In the space L_N with two non-symmetric affine connections* $\overset{1}{\nabla}$, $\overset{2}{\nabla}$, *linked by equation* (1.1), the third Ricci type identity for a vector *is given by equation* (3.1). *This identity can be written also in forms* (3.2), (3.5) *and* (3.11). *From* (2.8), *for* $\mu = 1, \nu = 2$, *one obtains* the third Ricci type identity for a covector. *The corresponding identities in coordinates are* (3.8), (3.9), (3.12) *and* (3.13).

3.4. In order to obtain an identity in which $\overset{5}{R}$ appears, let us start from the expression which appears in (1.7). So,

$$
\begin{aligned}
&(\overset{1}{\nabla}_Z\overset{1}{\nabla}_Y X + \overset{2}{\nabla}_Z\overset{2}{\nabla}_Y X - \overset{1}{\nabla}_Y\overset{2}{\nabla}_Z X - \overset{2}{\nabla}_Y\overset{1}{\nabla}_Z X)(\omega)\\
&\qquad\overset{(2.3)}{=} \big\{ZY[X(\omega)] - Z[X(\overset{1}{\nabla}_Y\omega)] - Y[X(\overset{1}{\nabla}_Z\omega)] + X(\overset{1}{\nabla}_Y\overset{1}{\nabla}_Z\omega)\\
&\qquad\quad + ZY[X(\omega)] - Z[X(\overset{2}{\nabla}_Y\omega)] - Y[X(\overset{2}{\nabla}_Z\omega)] + X(\overset{2}{\nabla}_Y\overset{2}{\nabla}_Z\omega)\\
&\qquad\quad - YZ[X(\omega)] + Y[X(\overset{2}{\nabla}_Z\omega)] + Z[X(\overset{1}{\nabla}_Y\omega)] - X(\overset{2}{\nabla}_Z\overset{1}{\nabla}_Y\omega)\\
&\qquad\quad - YZ[X(\omega)] + Y[X(\overset{1}{\nabla}_Z\omega)] + Z[X(\overset{2}{\nabla}_Y\omega)] - X(\overset{1}{\nabla}_Z\overset{2}{\nabla}_Y\omega)\big\}(\omega),
\end{aligned}
$$

that is

$$
\begin{aligned}
&\frac{1}{2}(\overset{1}{\nabla}_Z\overset{1}{\nabla}_Y X + \overset{2}{\nabla}_Z\overset{2}{\nabla}_Y X - \overset{1}{\nabla}_Y\overset{2}{\nabla}_Z X - \overset{2}{\nabla}_Y\overset{1}{\nabla}_Z X)(\omega)\\
&\qquad = [Z,Y][X(\omega)] + \frac{1}{2}X(\overset{1}{\nabla}_Y\overset{1}{\nabla}_Z\omega + \overset{2}{\nabla}_Y\overset{2}{\nabla}_Z\omega - \overset{1}{\nabla}_Z\overset{2}{\nabla}_Y\omega - \overset{2}{\nabla}_Z\overset{1}{\nabla}_Y\omega).
\end{aligned}
\tag{3.14}
$$

Definition 3.1. Equation (3.14) we call the *combined Ricci type identity for a vector* in L_N.

Using (1.7), from (3.14) it is obtained

$$
\begin{aligned}
&\overset{5}{R}(X;Y,Z)(\omega) + (\overset{0}{\nabla}_{[Z,Y]}X)(\omega)\\
&\qquad = [Z,Y][\omega(X)] + \frac{1}{2}(\overset{1}{\nabla}_Y\overset{1}{\nabla}_Z\omega + \overset{2}{\nabla}_Y\overset{2}{\nabla}_Z\omega - \overset{1}{\nabla}_Z\overset{2}{\nabla}_Y\omega - \overset{2}{\nabla}_Z\overset{1}{\nabla}_Y\omega)(X).
\end{aligned}
\tag{3.15}
$$

From

$$
(\overset{0}{\nabla}_{[Z,Y]}X)(\omega) = [Z,Y][X(\omega)] - X(\overset{0}{\nabla}_{[Z,Y]}\omega) = [Z,Y][\omega(X)] + (\overset{0}{\nabla}_{[Y,Z]}\omega)(X),
$$

we find the first addend of the right-hand side and substitute into (3.15). So,

$$
\begin{aligned}
&\overset{5}{R}(X;Y,Z)(\omega) + (\overset{0}{\nabla}_{[Z,Y]}X)(\omega)\\
&\qquad = (\overset{0}{\nabla}_{[Z,Y]}X)(\omega) + (\overset{0}{\nabla}_{[Z,Y]}\omega)(X)\\
&\qquad\quad + \frac{1}{2}(\overset{1}{\nabla}_Y\overset{1}{\nabla}_Z\omega + \overset{2}{\nabla}_Y\overset{2}{\nabla}_Z\omega - \overset{1}{\nabla}_Z\overset{2}{\nabla}_Y\omega - \overset{2}{\nabla}_Z\overset{1}{\nabla}_Y\omega)(X),
\end{aligned}
$$

where $\overset{5}{R}$ is given in (1.7). Denoting

(3.16)
$$\overline{\overset{5}{R}}(\omega;Y,Z)=\frac{1}{2}(\overset{1}{\nabla}_Z\overset{1}{\nabla}_Y\omega+\overset{2}{\nabla}_Z\overset{2}{\nabla}_Y\omega-\overset{1}{\nabla}_Y\overset{2}{\nabla}_Z\omega-\overset{2}{\nabla}_Y\overset{1}{\nabla}_Z\omega+\overset{1}{\nabla}_{[Y,Z]}\omega+\overset{2}{\nabla}_{[Y,Z]}\omega)$$

the previous equation gives

$$\overset{5}{R}(X;Y,Z)(\omega)=\overline{\overset{5}{R}}(\omega;Z,Y)(X). \tag{3.17}$$

Substituting here $X=\partial_j$, $Y=\partial_k$, $Z=\partial_l$, $\omega=dx^i$ and taking into consideration (3.16), for $\overset{5}{R}{}^i_{jkl}$ (1.12) is obtained.

Remark 3.1. We see that relation between $\overset{5}{R}$ and $\overline{\overset{5}{R}}$ is not of the form relating to $\overset{\theta}{R}$, $\overline{\overset{\theta}{R}}$, $\theta=1,2,3,4$. In fact using the corresponding values from [22]

$$\begin{aligned}
\overset{5}{R}&=\overset{0}{R}(X;Y,Z)+\tau(\tau(X,Y),Z)+\tau(\tau(X,Z),Y),\\
\overset{8}{R}&=\frac{1}{2}(\overset{1}{\nabla}_Z\overset{2}{\nabla}_YX+\overset{2}{\nabla}_Z\overset{1}{\nabla}_YX-\overset{1}{\nabla}_Y\overset{1}{\nabla}_ZX-\overset{2}{\nabla}_Y\overset{2}{\nabla}_ZX+\overset{1}{\nabla}_{[Y,Z]}X+\overset{2}{\nabla}_{[Y,Z]}X)\\
&=\overset{0}{R}(X;Y,Z)-\tau(\tau(X,Y),Z)-\tau(\tau(X,Z),Y).
\end{aligned}$$

We conclude that $\overset{5}{R}-2\overset{0}{R}=\overset{8}{R}$ and

$$\overset{5}{R}(X;Y,Z)(\omega)=\overline{\overset{5}{R}}(\omega;Z,Y)(X)=-\overline{\overset{8}{R}}(\omega;Y,Z)(X).$$

So, we have

Theorem 3.2. *In the space L_N with two non-symmetric affine connections $\overset{1}{\nabla}$, $\overset{2}{\nabla}$, linked according to* (1.1), *by equation* (3.14) *the combined Ricci type identity for a vector is given. Some other forms of* (3.14) *are* (3.15)–(3.17). *From* (3.15) *combined Ricci type identity for a covector is obtained:*

$$\begin{aligned}
&\frac{1}{2}(\overset{1}{\nabla}_Y\overset{1}{\nabla}_Z\omega+\overset{2}{\nabla}_Y\overset{2}{\nabla}_Z\omega-\overset{1}{\nabla}_Z\overset{2}{\nabla}_Y\omega-\overset{2}{\nabla}_Z\overset{1}{\nabla}_Y\omega)(X)\\
&\quad=\frac{1}{2}(\overset{1}{\nabla}_Z\overset{1}{\nabla}_YX+\overset{2}{\nabla}_Z\overset{2}{\nabla}_YX-\overset{1}{\nabla}_Y\overset{2}{\nabla}_ZX-\overset{2}{\nabla}_Y\overset{1}{\nabla}_ZX)(\omega)+[Z,Y][\omega(X)].
\end{aligned} \tag{3.18}$$

From (3.14) and (3.18) we obtain the corresponding combined Ricci type identities for a vector and covector respectively *local coordinates*:

$$\frac{1}{2}(X^i_{\underset{1}{|}kl}+X^i_{\underset{2}{|}kl}-X^i_{\underset{1}{|}l\underset{2}{|}k}-X^i_{\underset{2}{|}l\underset{1}{|}k})=\overset{5}{R}{}^i_{pkl}X^p,$$
$$\frac{1}{2}(\omega_{j\underset{1}{|}kl}+\omega_{j\underset{2}{|}kl}-\omega_{j\underset{1}{|}l\underset{2}{|}k}-\omega_{j\underset{2}{|}l\underset{1}{|}k})=(\overset{5}{R}-2\overset{0}{R})^p_{jkl}\omega_p,$$

where $\overset{0}{R}{}^i_{jkl}$ is defined by $\overset{0}{L}{}^i_{jk}=\frac{1}{2}(L^i_{jk}+L^i_{jk})$, i.e., by symmetric connection coefficients.

DEFINITION 3.2. The objects $\overset{\theta}{\overline{R}}$, $(\theta = 1, \dots, 5)$, defined by $\overset{\theta}{\overline{R}} : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}^*$ on $\mathcal{M}_N$, we call *dual curvature tensors* in relation to $\overset{\theta}{R}$.

4. Identities for a tensor field t of the type (r, s)

4.1. Let us consider a tensor field of the type (r, s), which will be denoted $\underset{s}{\overset{r}{t}} \equiv t$, i.e., consider a mapping $\underset{s}{\overset{r}{t}} : (\mathcal{X}^*)^r \times (\mathcal{X})^s \mapsto \mathcal{F}(\mathcal{M}^{\mathcal{N}})$. So,

$$\underset{s}{\overset{r}{t}}(\omega^1, \dots, \omega^r, X_1, \dots, X_s) \in \mathcal{F}(\mathcal{M}^{\mathcal{N}}),$$

is a differentiable function on $\mathcal{M}_{\mathcal{N}}$.

As known, a covariant derivative $\nabla_Y \underset{s}{\overset{r}{t}}$ is also of a type (r, s). As in (2.1), one can consider the expression $(\overset{\nu}{\nabla}_Z \overset{\mu}{\nabla}_Y \underset{s}{\overset{r}{t}} - \overset{\mu}{\nabla}_Y \overset{\nu}{\nabla}_Z \underset{s}{\overset{r}{t}})(\omega^1, \dots, \omega^r, X_1, \dots, X_s)$.

4.2. Let us examine nearer the case $(r, s) = (2, 1)$, i.e., $\underset{1}{\overset{2}{t}} \equiv t$. We have

$$\begin{aligned}(\overset{\mu}{\nabla}_Y \underset{1}{\overset{2}{t}})(\omega^1, \omega^2; X) = \overset{\mu}{\nabla}_Y [t(\omega^1, \omega^2; X] &- t(\overset{\mu}{\nabla}_Y \omega^1, \omega^2; X) \\ &- t(\omega^1, \overset{\mu}{\nabla}_Y \omega^2; X) - t(\omega^1, \omega^2; \overset{\mu}{\nabla}_Y X).\end{aligned}$$

Denoting $\overset{\mu}{\nabla}_Y t = \overline{t}$, we have

$$\begin{aligned}
&(\overset{\nu}{\nabla}_Z \overset{\mu}{\nabla}_Y t)(\omega^1, \omega^2; X) = (\overset{\nu}{\nabla}_Z \overline{t})(\omega^1, \omega^2; X) \\
&\underset{(4)}{=} \overset{\nu}{\nabla}_Z [\overline{t}(\omega^1, \omega^2; X] - \overline{t}(\overset{\nu}{\nabla}_Z \omega^1, \omega^2; X) - \overline{t}(\omega^1, \overset{\nu}{\nabla}_Z \omega^2; X) - \overline{t}(\omega^1, \omega^2; \overset{\nu}{\nabla}_Z X) \\
&= \overset{\nu}{\nabla}_Z [(\overset{\mu}{\nabla}_Y t)(\omega^1, \omega^2; X] - (\overset{\mu}{\nabla}_Y t)(\overset{\nu}{\nabla}_Z \omega^1, \omega^2; X) - (\overset{\mu}{\nabla}_Y t)(\omega^1, \overset{\nu}{\nabla}_Z \omega^2; X) - (\overset{\mu}{\nabla}_Y t)(\omega^1, \omega^2; \overset{\nu}{\nabla}_Z X) \\
&\underset{(4)}{=} \overset{\nu}{\nabla}_Z \{\overset{\mu}{\nabla}_Y [t(\omega^1, \omega^2; X] - t(\overset{\mu}{\nabla}_Y \omega^1, \omega^2; X) - t(\omega^1, \overset{\mu}{\nabla}_Y \omega^2; X) - t(\omega^1, \omega^2; \overset{\mu}{\nabla}_Y X)\} \\
&- \{\overset{\mu}{\nabla}_Y [t(\overset{\nu}{\nabla}_Z \omega^1, \omega^2; X] - t(\overset{\mu}{\nabla}_Y \overset{\nu}{\nabla}_Z \omega^1, \omega^2; X) - t(\overset{\nu}{\nabla}_Z \omega^1, \overset{\mu}{\nabla}_Y \omega^2; X) - t(\overset{\nu}{\nabla}_Z \omega^1, \omega^2; \overset{\mu}{\nabla}_Y X)\} \\
&- \{\overset{\mu}{\nabla}_Y [t(\omega^1, \overset{\nu}{\nabla}_Z \omega^2; X] - t(\overset{\mu}{\nabla}_Y \omega^1, \overset{\nu}{\nabla}_Z \omega^2; X) - t(\omega^1, \overset{\mu}{\nabla}_Y \overset{\nu}{\nabla}_Z \omega^2; X) - t(\omega^1, \overset{\nu}{\nabla}_Z \omega^2; \overset{\mu}{\nabla}_Y X)\} \\
&- \{\overset{\mu}{\nabla}_Y [t(\omega^1, \omega^2; \overset{\nu}{\nabla}_Z X] - t(\overset{\mu}{\nabla}_Y \omega^1, \omega^2; \overset{\nu}{\nabla}_Z X) - t(\omega^1, \overset{\mu}{\nabla}_Y \omega^2; \overset{\nu}{\nabla}_Z X) - t(\omega^1, \omega^2; \overset{\mu}{\nabla}_Y \overset{\nu}{\nabla}_Z X)\}.
\end{aligned}$$

wherefrom

$$\begin{aligned}
(4.1) \qquad (\overset{\nu}{\nabla}_Z \overset{\mu}{\nabla}_Y \underset{1}{\overset{2}{t}} - \overset{\mu}{\nabla}_Y \overset{\nu}{\nabla}_Z \underset{1}{\overset{2}{t}})(\omega^1, \omega^2; X) &= [Z, Y][\underset{1}{\overset{2}{t}}(\omega^1, \omega^2; X)] \\
&= -\underset{1}{\overset{2}{t}}(\overset{\nu}{\nabla}_Z \overset{\mu}{\nabla}_Y \omega^1 - \overset{\mu}{\nabla}_Y \overset{\nu}{\nabla}_Z \omega^1, \omega^2; X) \\
&\quad - \underset{1}{\overset{2}{t}}(\omega^1, \overset{\nu}{\nabla}_Z \overset{\mu}{\nabla}_Y \omega^2 - \overset{\mu}{\nabla}_Y \overset{\nu}{\nabla}_Z \omega^2; X) \\
&\quad - \underset{1}{\overset{2}{t}}(\omega^1, \omega^2; \overset{\nu}{\nabla}_Z \overset{\mu}{\nabla}_Y X - \overset{\mu}{\nabla}_Y \overset{\nu}{\nabla}_Z X).
\end{aligned}$$

4.3. In the general case, starting from

$$(\overset{\mu}{\nabla}_Y \overset{r}{\underset{s}{t}})(\omega^1,\dots,\omega^r;X_1,\dots,X_s)=\overset{\mu}{\nabla}_Y[\overset{r}{\underset{s}{t}}(\omega^1,\dots,\omega^r;X_1,\dots,X_s)] \tag{4.2}$$
$$-\sum_{i=1}^{r}\overset{r}{\underset{s}{t}}(\omega^1,\dots,\omega^{i-1},\overset{\mu}{\nabla}_Y\omega^i,\omega^{i+1},\dots,\omega^r;X_1,\dots,X_s)$$
$$-\sum_{j=1}^{s}\overset{r}{\underset{s}{t}}(\omega^1,\dots,\omega^r;X_1,\dots,X_{j-1},\overset{\mu}{\nabla}_Y X_j,X_{j+1},\dots,X_s),$$

we get

$$(\overset{\nu}{\nabla}_Z\overset{\mu}{\nabla}_Y\overset{r}{\underset{s}{t}}-\overset{\mu}{\nabla}_Y\overset{\nu}{\nabla}_Z\overset{r}{\underset{s}{t}})(\omega^1,\dots,\omega^r;X_1,\dots,X_s)=[Z,Y][\overset{s}{\underset{r}{t}}(\omega^1,\dots,\omega^r;X_1,\dots,X_s)]$$
$$-\sum_{i=1}^{r}\overset{r}{\underset{s}{t}}(\omega^1,\dots,\omega^{i-1},\overset{\nu}{\nabla}_Z\overset{\mu}{\nabla}_Y\omega^i-\overset{\mu}{\nabla}_Y\overset{\nu}{\nabla}_Z\omega^i,\omega^{i+1},\dots,\omega^r;X_1,\dots,X_s) \tag{4.3}$$
$$-\sum_{j=1}^{s}\overset{r}{\underset{s}{t}}(\omega^1,\dots,\omega^r;X_1,\dots,X_{j-1},\overset{\nu}{\nabla}_Z\overset{\mu}{\nabla}_Y X_j-\overset{\mu}{\nabla}_Y\overset{\nu}{\nabla}_Z X_j,X_{j+1},\dots,X_s).$$

Consider some particular cases, obtained from (4.2). For example:

1) For $\mu=\nu=1, r=1, s=0$ that is for $\overset{1}{\underset{0}{t}}=X$, from (2.4) corresponding identity is obtained, i.e., in coordinates, (2.10), and for $r=1,\ s=0$ it follows (2.11), respectively (2.12)
2) For $\mu=\nu=2,\ r=1,\ s=0$ analogous relation (2.13), and equations corresponding to (2.11) and (2.12) are obtained.
3) For $\mu=1,\ \nu=2,\ r=1,\ s=0$ we obtain (3.1) and for $r=0,\ s=1$ the corresponding equation follows, where the roles of X and ω are exchanged.
4) For $r=2,\ s=1$, relation (4.1) follows.

4.4. Identities (4.3) can be written so that in them curvature tensors figure explicitly. For example, for $\mu=\nu=r=s=1$, $\overset{1}{\nabla}\equiv\nabla$ we have

$$(\nabla_Z\nabla_Y\overset{1}{\underset{1}{t}}-\nabla_Y\nabla_Z\overset{1}{\underset{1}{t}})(\omega;X)=[Z,Y][\overset{1}{\underset{1}{t}}(\omega;X)]$$
$$-\overset{1}{\underset{1}{t}}(\nabla_Z\nabla_Y\omega-\nabla_Y\nabla_Z\omega;X)-\overset{1}{\underset{1}{t}}(\omega;\nabla_Z\nabla_Y X-\nabla_Y\nabla_Z X)$$
$$=\nabla_{[Z,Y]}[\overset{1}{\underset{1}{t}}(\omega;X)]-\overset{1}{\underset{1}{t}}(\overline{\overset{1}{R}}(\omega;Y,Z)-\nabla_{[Y,Z]}\omega;X)-\overset{1}{\underset{1}{t}}(\omega;\overset{1}{R}(X;Y,Z)-\nabla_{[Y,Z]}X)$$
$$=\nabla_{[Z,Y]}[\overset{1}{\underset{1}{t}}(\omega;X)]-\overset{1}{\underset{1}{t}}(\overline{\overset{1}{R}}(\omega;Y,Z);X)+\overset{1}{\underset{1}{t}}(\nabla_{[Y,Z]}\omega;X)-\overset{1}{\underset{1}{t}}(\omega;\overset{1}{R}(X;Y,Z))+\overset{1}{\underset{1}{t}}(\omega;\nabla_{[Y,Z]}X).$$

In view of (4.2) it is

$$(\nabla_{[Z,Y]}\overset{1}{\underset{1}{t}})(\omega;X)=\nabla_{[Z,Y]}[\overset{1}{\underset{1}{t}}(\omega;X)]+\overset{1}{\underset{1}{t}}(\nabla_{[Y,Z]}\omega;X)+\overset{1}{\underset{1}{t}}(\omega;\nabla_{[Y,Z]}X),$$

the previous equation gives the identity

$$
\begin{aligned}
(4.4)\qquad &(\overset{1}{\nabla}_Z\overset{1}{\nabla}_Y \underset{1}{\overset{1}{t}} - \overset{1}{\nabla}_Y\overset{1}{\nabla}_Z \underset{1}{\overset{1}{t}})(\omega; X)\\
&\quad = (\overset{1}{\nabla}_{[Z,Y]}\underset{1}{\overset{1}{t}})(\omega; X)] - \underset{1}{\overset{1}{t}}(\overline{\overset{1}{R}}(\omega; Y, Z); X) - \underset{1}{\overset{1}{t}}(\omega; \overset{1}{R}(X; Y, Z)).
\end{aligned}
$$

Herefrom, in the local coordinates one obtains

$$
t^i_{j\underset{1}{|}kl} - t^i_{j\underset{1}{|}lk} = \overset{1}{R}{}^i_{pkl} t^p_j - \overset{1}{R}{}^p_{jkl} t^i_p - T^p_{kl} t^i_{j\underset{1}{|}p}.
$$

Finally, from the exposed, the following theorem is valid.

THEOREM 4.1. *In the space* L_N *with two non-symmetric connections* $\overset{1}{\nabla}$, $\overset{2}{\nabla}$, *linked with equation* (1.1)*, equation* (4.3) *represents general Ricci type identity for a tensor* $\underset{s}{\overset{r}{t}}$ *of the type* (r, s)*. The equations obtained previously for a vector and a covector, also* (4.4)*, are particular cases of* (4.3)*.*

In (4.3) we see how the quantities $\overset{\theta}{R}$ and $\overline{\overset{\theta}{R}}$ can be introduced.

References

1. L. S. Das, R. Nivas, S. Ali, M. Ahmad, *Study of submanifolds immersed in a manifolds with quater symmetric semi symmetric connection*, Tensor N.S. **65** (2004), 250–260.
2. T. Imai, *Notes on semi-symmetric metric connections*, Tensor N.S. **24** (1972), 293-296.
3. S. M. Minčić, *Ricci identities in the space of non-symmetric affine connexion*, Mat. Vesnik **10(25)**(2) (1973), 161–172.
4. ______, *On curvature tensors and pseudotensors of the space with non-symmetric affine connection*, Math. Balkan. **??** (1974), 427–430 (in Russian).
5. ______, *Curvature tensors of the space of non-symmetric affine connexion, obtained from the curvature pseudotensors*, Mat. Vesnik **13(28)** (1976), 421–435.
6. ______, *New commutation formulas in the non-symetric affine connexion space*, Publ. Inst. Math. (Beograd) (N.S) **22(36)** (1977), 189–199.
7. ______, *Independent curvature tensors and pseudotensors of spaces with non-symmetric affine connexion*, Coll. Math. Soc. János Bolyai, 31 Dif. Geom., Budapest (Hungary), (1979), 445–460.
8. ______, *On curvature tensors of non-symmetric affine connection*, Acta Comment. Univ. Tartu. Math. **9** (2005), 13–20.
9. ______, *Some characteristics of curvature tensors of nonsymmetric affine connexion*, Novi Sad J. Math. **29**(3) (1999), 169–186.
10. S. M. Minčić, Lj. S. Velimirović, *Differential Geometry of Manifolds*, Faculty of Science and Mathematics, Univ. of Niš, 2011 (in Serbian).
11. C. Nitescu, *Bianchi identities in a non symmetric connection space*, Bul. Inst. Polit. Iasi Sec.1, Mat. Mec. teor. Fiz. **20(24)**(1-2) (1974),
12. H. D. Pande, *Decomposition of curvature tensor in* $R - \oplus$ *recurent Finsler space with non-symmetric connection (I)*, Prog. Math. **22**(1,2) (1988), 61–68.
13. M. Prvanović, *Une connexion non-symmetrique associé á l' éspace Riemannien*, Publ. Inst. Math. **10(24)** (1970), 53–64.
14. ______, *Four curvature tensors of non-symmetric affine connexion*, Proc. Conf. “150 years of Lobachevski geometry", Kazan 1976, Moscow 1997, 199–205 (in Russian).
15. U. P. Singh, *On relative curvature tensors in the subspace of a Riemannian space*, Revne Fac. Sci. Instambul **33** (1968), 69–75.

16. K. Yano, *On semi-symmetric metric connection*, Revne Roum. Math. Pures Appl. **15** (1970), 1579–1581.
17. N. Youssef, A. Sid-Ahmed, *Linear connection and curvature tensors of the geometry of parallelisable manifolds*, Reports Math. Phys. **60**(1) (2007), 39.
18. S. M. Minčić, *On curvature tensors obtained by two non-symmetric affine connections*, submitted.

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