PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 93 (107) (2013), 69–93 DOI: 10.2298/PIM1307069N

# FIXED POINT THEOREMS VIA VARIOUS CYCLIC CONTRACTIVE CONDITIONS IN PARTIAL METRIC SPACES

**Hemant Kumar Nashine, Zoran Kadelburg, and Stojan Radenović**

*Communicated by Stevan Pilipović*

Abstract. We present some fixed point results for mappings which satisfy Hardy–Rogers rational type, quasicontraction type, weak contraction type and generalized $f_\psi$ type cyclic conditions in 0-complete partial metric spaces. Presented results generalize or improve many existing fixed point theorems in the literature. To demonstrate our results, we give throughout the paper some examples. One of the possible applications of our results to well-posed and limit shadowing property of fixed point problems is also presented.

## 1. Introduction

The Banach Contraction Principle is a very popular tool in solving existence problems in many branches of Mathematical Analysis and its applications. It is no surprise that there is a great number of generalizations of this fundamental theorem. They go in several directions—modifying the basic contractive condition or changing the ambiental space.

Concerning the first direction we mention Hardy–Rogers and Ćirić quasicontraction type conditions (see [**40**]), so called weakly contractive conditions of Alber and Guerre-Delabriere [**6**] and Rhoades [**41**], and altering distance functions used by Khan et al. [**23**] and Boyd and Wong [**10**].

Cyclic representations and cyclic contractions were introduced by Kirk et al. [**25**] and further used by several authors to obtain various fixed point results for not necessarily continuous mappings (see, e.g., [**12**, **20**, **22**, **32**]).

On the other hand, Matthews [**26**] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks.

2010 *Mathematics Subject Classification:* Primary 47H10; Secondary 54C60, 54H25, 55M20.
*Key words and phrases:* cyclic contraction; partial metric space; control function.
The second and third authors are thankful to the Ministry of Education, Science and Technological Development of Serbia.

In partial metric spaces, self-distance of an arbitrary point need not be equal to zero. Several authors obtained many useful fixed point results in these spaces—we mention just [**7**, **11**, **15**, **29**, **31**, **42**, **44**].

Some results for cyclic contractions in partial metric spaces were very recently obtained in [**1**, **2**, **5**, **8**, **28**].

In this paper, we introduce various types of cyclic contraction conditions, named as Hardy–Rogers rational type cyclic contraction, quasicontraction type cyclic contraction, weak contraction type cyclic contraction, and cyclic generalized $f_\psi$-contraction in partial metric spaces and develop new fixed point results for such cyclic contraction mappings in 0-complete partial metric spaces. Our results are extensions or refinements of recent fixed point theorems of Abbas et al. [**1**], Agarwal et al. [**5**], di Bari and Vetro [**8**], Karapinar [**20**], Karapinar et al. [**22**], and some other papers. Examples are given to support the usability of the results and to show that some of these extensions are proper. At the end one of the possible applications of our results to well-posed and limit shadowing property of fixed point problems is also presented.

We note that in very recent papers [**14**, **16**, **43**], it was shown that in some cases partial metric fixed point results can be obtained directly from their standard metric counterparts. However, some conclusions important for applications of partial metrics in information sciences cannot be obtained in this way. For example, using the method from [**14**] one cannot conclude that $p(x,x) = 0 = p(fx,fx)$ when $x$ is a fixed point of $f$; and using methods of [**43**] 0-completeness cannot be used. Moreover, some of our results are new even in the standard metric context, and we decided to treat uniformly all cases.

## 2. Preliminaries

A very powerful tool in solving existence problems in many branches of analysis is the Banach fixed point theorem (or Banach's contraction principle), which assures that every contraction from a complete metric space into itself has a unique fixed point. Recall that a self-mapping $T : X \to X$, where $(X,d)$ is a metric space, is said to be a contraction if there exists $0 < k < 1$ such that for all $x, y \in X$,

$$d(Tx,Ty) \leqslant kd(x,y). \tag{2.1}$$

Inequality (2.1) implies continuity of $T$. A natural question is whether we can find contractive conditions which will imply the existence of a fixed point in a complete metric space but will not imply continuity.

One of the remarkable generalizations of the Banach's contraction principle was reported by Kirk, Srinivasan and Veeramani [**25**] via cyclic contraction.

Definition 2.1. [**25**] Let $X$ be a nonempty set, $m \in \mathbb{N}$ and let $f : X \to X$ be a self-mapping. Then $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of $X$ with respect to $f$ if

(a) $A_i$, $i = 1, \dots, m$ are nonempty subsets of $X$;
(b) $f(A_1) \subset A_2$, $f(A_2) \subset A_3, \dots, f(A_{m-1}) \subset A_m$, $f(A_m) \subset A_1$.

They proved the following fixed point result.

Theorem 2.1. [**25**] *Let $(X,d)$ be a complete metric space, $f : X \to X$ and let $X = \bigcup_{i=1}^{m} A_i$ be a cyclic representation of $X$ with respect to $f$. Suppose that $f$ satisfies the following condition*

$$d(fx, fy) \leqslant \psi(d(x,y)), \text{ for all } x \in A_i,\ y \in A_{i+1},\ i \in \{1,2,\dots,m\},$$

*where $A_{m+1} = A_1$ and $\psi : [0,1) \to [0,1)$ is a function, upper semi-continuous from the right and $0 \leqslant \psi(t) < t$ for $t > 0$. Then, $f$ has a fixed point $z \in \bigcap_{i=1}^{m} A_i$.*

Notice that although a contraction is continuous, cyclic contraction need not be. This is one of the important gains of this theorem.

In 2010, Păcurar and Rus introduced the following notion of cyclic weaker $\varphi$-contraction.

Definition 2.2. [**32**] Let $(X,d)$ be a metric space, $m \in N$, $A_1, A_2, \dots, A_m$ be closed nonempty subsets of $X$ and $X = \bigcup_{i=1}^{m} A_i$. An operator $f : X \to X$ is called a cyclic weaker $\varphi$-contraction if

(1) $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of $X$ with respect to $f$;

(2) there exists a continuous, nondecreasing function $\varphi : [0,1) \to [0,1)$ with $\varphi(t) > 0$ for $t \in (0,1)$ and $\varphi(0) = 0$ such that $d(fx,fy) \leqslant d(x,y) - \varphi(d(x,y))$, for any $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\dots,m$, where $A_{m+1} = A_1$.

They proved the following result.

Theorem 2.2. [**32**] *Suppose that $f$ is a cyclic weaker $\varphi$-contraction on a complete metric space $(X,d)$. Then, $f$ has a fixed point $z \in \bigcap_{i=1}^{m} A_i$.*

Recently, Petric [**33**] established metrical fixed point theorems for some contractive orbital mappings involving a cyclic condition.

The following definitions and properties can be seen, e.g., in [**7, 11, 15, 26, 31, 42, 44**].

Definition 2.3. A partial metric on a nonempty set $X$ is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

($\mathrm{p}_1$) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
($\mathrm{p}_2$) $p(x,x) \leqslant p(x,y)$,
($\mathrm{p}_3$) $p(x,y) = p(y,x)$,
($\mathrm{p}_4$) $p(x,y) \leqslant p(x,z) + p(z,y) - p(z,z)$.

The pair $(X,p)$ is called a partial metric space.

It is clear that, if $p(x,y) = 0$, then from ($\mathrm{p}_1$) and ($\mathrm{p}_2$) $x = y$. But if $x = y$, $p(x,y)$ may not be 0.

Each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$ which has as a base the family of open $p$-balls $\{B_p(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in $(X,p)$ converges to a point $x \in X$ (in the sense of $\tau_p$) if $\lim_{n\to\infty} p(x,x_n) = p(x,x)$. This will be denoted as $x_n \to x$ $(n \to \infty)$ or $\lim_{n\to\infty} x_n = x$. Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(\cdot,\cdot)$ need not be continuous in the sense that $x_n \to x$ and $y_n \to y$ imply $p(x_n,y_n) \to p(x,y)$.

If $p$ is a partial metric on $X$, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on $X$. It is called the associated metric with the partial metric $p$.

EXAMPLE 2.1. (1) A paradigmatic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x,y \in \mathbb{R}^+$. The associated metric is

$$p^s(x,y) = 2\max\{x,y\} - x - y = |x-y|.$$

(2) If $(X,d)$ is a metric space and $c \geqslant 0$ is arbitrary, then $p(x,y) = d(x,y) + c$ defines a partial metric on $X$ and the corresponding metric is $p^s(x,y) = 2d(x,y)$.

Other examples of partial metric spaces which are interesting from the computational point of view may be found in [**13**, **26**].

DEFINITION 2.4. Let $(X,p)$ be a partial metric space. Then:

(1) A sequence $\{x_n\}$ in $(X,p)$ is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and is finite). The space $(X,p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges, with respect to $\tau_p$, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.
(2) [**42**] A sequence $\{x_n\}$ in $(X,p)$ is called 0-Cauchy if $\lim_{n,m\to\infty} p(x_n,x_m)=0$. The space $(X,p)$ is said to be 0-complete if every 0-Cauchy sequence in $X$ converges (in $\tau_p$) to a point $x \in X$ such that $p(x,x) = 0$.

LEMMA 2.1. *Let $(X,p)$ be a partial metric space.*

(a) [**4**, **21**] *If $p(x_n,z) \to p(z,z) = 0$ as $n \to \infty$, then $p(x_n,y) \to p(z,y)$ as $n \to \infty$ for each $y \in X$.*
(b) [**42**] *If $(X,p)$ is complete, then it is 0-complete.*

The converse assertion of (b) does not hold as the following easy example shows.

EXAMPLE 2.2. [**42**] The space $X = [0,+\infty) \cap \mathbb{Q}$ with the partial metric $p(x,y) = \max\{x,y\}$ is 0-complete, but is not complete. Moreover, the sequence $\{x_n\}$ with $x_n = 1$ for each $n \in \mathbb{N}$ is a Cauchy sequence in $(X,p)$, but it is not a 0-Cauchy sequence.

It is easy to see that every closed subset of a 0-complete partial metric space is 0-complete.

Let $(X,d)$ be a metric space and $f : X \to X$ be a mapping. Then it is said that $f$ satisfies the orbital condition if there exists a constant $k \in (0,1)$ such that

$$d(fx, f^2x) \leqslant kd(x,fx), \tag{2.2}$$

for all $x \in X$.

Condition (2.2) in metric spaces was used in [**18**] (the term was introduced in [**2**], see also [**3**]). The respective result in the partial metric case can be stated as follows:

THEOREM 2.3. *Let $(X, p)$ be a 0-complete partial metric space and $f : X \to X$ be continuous such that*

$$p(fx, f^2x) \leqslant k\, p(x, fx) \tag{2.3}$$

*holds for all $x \in X$, where $k \in (0, 1)$. Then there exists $z \in X$ such that $p(z, z) = 0$ and $p(Tz, z) = p(Tz, Tz)$.*

Recall that (see [**18**]) a map $f : X \to X$ is said to have property (P) if it satisfies $\mathrm{Fix}(f) = \mathrm{Fix}(f^n)$ for each $n \in \mathbb{N}$ (here, $X$ is a nonempty set and $\mathrm{Fix}(f)$ is the set of fixed points of $f$). It is known that the orbital condition implies property $(P)$ in metric spaces [**18**], as well as in partial metric space [**19**]. We state explicitly the last result:

LEMMA 2.2. *Let $(X, p)$ be a partial metric space, $f : X \to X$ be a selfmap such that* $\mathrm{Fix}(f) \neq \emptyset$. *Then $f$ has the property $(P)$ if* (2.3) *holds for some $k \in (0, 1)$ and either* (i) *for all $x \in X$, or* (ii) *for all $x \neq fx$.*

Recently, Di Bari and Vetro [**8**] proved fixed point theorems for cyclic weaker $\varphi$-contractions in partial metric spaces. Karapinar, Erhan and Ulus [**22**] also proved fixed point results using weakly contractive type cyclic contractive condition.

**2.1. Contractive conditions.** Let $(X, p)$ be a partial metric space and $f : X \to X$ be a selfmap. When constructing various contractive conditions, usually one of the following sets is used:

$$\begin{aligned}
M_f^5(x, y) &= \big\{p(x, y), p(x, fx), p(y, fy), p(x, fy), p(y, fx)\big\},\\
M_f^4(x, y) &= \big\{p(x, y), p(x, fx), p(y, fy), \tfrac{1}{2}(p(x, fy) + p(y, fx))\big\},\\
M_f^3(x, y) &= \big\{p(x, y), \tfrac{1}{2}(p(x, fx) + p(y, fy)), \tfrac{1}{2}(p(x, fy) + p(y, fx))\big\}.
\end{aligned}$$

Then, the contractive condition takes the form

$$p(fx, fy) \leqslant \lambda \max M_f^j(x, y), \quad x, y \in X, \tag{2.4}$$

where $\lambda \in [0, 1)$ (in some cases $\lambda \in [0, \frac{1}{2})$) and $j \in \{3, 4, 5\}$. Mappings $f$ satisfying (2.4) with $j = 5$ for all $x, y \in X$ (in metric case) are usually called quasicontractions (Ćirić, see relation (24) in [**40**]).

On the other hand, Hardy–Rogers type contractive conditions (relation (18) in [**40**]) use the expression

$$\Theta_f(x, y) = Ap(x, y) + Bp(x, fx) + Cp(y, fy) + Dp(x, fy) + Ep(y, fx),$$

where nonnegative constants $A, B, C, D, E$ satisfy some additional condition, depending on the concrete situation. The contractive condition is of the form

$$p(fx, fy) \leqslant \Theta_f(x, y).$$

Weak contractive conditions in Banach spaces were first used by Rhoades in [**41**]. Subsequently, a lot of variations of these conditions were introduced (see, e.g., results in metric spaces in [**38, 45**] and in partial metric spaces in [**27**]). These conditions are (in the metric case) of the form $d(fx, fy) \leqslant m(x, y) - \varphi(m(x, y))$, where $m(x, y)$ depends in a certain way on the elements of the set $M_f^5(x, y)$. Here

$\varphi : [0, +\infty) \to [0, +\infty)$ is a lower-semicontinuous function such that $\varphi(t) = 0$ if and only if $t = 0$. Sometimes an additional function $\psi$ is introduced, but it was showed [**17**] that its usage can be avoided.

## 3. Hardy–Rogers rational type cyclic contractions in partial metric spaces

We will prove some fixed point theorems for self-mappings defined on a 0-complete partial metric space and satisfying certain Hardy–Rogers rational type cyclic contractive conditions. To achieve our goal, we introduce the notion of a Hardy–Rogers rational type cyclic contraction.

DEFINITION 3.1. Let $(X, p)$ be a partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty subsets of $X$ and $X = \bigcup_{i=1}^m A_i$. An operator $f : X \to X$ is called a Hardy–Rogers rational type cyclic contraction if:

(1) $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of $X$ with respect to $f$ and $A_{m+1} = A_1$;

(2) there exist nonnegative constants $A, B, C, D, E, F$ with

$$A + B + C + D + E + F < 1 \tag{3.1}$$

such that

$$p(fx, fy) \leqslant \Phi_f(x, y), \tag{3.2}$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$, where

$$\begin{aligned}\Phi_f(x, y) = Ap(x, y) + Bp(x, fx) + Cp(y, fy) + Dp(x, fy)\\ + Ep(y, fx) + F\frac{p(x, fx) \cdot p(y, fy)}{1 + p(x, y)}.\end{aligned}$$

The main result of this section is the following:

THEOREM 3.1. *Let $(X, p)$ be a 0-complete partial metric space, $m \in \mathbb{N}$, $A_1$, $A_2, \ldots, A_m$ be nonempty closed subsets of $(X, p)$ and $Y = \bigcup_{i=1}^m A_i$. Suppose that $f : Y \to Y$ is a Hardy–Rogers rational type cyclic contraction. Then, $f$ has a unique fixed point $z \in Y$. Moreover, $p(z, z) = 0$ and $z \in \bigcap_{i=1}^m A_i$. Each Picard sequence $x_n = f^n x_0$, $x_0 \in Y$ converges to $z$ in topology $\tau_p$.*

PROOF. Let $x_0$ be an arbitrary point of $Y$. Then there exists some $i_0$ such that $x_0 \in A_{i_0}$. Now $x_1 = fx_0 \in A_{i_0+1}$ and, similarly, $x_n := fx_{n-1} = f^n x_0 \in A_{i_0+n}$ for $n \in \mathbb{N}$, where $A_{m+k} = A_k$. In the case $p(x_{n_0}, x_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, it is clear that $x_{n_0}$ is a fixed point of $f$. Now assume that $p(x_n, x_{n+1}) > 0$ for all $n$. Since $f : Y \to Y$ is a Hardy–Rogers rational type cyclic contraction, we have that for all $n \in \mathbb{N}$

$$\begin{aligned}&p(x_n, x_{n+1})\\ &\quad= p(fx_{n-1}, fx_n) \leqslant Ap(x_{n-1}, x_n) + Bp(x_{n-1}, x_n) + Cp(x_n, x_{n+1})\\ &\qquad+ Dp(x_{n-1}, x_{n+1}) + Ep(x_n, x_n) + F\frac{p(x_{n-1}, x_n) \cdot p(x_n, x_{n+1})}{1 + p(x_{n-1}, x_n)}\\ &\quad\leqslant (A + B + D)p(x_{n-1}, x_n) + (C + D + F)p(x_n, x_{n+1}) + (E - D)p(x_n, x_n)\end{aligned}$$

(it was used that $p(x_{n-1},x_n) < 1 + p(x_{n-1},x_n)$). Similarly, starting from $p(x_{n+1},x_n) = p(fx_n, fx_{n-1})$,

$$p(x_{n+1},x_n) \leqslant (A+C+E)p(x_n,x_{n-1}) + (B+E+F)p(x_{n+1},x_n) + (D-E)p(x_n,x_n)$$

is obtained. Adding up, it follows that $p(x_{n+1},x_n) \leqslant \lambda p(x_n,x_{n-1})$, with $\lambda = \frac{2A+B+C+D+E}{2-B-C-D-E-2F} < 1$, which is a consequence of (3.1).

It follows that

$$p(x_n,x_{n+1}) \leqslant \lambda^n p(x_0,x_1) \quad \text{and} \quad \lim_{n\to\infty} p(x_n,x_{n+1}) = 0. \tag{3.3}$$

Then $p(x_n,x_n) \leqslant p(x_n,x_{n+1})$ implies that $\lim_{n\to\infty} p(x_n,x_n) = 0$. Also, for $n > m$,

$$p(x_n,x_m) \leqslant (\lambda^m + \cdots + \lambda^{n-1})p(x_0,x_1),$$

and so $\lim_{m,n\to\infty} p(x_n,x_m) = 0$. Hence, $\{x_n\}$ is a 0-Cauchy sequence. Since $Y$ is closed in $(X,p)$, then $(Y,p)$ is also 0-complete and there exists $z \in Y = \bigcup_{i=1}^m A_i$ such that $\lim_{n\to\infty} p(x_n,z) = 0 = p(z,z)$. Notice that the iterative sequence $\{x_n\}$ has an infinite number of terms in $A_i$ for each $i = 1,\dots,m$. Hence, in each $A_i$, $i = 1,\dots,m$, we can construct a subsequence of $\{x_n\}$ that converges to $z$. Using that each $A_i$, $i = 1,\dots,m$, is closed, we conclude that $z \in \bigcap_{i=1}^m A_i$ and thus $\bigcap_{i=1}^m A_i \neq \emptyset$.

In order to prove that $z$ is a fixed point of $f$, use ($\mathrm{p}_4$) and (3.2) (which is possible since $z$ belongs to each $A_i$) to obtain

$$\begin{aligned} p(z,fz) &\leqslant p(z,x_{n+1}) + p(x_{n+1},fz) - p(x_{n+1},x_{n+1}) \\ &\leqslant p(z,x_{n+1}) + p(fx_n,fz) \\ &\leqslant p(z,x_{n+1}) + Ap(x_n,z) + Bp(x_n,x_{n+1}) + Cp(z,fz) + Dp(x_n,fz) \\ &\qquad + Ep(x_{n+1},z) + F\frac{p(x_n,x_{n+1})\cdot p(z,fz)}{1+p(x_n,z)}. \end{aligned} \tag{3.4}$$

Using Lemma 2.1.(a) and passing to the limit when $n \to \infty$ in (3.4), we obtain that

$$(1-C-D)p(z,fz) \leqslant 0,$$

and hence $p(z,fz) = 0$. We have proved that $fz = z$ and $p(z,z) = 0$.

Finally, to prove the uniqueness of the fixed point, let $u$ be another fixed point of $f$ in $Y$, with $p(u,z) \neq 0$. By the cyclic character of $f$, we have $u,z \in \bigcap_{i=1}^m A_i$. Since $f$ is a Hardy–Rogers rational type cyclic contraction, we have

$$\begin{aligned} p(u,z) = p(fu,fz) &\leqslant Ap(u,z) + Bp(u,u) + Cp(z,z) + Dp(u,z) \\ &\qquad + Ep(u,z) + F\frac{p(u,fu)\cdot p(z,fz)}{1+p(u,z)} \\ &\leqslant (A+B+C+D+E)p(u,z) < p(u,z), \end{aligned}$$

a contradiction. It follows that $p(u,z) = 0$ and $u = z$. Thus $z$ is a unique fixed point of $f$. □

As corollaries we obtain partial metric versions of well-known Banach [**26**], Kannan and Chatterjea fixed point results (relations (1), (4) and (11) in [**40**]) in the cyclic variant.

Corollary 3.1. *Let $(X,p)$ be a 0-complete partial metric space, $m\in\mathbb{N}$, $A_1$, $A_2,\dots,A_m$ be nonempty closed subsets of $(X,p)$ and $Y=\bigcup_{i=1}^m A_i$. Let $f:Y\to Y$ be such that:*

*(1) $Y=\bigcup_{i=1}^m A_i$ is a cyclic representation of $Y$ with respect to $f$;*

*(2) there exists $\lambda\in[0,1)$ such that one of the following conditions hold for all $x\in A_i$, $y\in A_{i+1}$, $i=1,2,\dots,m$:*

$$\begin{aligned}p(fx,fy)&\leqslant \lambda p(x,y),\\ p(fx,fy)&\leqslant \frac{\lambda}{2}\big[p(x,fx)+p(y,fy)\big],\\ p(fx,fy)&\leqslant \frac{\lambda}{2}\big[p(x,fy)+p(y,fx)\big].\end{aligned}$$

*where $A_{m+1}=A_1$. Then, $f$ has a unique fixed point $z\in Y$. Moreover, $p(z,z)=0$ and $z\in\bigcap_{i=1}^m A_i$.*

Example 3.1. Let $X=\mathbb{R}$ be equipped with the usual partial metric $p(x,y)=\max\{x,y\}$. The partial metric space $(X,p)$ is 0-complete. Suppose $A_1=[0,1]$, $A_2=[0,\frac12]$, $A_3=[0,\frac14]$, $A_4=[0,\frac18]$ and $Y=\bigcup_{i=1}^4 A_i=[0,1]$. Define $f:Y\to Y$ such that $fx=\frac{x}{2}$ for all $x\in Y$. It is clear that $\bigcup_{i=1}^4 A_i$ is a cyclic representation of $Y$ with respect to $f$.

Take $A=\frac12$, $B=C=0$ and $D=E=F=\frac18$, i.e.

$$\Phi_f(x,y)=\frac12 p(x,y)+\frac18 p(x,fy)+\frac18 p(y,fx)+\frac18\frac{p(x,fx)\cdot p(y,fy)}{1+p(x,y)}$$

(the condition (3.1) on coefficients is fulfilled). Consider the following cases:

1° $y\leqslant x$. Then $p(fx,fy)=\max\{\frac{x}{2},\frac{y}{2}\}=\frac{x}{2}$, and

$$\Phi_f(x,y)=\frac12 x+\frac18 x+\frac18\max\Big\{y,\frac{x}{2}\Big\}+\frac18\frac{x\cdot y}{1+x}\geqslant\frac58 x.$$

Hence, $p(fx,fy)=\frac12 x\leqslant\frac58 x\leqslant\Phi_f(x,y)$ is fulfilled.

2° $x<y$. Then $p(fx,fy)=\frac{y}{2}$ and

$$\Phi_f(x,y)=\frac12 y+\frac18\max\Big\{x,\frac{y}{2}\Big\}+\frac18 y+\frac18\frac{x\cdot y}{1+y}\geqslant\frac58 y.$$

Hence, $p(fx,fy)=\frac{y}{2}\leqslant\frac58 y\leqslant\Phi_f(x,y)$.

All conditions of Theorem 3.1 are satisfied and we deduce that $f$ has a unique fixed point $z=0\in A_1\cap A_2\cap A_3\cap A_4$ and $p(z,z)=0$ holds true.

In the next example we show that Theorem 3.1 can be used when its standard metric counterpart cannot be used.

Example 3.2. Let $X$, $p$, $A_i$ $(i=1,2,3,4)$ and $Y$ be as in the previous example. Consider the mapping $g:Y\to Y$ given by $gx=\frac{x^2}{1+x}$, and take $A=\frac23$, $B=C=D=E=F=0$, i.e., $\Phi_g(x,y)=\frac23 p(x,y)$. Then it is easy to see that $g$ is a Hardy–Rogers rational type cyclic contraction, since (for, say, $x\geqslant y$)

$$p(gx,gy)=\frac{x^2}{1+x}\leqslant\frac23 x=\Phi_g(x,y).$$

However, if we consider the same problem in the standard metric space $(X,d)$, where $d(x,y)=|x-y|=p^s(x,y)$, and take the respective $\Phi_g'(x,y)=\frac{2}{3}d(x,y)=\frac{2}{3}|x-y|$, then contractive condition (3.2) takes the form

$$d(gx,gy)=\frac{x^2}{1+x}-\frac{y^2}{1+y}\leqslant\frac{2}{3}(x-y).$$

Putting $x=1$, $y=1-\alpha$, $0<\alpha<1$, this reduces to $\frac{1}{2}-\frac{(1-\alpha)^2}{2-\alpha}\leqslant\frac{2}{3}\alpha$, which is easy shown to be equivalent to $\alpha(2\alpha-1)\geqslant 0$, which is impossible for $0<\alpha<\frac{1}{2}$.

We state a more involved example that is inspired with the one from [**34**].

EXAMPLE 3.3. Let $X\subset\ell^1$, $X\ni x=(x_n)_{n=1}^\infty$ iff $x_n\geqslant 0$ for each $n\in\mathbb{N}$. Define a partial metric $p$ on $X$ by $p((x_n),(y_n))=\sum_{n=1}^\infty\max\{x_n,y_n\}$ (it is easy to check that axioms $(p_1)$–$(p_4)$ hold true). Let $\alpha\in(0,1)$ be fixed, denote $\mathbf{0}=(0)_{n=1}^\infty$ and consider the subsets $A_1$ and $A_2$ of $X$ defined by $A_1=A'\cup\{\mathbf{0}\}$, $A_2=A''\cup\{\mathbf{0}\}$, where

$$A'\ni x^l=(x_n^l)_{n=1}^\infty\quad\text{iff}\quad x_n^l=\begin{cases}0, & n<2l\vee n=2k-1, k\in\mathbb{N},\\ \alpha^n, & n=2k\geqslant 2l\end{cases}\qquad l=1,2,\dots$$

and

$$A''\ni x^l=(x_n^l)_{n=1}^\infty\quad\text{iff}\quad x_n^l=\begin{cases}0, & n<2l-1\vee n=2k, k\in\mathbb{N},\\ \alpha^n, & n=2k-1\geqslant 2l-1\end{cases}\qquad l=1,2,\dots$$

Denote $Y=A_1\cup A_2$ (obviously $A_1\cap A_2=\{\mathbf{0}\}$).

Consider the mapping $f:Y\to Y$ given by:

$$\begin{gathered}f(\mathbf{0})=\mathbf{0},\\ f((\underbrace{0,\dots,0}_{2l-1},\alpha^{2l},0,\alpha^{2l+2},0,\dots))=(\underbrace{0,\dots,0}_{2l},\alpha^{2l+1},0,\alpha^{2l+3},0,\dots),\\ f((\underbrace{0,\dots,0}_{2l},\alpha^{2l+1},0,\alpha^{2l+3},0,\dots))=(\underbrace{0,\dots,0}_{2l+1},\alpha^{2l+2},0,\alpha^{2l+4},0,\dots).\end{gathered}$$

Obviously, $f(A_1)\subset A_2$ and $f(A_2)\subset A_1$, hence $Y=A_1\cup A_2$ is a cyclic representation of $Y$ with respect to $f$.

Take $A=\alpha$, $0\leqslant B=C=D=E=F<\frac{1-\alpha}{5}$. Then $A+B+C+D+E+F<1$. Let us check the contractive condition (3.2) of Theorem 3.1. Take $x=(\underbrace{0,\dots,0}_{2l-1},\alpha^{2l},0,\alpha^{2l+2},0,\dots)\in A_1$ and $y=(\underbrace{0,\dots,0}_{2m},\alpha^{2m+1},0,\alpha^{2m+3},0,\dots)\in A_2$ and assume, e.g., that $l\leqslant m$ (the case $l>m$ is treated similarly, as well as the case when $x$ or $y$ is equal to $\mathbf{0}$). Then

$$\begin{aligned}p(x,y)&=\alpha^{2l}+\dots+\alpha^{2m-2}+\frac{\alpha^{2m}}{1-\alpha},\\ p(fx,fy)&=\alpha^{2l+1}+\dots+\alpha^{2m-1}+\frac{\alpha^{2m+1}}{1-\alpha}.\end{aligned}$$

Hence,

$$p(fx,fy)=\alpha p(x,y)\leqslant Ap(x,y)+Bp(x,fx)+Cp(y,fy)$$
$$+Dp(x,fy)+Ep(y,fx)+F\frac{p(x,fx)p(y,fy)}{1+p(x,y)}.$$

Obviously, $f$ has a unique fixed point $\mathbf{0}$. Moreover, $p(\mathbf{0},\mathbf{0})=0$.

Another consequence of Theorem 3.1 is the following:

Theorem 3.2. *Let $(X,p)$ and $f$ satisfy conditions of Theorem 3.1. Then $f$ satisfies orbital condition (2.3). In particular, there exists $z\in Y$ such that $p(z,z)=0$ and $p(fz,z)=p(fz,fz)$; also, $f$ has the property $(P)$.*

Proof. By Theorem 3.1, the set $\mathrm{Fix}(f)$ of fixed points of $f$ is not empty. We will prove that $f$ satisfies condition (2.3) of Theorem 2.3. Let $x\in Y$ be arbitrary. Putting $x=x$ and $y=fx$ in condition (3.2) of Theorem 3.1, we get that

$$\begin{aligned}p(fx,f^2x)&\leqslant Ap(x,fx)+Bp(x,fx)+Cp(fx,f^2x)+Dp(x,f^2x)+Ep(fx,fx)\\&\quad+F\frac{p(x,fx)\cdot p(fx,f^2x)}{1+p(x,fx)}\\&\leqslant (A+B+D)p(x,fx)+(C+D+F)p(fx,f^2x)+(E-D)p(fx,fx),\end{aligned}$$

as $p(x,fx)\leqslant 1+p(x,fx)$ and, symmetrically,

$$\begin{aligned}p(f^2x,fx)&\leqslant Ap(fx,x)+Bp(fx,f^2x)+Cp(x,fx)+Dp(fx,fx)+Ep(x,f^2x)\\&\quad+F\frac{p(fx,f^2x)\cdot p(x,fx)}{1+p(x,fx)}\\&\leqslant (A+C+E)p(x,fx)+(B+E+F)p(fx,f^2x)+(D-E)p(fx,fx).\end{aligned}$$

Adding up, it follows that $p(fx,f^2x)\leqslant\frac{2A+B+C+D+E}{2-(B+C+D+E+2F)}p(x,fx)=\lambda p(x,fx)$, where $0\leqslant\lambda<1$ by assumption (3.1) of Theorem 3.1. Thus, $f$ satisfies the orbital condition. By Theorem 2.3, there exists $z\in Y$ such that $p(z,z)=0$ and $p(fz,z)=p(fz,fz)$. By Lemma 2.2, $f$ has the property $(P)$. □

## 4. Quasicontraction cyclic type mappings in partial metric spaces

Theorem 4.1. *Let $(X,p)$ be a 0-complete partial metric space, $m\in\mathbb{N}$, $A_1$, $A_2,\dots,A_m$ nonempty closed subsets of $X$, $Y=\bigcup_{i=1}^m A_i$ and $f:Y\to Y$. Suppose that:*

(a) *$Y=\bigcup_{i=1}^m A_i$ is a cyclic representation of $Y$ with respect to $f$;*
(b) *there exists $\lambda\in[0,1)$ such that for any $(x,y)\in A_i\times A_{i+1}$, $i=1,2,\dots,m$ (with $A_{m+1}=A_1$),*

$$p(fx,fy)\leqslant\lambda\max M_f^j(x,y)\tag{4.1}$$

*for $j=3$, $j=4$ or $j=5$.*

*Then $f$ has a unique fixed point $z$. Moreover, $p(z,z)=0$ and $z\in\bigcap_{i=1}^m A_i$.*

PROOF. We will prove the theorem in the case $j = 5$; the cases $j = 3$ and $j = 4$ will follow as consequences (note that, however, in these cases the proof could be shorter).

For arbitrary $x_0 \in Y$ (and so $x \in A_i$ for some $i \in \{1, \dots, m\}$), form the sequence $\{x_n\}$ in $Y$ by $x_{n+1} = fx_n$, $n \in \mathbb{N}_0$. Denote by $O_f(x_0; n) = \{x_1, x_2, \dots, x_n\}$ the $n$-th orbit of $x_0$ and by $O_f(x_0; \infty) = \{x_1, x_2, \dots\}$ its orbit. Also, denote by $\operatorname{diam} A = \sup\{\, p(x, y) \mid x, y \in A\,\}$ the diameter of a nonempty set $A \subset X$. Note that $\operatorname{diam} A = 0$ implies that $A$ is a singleton, but the converse is not true.

If $p(x_n, x_{n+1}) = 0$ for some $n \in \mathbb{N}_0$, it follows that $fx_n = x_{n+1} = x_n$, i.e., $x_n$ is a fixed point of $f$ satisfying $p(x_n, x_n) = 0$. Suppose further that $p(x_n, x_{n+1}) > 0$ for each $n \in \mathbb{N}_0$.

*Claim 1.* $\operatorname{diam} O_f(x_0; \infty) \leqslant \frac{1}{1-\lambda} p(x_1, x_2)$. Indeed, let $1 \leqslant i, j \leqslant n$. Then

$$
\begin{aligned}
(4.2) \quad p(x_{i+1}, x_{j+1}) &= p(fx_i, fx_j) \\
&= \lambda \max\big\{p(x_i, x_j), p(x_i, x_{i+1}), p(x_j, x_{j+1}), p(x_i, x_{j+1}), p(x_j, x_{i+1})\big\}.
\end{aligned}
$$

Since the points $x_i, x_{i+1}, x_j, x_{j+1}$ belong to the set $O_f(x_0; n)$, it follows that

$$p(x_{i+1}, x_{j+1}) \leqslant \lambda \operatorname{diam} O_f(x_0; n) < \operatorname{diam} O_f(x_0; n).$$

Hence, there exists $k \leqslant n$ such that $\operatorname{diam} O_f(x_0; n) = p(x_1, x_k)$. Since, by ($p_4$),

$$p(x_1, x_k) \leqslant p(x_1, x_2) + p(x_2, x_k) - p(x_2, x_2) \leqslant p(x_1, x_2) + p(x_2, x_k),$$

we have

$$\operatorname{diam} O_f(x_0; n) \leqslant p(x_1, x_2) + \lambda \operatorname{diam} O_f(x_0; n),$$

i.e., $\operatorname{diam} O_f(x_0; n) \leqslant \frac{1}{1-\lambda} p(x_1, x_2)$. Taking the supremum in this inequality, proof of Claim 1 is obtained.

*Claim 2.* Let $m > n \geqslant 1$. Then $p(x_{m+1}, x_{n+1}) \leqslant \frac{\lambda^n}{1-\lambda} p(x_1, x_2)$.

Similarly as in (4.2), we have that

$$
\begin{aligned}
p(x_{m+1}, x_{n+1}) \leqslant \lambda \max\big\{&p(x_m, x_n), p(x_m, x_{m+1}), p(x_n, x_{n+1}), \\
&p(x_m, x_{n+1}), p(x_n, x_{m+1})\big\}.
\end{aligned}
$$

Since $x_m, x_{m+1}, x_n, x_{n+1} \in O_f(x_{n-1}; m - n + 2)$, we have

$$(4.3) \qquad p(x_{m+1}, x_{n+1}) \leqslant \lambda \operatorname{diam} O_f(x_{n-1}; m - n + 2) = \lambda p(x_n, x_{k_1})$$

for some $k_1$, $n + 1 \leqslant k_1 \leqslant m + 1$. Now, similarly,

$$
\begin{aligned}
p(x_n, x_{k_1}) &\leqslant \lambda \max\big\{p(x_{n-1}, x_{k_1-1}), p(x_{n-1}, x_n), p(x_{k_1-1}, x_{k_1}), \\
&\qquad\qquad p(x_{n-1}, x_{k_1}), p(x_n, x_{k_1-1})\big\} \\
&\leqslant \lambda \operatorname{diam} O_f(x_{n-2}; m - n + 3),
\end{aligned}
$$

which, together with (4.3), gives $p(x_{m+1}, x_{n+1}) \leqslant \lambda^2 p(x_{n-1}, x_{k_2})$ for some $k_2 \leqslant m + 1$. Continuing the process, we obtain that

$$
\begin{aligned}
p(x_{m+1}, x_{n+1}) &\leqslant \lambda^{n-1} \operatorname{diam} O_f(x_0; m - 1) = \lambda^{n-1} p(x_1, y_{k_{n-1}}) \\
&\leqslant \lambda^{n-1} \cdot \lambda \operatorname{diam} O_f(x_0; m) \leqslant \frac{\lambda^n}{1-\lambda} p(x_1, x_2)
\end{aligned}
$$

and Claim 2 is proved.

It follows that $p(x_m, x_n) \to 0$ as $m, n \to \infty$, i.e., $\{x_n\}$ is a 0-Cauchy sequence. Since $(X, p)$ is 0-complete and $Y$ is closed, there exists $z \in Y$ such that

$$\lim_{n\to\infty} p(x_n, z) = 0 = p(z, z).$$

Moreover, $z \in \bigcap_{i=1}^m A_i$. Now we prove that also $fz = z$. Using property ($p_4$) and condition (4.1) with $j = 5$ (which is possible since $z$ belongs to each $A_i$), we have

$$\begin{aligned} p(fz, z) &\leqslant p(fz, fx_n) + p(fx_n, z) \\ &\leqslant \lambda \max\big\{p(z, x_n), p(z, fz), p(x_n, x_{n+1}), p(z, x_{n+1}), p(x_n, fz)\big\} + p(x_{n+1}, z). \end{aligned}$$

Since $p(z, x_n)$, $p(x_n, x_{n+1})$ and $p(z, x_{n+1})$ tend to 0 as $n \to \infty$, and since also $p(x_n, fz) \to p(z, fz)$ (by Lemma 2.1.(a)), if we suppose that $p(fz, z) > 0$, we get a contradiction $p(fz, z) \leqslant \lambda p(fz, z)$. Hence, $p(fz, z) = 0$ and so $fz = z$.

Suppose that there exists $z_1 \in Y$ such that $fz_1 = z_1$. Then

$$\begin{aligned} p(z, z_1) &= p(fz, fz_1) \\ &\leqslant \lambda \max\big\{p(z, z_1), p(z, fz), p(zv_1, fz_1), p(z, fz_1), p(z_1, fz)\big\} \\ &= \lambda \max\big\{p(z, z_1), p(z, z), p(z_1, z_1), p(z, z_1), p(z_1, z)\big\} \\ &= \lambda p(z, z_1) \qquad \text{by } (\mathrm{p}_2), \end{aligned}$$

which is possible only if $p(z, z_1) = 0$, and hence $z = z_1$. Thus, we have proved that the fixed point of $f$ is unique. □

According to the well-known classification of Rhoades [**40**] (which obviously holds for partial as well as for standard metric), Theorem 4.1 implies several other fixed point results, e.g., those of Banach, Kannan, Chatterjea, Bianchini, Hardy–Rogers and Zamfirescu. For example, we state Bianchini's fixed point result (relation (5) in [**40**]) in its cyclic variant.

Corollary 4.1. *Let $(X, p)$ be a 0-complete partial metric space, $m \in \mathbb{N}$, $A_1$, $A_2, \dots, A_m$ nonempty closed subsets of $X$, $Y = \bigcup_{i=1}^m A_i$ and $f : Y \to Y$. Suppose that:*

(a) *$Y = \bigcup_{i=1}^p A_i$ is a cyclic representation of $Y$ with respect to $f$;*
(b) *there exists $\lambda \in [0, 1)$ such that for any $(x, y) \in A_i \times A_{i+1}$, $i = 1, 2, \dots, m$ (with $A_{m+1} = A_1$),*

$$p(fx, fy) \leqslant \lambda \max\{p(x, fx), p(y, fy)\} \quad \text{or}$$
$$p(fx, fy) \leqslant \lambda \max\{p(x, y), p(x, fx), p(y, fy)\}.$$

*Then $f$ has a unique fixed point $z$. Moreover, $p(z, z) = 0$ and $z \in \bigcap_{i=1}^m A_i$.*

The following example illustrate the validity of Theorem 4.1.

Example 4.1. Let $X = [0, 1]$ and a partial metric $p : X \times X \to \mathbb{R}^+$ be given by

$$p(x, y) = \begin{cases} |x - y|, & \text{if } x, y \in [0, 1), \\ 1, & \text{if } x = 1 \text{ or } y = 1. \end{cases}$$

If a mapping $f : X \to X$ is given by

$$fx = \begin{cases} 1/2, & \text{if } x \in [0,1), \\ 1/6, & \text{if } x = 1, \end{cases}$$

and $A_1 = [0, \frac{1}{2}]$, $A_2 = [\frac{1}{2}, 1]$, then $A_1 \cup A_2 = X$ is a cyclic representation of $X$ with respect to $f$. Moreover, mapping $f$ is a cyclic contraction of type (4.1). Indeed, consider the following cases:

$1^\circ$ $x \in [0, \frac{1}{2}]$, $y \in [\frac{1}{2}, 1)$ or $y \in [0, \frac{1}{2}]$, $x \in [\frac{1}{2}, 1)$. Then $p(fx, fy) = p(\frac{1}{2}, \frac{1}{2}) = 0$ and relation (4.1) is trivially satisfied.

$2^\circ$ $x \in [0, \frac{1}{2}]$, $y = 1$ or $y \in [0, \frac{1}{2}]$, $x = 1$. Then $p(fx, fy) = p(\frac{1}{2}, \frac{1}{6}) = \frac{1}{3}$ and $\max M_f^5(x, y) = p(x, y) = 1$. Relation (4.1) holds for any $\lambda \in (\frac{1}{3}, 1)$.

Therefore, all conditions of Theorem 4.1 are satisfied (with $m = 2$), and so $f$ has a fixed point (which is $z = \frac{1}{2} \in \bigcap_{i=1}^{2} A_i$).

We give an easy example of a partial metric space which is not a metric space, and a selfmap in it which is a quasicontraction and not a contraction.

Example 4.2. Consider the set $X = \{a, b, c\} = \{b, c\} \cup \{a, b\} = A_1 \cup A_2$, equipped with the function $p : X \times X \to \mathbb{R}^+$ given by $p(a, b) = p(b, c) = 1$, $p(a, c) = \frac{3}{2}$, $p(x, y) = p(y, x)$, $p(a, a) = p(c, c) = \frac{1}{2}$ and $p(b, b) = 0$. Obviously, $p$ is a partial metric on $X$, not being a metric (since $p(x, x) \neq 0$ for $x = a$ or $x = c$). Define a selfmap $f$ on $X$ by $f : \left(\begin{smallmatrix} a & b & c \\ b & b & a \end{smallmatrix}\right)$. Then $A_1 \cup A_2 = X$ is a cyclic representation of $X$ with respect to $f$ and $f$ is not a (Banach)-contraction since $p(fc, fc) = p(a, a) = \frac{1}{2} = p(c, c)$ and there is no $\lambda \in [0, 1)$ such that $p(fc, fc) \leqslant \lambda p(c, c)$. We will check that $f$ is a cyclic contraction of type (4.1) with $\lambda = \frac{2}{3}$.

If $x \in \{a, b\}$ and $y \in \{a, b\}$, then $p(fx, fy) = p(b, b) = 0$ and (4.1) trivially holds.

Let $x \in \{a, b\}$ and $y = c$; then we have the following two cases:

$$p(fa, fc) = p(b, a) = 1 \leqslant \tfrac{2}{3} \cdot \tfrac{3}{2} = \lambda \max \big\{p(a, c), p(a, fa), p(c, fc), p(a, fc), p(c, fa)\big\},$$

$$p(fb, fc) = p(b, a) = 1 \leqslant \tfrac{2}{3} \cdot \tfrac{3}{2} = \lambda \max \big\{p(b, c), p(b, fb), p(c, fc), p(b, fc), p(c, fb)\big\}.$$

Finally, if $x = y = c$, then

$$p(fc, fc) = p(a, a) = \frac{1}{2} < \frac{2}{3} \cdot \frac{3}{2} = \lambda \max\{p(c, c), p(c, fc)\}.$$

Thus, conditions of Theorem 4.1 are satisfied (with $m = 2$) and the existence of a fixed point of $f$ ($z = b \in \bigcap_{i=1}^{2} A_i$) follows. The same conclusion cannot be obtained by Banach-type fixed point results from [**26**, **44**].

We present another example showing the use of Theorem 4.1. It also shows that there are situations when standard completeness of the $p$-metric, as well as usual metric arguments cannot be used to obtain the existence of a fixed point.

Example 4.3. Let $X = [0, 1] \cap \mathbb{Q}$ be equipped with the partial metric $p$ defined by $p(x, y) = \max\{x, y\}$ for $x, y \in X$. Let $f : X \to X$ be given by $fx = \frac{x^2}{1+x}$, $x \in X$. By Example 2.2, the space $(X, p)$ is 0-complete (but not complete). Take $\lambda = \frac{1}{2}$. The contractive condition (4.1) for (say) $x \geqslant y$ takes the form

$$\begin{aligned} p(fx,fy) &= \max\Big\{\frac{x^2}{1+x},\frac{y^2}{1+y}\Big\} = \frac{x^2}{1+x} \\ &\leqslant \frac{1}{2}\max\big\{p(x,y),p\big(x,\tfrac{x^2}{1+x}\big),p\big(y,\tfrac{y^2}{1+y}\big),p\big(x,\tfrac{y^2}{1+y}\big),p\big(y,\tfrac{x^2}{1+x}\big)\big\} \\ &= \frac{1}{2}\max\big\{x,x,y,x,\max\big\{y,\tfrac{x^2}{1+x}\big\}\big\} = \frac{1}{2}x, \end{aligned}$$

and it is satisfied for all $x,y\in X$, since $0\leqslant x\leqslant 1$. Hence, all the conditions of Theorem 4.1 are satisfied and $f$ has a unique common fixed point ($z=0$).

Since $(X,p)$ is not complete, nor is the space $(X,d)$, where $d=p^s$ is the Euclidean metric, the existence of a (common) fixed point cannot be deduced using known results.

It is well known that a quasicontraction $f$ in a metric space need not be continuous, but that it is continuous at a fixed point of $f$. The same is true in the partial metric case:

Corollary 4.2. *Let $(X,p)$ be a partial metric space and $f:X\to X$. Let $f$ satisfy (4.1) for $\lambda\in[0,1)$ and all $x,y\in X$ and let $fz=z$. Then $f$ is continuous at the point $z$, i.e., $x_n\to z$ in $\tau_p \Rightarrow fx_n\to fz=z$ in $\tau_p$.*

Proof. By Theorem 4.1, there is a unique fixed point $z$ of $f$ and it satisfies that $p(z,z)=p(fz,fz)=0$. Suppose that $\{x_n\}$ is a sequence in $X$ such that $x_n\to z$ in $\tau_p$ when $n\to\infty$, i.e., $\lim_{n\to\infty}p(x_n,z)=p(z,z)$. We have to show that $fx_n\to fz=z$ in $\tau_p$ when $n\to\infty$, i.e., that

$$\lim_{n\to\infty}p(fx_n,fz)=p(fz,fz)=p(z,z)=0. \tag{4.4}$$

We have that

$$\begin{aligned} p(fx_n,fz) &\leqslant \lambda\max\big\{p(x_n,z),p(x_n,fx_n),p(z,fz),p(x_n,fz),p(z,fx_n)\big\} \\ &\leqslant \lambda\max\big\{p(x_n,z),p(x_n,z)+p(fz,fx_n),0,p(x_n,z),p(fz,fx_n)\big\} \\ &= \lambda\big(p(x_n,z)+p(fz,fx_n)\big). \end{aligned}$$

It follows that $p(fx_n,fz)\leqslant \frac{\lambda}{1-\lambda}p(x_n,z)\to 0$ when $n\to\infty$, and (4.4) is proved. $\square$

Theorem 4.2. *Let $(X,p)$ and $f:Y\to Y$ satisfy conditions of Theorem 4.1 with $\lambda\in[0,\frac12)$. Then $f$ satisfies orbital condition (2.3). In particular, there exists $z\in X$ such that $p(z,z)=0$ and $p(fz,z)=p(fz,fz)$; also, $f$ has the property $(P)$.*

Proof. According to Theorem 4.1, $\mathrm{Fix}(f)\neq\emptyset$. We will prove condition (2.3) of Theorem 2.3. Let $x\in Y$ be such that $x\neq fx$. Then

$$\begin{aligned} p(fx,f^2x) &\leqslant \lambda\max\big\{p(x,fx),p(x,fx),p(fx,f^2x),p(x,f^2x),p(fx,fx)\big\} \\ &\leqslant \lambda\max\big\{p(x,fx),p(fx,f^2x),p(x,fx)+p(fx,f^2x),p(fx,f^2x)\big\} \\ &= \lambda\big(p(x,fx)+p(fx,f^2x)\big). \end{aligned}$$

and it follows that $p(fx,f^2x)\leqslant\frac{\lambda}{1-\lambda}p(x,fx)$, where $\frac{\lambda}{1-\lambda}<1$. According to Theorem 2.3, $f$ satisfies the orbital condition and there exists $z\in Y$ such that $p(z,z)=0$ and $p(fz,z)=p(fz,fz)$. The last assertion follows from Lemma 2.2. $\square$

## 5. Weak contraction cyclic type mappings in partial metric spaces

Assertions similar to the following lemma (see, e.g., [**38**]) were used (and proved) in the course of proofs of several fixed point results in various papers.

LEMMA 5.1. *Let $(X,p)$ be a partial metric space and let $\{x_n\}$ be a sequence in $X$ such that $\{p(x_{n+1},x_n)\}$ is nonincreasing and that $\lim_{n\to\infty} p(x_{n+1},x_n)=0$. If $\{x_n\}$ is not a 0-Cauchy sequence in $(X,p)$, then there exist $\varepsilon>0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to $\varepsilon$ when $k\to\infty$:*

$$p(x_{m_k},x_{n_k}),\quad p(x_{m_k},x_{n_k+1}),\quad p(x_{m_k-1},x_{n_k}),\quad p(x_{m_k-1},x_{n_k+1}). \tag{5.1}$$

THEOREM 5.1. *Let $(X,p)$ be a 0-complete partial metric space, $m\in\mathbb{N}$, $A_1$, $A_2,\dots,A_m$ nonempty closed subsets of $X$, $Y=\bigcup_{i=1}^m A_i$ and $f:Y\to Y$. Suppose that:*

(a) *$Y=\bigcup_{i=1}^m A_i$ is a cyclic representation of $Y$ with respect to $f$;*
(b) *there exists a function $\varphi:[0,+\infty)\to[0,+\infty)$ which is lower-semi-continuous, $\varphi(t)=0$ if and only if $t=0$, and such that for any $(x,y)\in A_i\times A_{i+1}$, $i=1,2,\dots,m$ (with $A_{m+1}=A_1$),*

$$p(fx,fy)\leqslant \max M_f^4(x,y)-\varphi(\max M_f^4(x,y)). \tag{5.2}$$

*Then $f$ has a unique fixed point $z$. Moreover, $p(z,z)=0$ and $z\in\bigcap_{i=1}^m A_i$.*

PROOF. Let $x_0\in A_1$ (such a point exists since $A_1\neq\emptyset$). Define the sequence $\{x_n\}$ in $X$ by $x_{n+1}=fx_n$, $n=0,1,2,\dots$ We shall prove that

$$\lim_{n\to\infty} p(x_n,x_{n+1})=0. \tag{5.3}$$

If for some $k$, we have $p(x_{k+1},x_k)=0$, then (5.3) follows immediately. So, we can suppose that $p(x_n,x_{n+1})>0$ for all $n$. From the condition (a), we observe that for all $n$, there exists $i=i(n)\in\{1,2,\dots,m\}$ such that $(x_n,x_{n+1})\in A_i\times A_{i+1}$. Then, apply assumption (5.2) for $x=x_n$ and $y=x_{n+1}$ to obtain

$$\begin{aligned} p(x_{n+1},x_{n+2})&=p(fx_n,fx_{n+1})\\ &\leqslant \max M_f^4(x_n,x_{n+1})-\varphi(\max M_f^4(x_n,x_{n+1})), \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} \max M_f^4(x_n,x_{n+1})&=\max\big\{p(x_n,x_{n+1}),p(x_n,x_{n+1}),p(x_{n+1},x_{n+2}),\\ &\qquad\qquad \tfrac12(p(x_n,x_{n+2})+p(x_{n+1},x_{n+1}))\big\}\\ &=\max\big\{p(x_n,x_{n+1}),p(x_{n+1},x_{n+2})\big\}. \end{aligned}$$

Suppose that $p(x_{n+1},x_{n+2})>p(x_n,x_{n+1})$ for some $n\in\mathbb{N}$. Then (5.4) implies that

$$p(x_{n+1},x_{n+2})\leqslant p(x_{n+1},x_{n+2})-\varphi(p(x_{n+1},x_{n+2})).$$

By the properties of function $\varphi$ it follows that $p(x_{n+1},x_{n+2})=0$, which is already excluded. Hence, $p(x_{n+1},x_{n+2})\leqslant p(x_n,x_{n+1})$, for all $n\in\mathbb{N}_0$. Thus, the sequence

$\{p(x_{n+1}, x_n)\}$ is nonincreasing. Since it is bounded from below, there exists $r \geqslant 0$ such that $\lim_{n\to\infty} p(x_{n+1}, x_n) = r$. Passing to the (upper) limit in

$$p(x_{n+1}, x_n) \leqslant p(x_n, x_{n-1}) - \varphi(p(x_n, x_{n-1}),$$

we get that $r \leqslant r - \varphi(r)$, and using the properties of $\varphi$, that $r = 0$.

Next, we claim that $\{x_n\}$ is a 0-Cauchy sequence in the space $(X, p)$. Suppose that this is not the case. Then, using Lemma 5.1 we get that there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and sequences (5.1) tend to $\varepsilon$ when $k \to \infty$.

Elements $x_{m(k)}$ and $x_{n(k)-1}$ might not lie in adjacently labelled sets $A_i$ and $A_{i+1}$. However, for all $k$, there exists $j(k) \in \{1, \dots, p\}$ such that $n(k)-1-m(k)+j(k) \equiv 1[p]$. Then $x_{m(k)-j(k)}$ (for $k$ large enough, $m(k) > j(k)$) and $x_{n(k)-1}$ lie in adjacently labelled sets $A_i$ and $A_{i+1}$ for certain $i \in \{1, \dots, p\}$. To simplify the procedure, we will suppose that already $(x_{m(k)}, x_{n(k)-1}) \in A_i \times A_{i+1}$. Applying condition (5.2) to elements $x = x_{m_k}$ and $y = x_{n_k-1}$ we get that

$$\begin{aligned} p(x_{m_k+1}, x_{n_k}) &= p(fx_{m_k}, fx_{n_k-1}) \\ &\leqslant \max M_f^4(x_{m_k}, x_{n_k-1}) - \varphi(\max M_f^4(x_{m_k}, x_{n_k-1})), \end{aligned}$$

where

$$\begin{aligned} \max M_f^4(x_{m_k}, x_{n_k-1}) &= \max\big\{p(x_{m_k}, x_{n_k-1}), p(x_{m_k}, x_{m_k+1}), p(x_{n_k-1}, x_{n_k}), \\ &\qquad \tfrac{1}{2}(p(x_{m_k}, x_{n_k}) + p(x_{n_k-1}, x_{m_k+1}))\big\} \\ &\to \varepsilon, \text{ as } k \to \infty. \end{aligned}$$

It follows that $\varepsilon \leqslant \varepsilon - \liminf_{k\to\infty} \varphi(\max M_f^4(x_{m_k}, x_{n_k-1}))$. Using the properties of function $\varphi$ we conclude that $\varepsilon = 0$, which is a contradiction.

Thus $\{x_n\}$ is a 0-Cauchy sequence. Since $(X, p)$ is 0-complete and $Y$ is closed, it follows that the sequence $\{x_n\}$ converges to some $z \in Y$. Moreover, $p(z, z) = 0$ and $z \in \bigcap_{i=1}^m A_i$. We will prove that $z$ is a fixed point of $f$.

Using condition (5.2) with $x = z$ and $y = x_{n+1}$ (which is possible since $z$ belongs to each $A_i$) we obtain that

$$\begin{aligned} p(z, fz) &\leqslant p(z, x_{n+2}) + p(x_{n+2}, fz) = p(z, x_{n+2}) + p(fz, fx_{n+1}) \\ &\leqslant p(z, x_{n+2}) + \max M_f^4(z, x_{n+1}) - \varphi(\max M_f^4(z, x_{n+1})), \end{aligned}$$

where

$$\begin{aligned} \max M_f^4(z, x_{n+1}) = \max\big\{&p(z, x_{n+1}), p(z, fz), p(x_{n+1}, x_{n+2}), \\ &\tfrac{1}{2}(p(z, x_{n+2}) + p(x_{n+1}, fz))\big\}. \end{aligned}$$

The first three terms of the previous set tend, respectively, to: $p(z, z) = 0$, $p(z, fz)$, 0, while the fourth is less than $\frac{1}{2}(p(z, x_{n+2}) + p(x_{n+1}, z) + p(z, fz))$ which tends to $\frac{1}{2}p(z, fz)$. Thus, passing to the (upper) limit, we get that

$$p(z, fz) \leqslant p(z, fz) - \varphi(p(z, fz)),$$

which implies (using the properties of function $\varphi$) that $p(z, fz) = 0$ and $fz = z$.

The final assertion can be proved in the same way as in some previous theorems. $\square$

We will show in the next example that Theorem 5.1 is more general than some other known fixed point results.

EXAMPLE 5.1. Let $X = \mathbb{R}^+$ be equipped with the usual partial metric $p(x,y) = \max\{x,y\}$. Suppose $A_1 = [0,1]$, $A_2 = [0,\frac{1}{4}], A_3 = [0,\frac{1}{28}], A_4 = [0,\frac{1}{868}]$ and $Y = \bigcup_{i=1}^4 A_i$. Consider the mapping $f : Y \to Y$ defined by $fx = \frac{x^2}{1+3x}$. It is clear that $\bigcup_{i=1}^4 A_i$ is a cyclic representation of $Y$ with respect to $f$. Further, consider the function $\varphi : [0,+\infty) \to [0,+\infty)$ given by $\varphi(t) = \frac{t}{1+2t}$. Take arbitrary elements, say $y \leqslant x$, from $Y$. Then

$$p(fx,fy) = \max\Big\{\frac{x^2}{1+3x}, \frac{y^2}{1+3y}\Big\} = \frac{x^2}{1+3x}.$$

On the other hand,

$$\begin{aligned}\max M_f^4(x,y) &= \max\big\{p(x,y), p\big(x,\tfrac{x^2}{1+3x}\big), p\big(y,\tfrac{y^2}{1+3y}\big), \tfrac{1}{2}\big(p\big(x,\tfrac{y^2}{1+3y}\big) + p\big(y,\tfrac{x^2}{1+3x}\big)\big)\big\}\\ &= \max\big\{x,x,y,\tfrac{1}{2}(x+y)\big\} = x,\end{aligned}$$

and

$$\max M_f^4(x,y) - \varphi(\max M_f^4(x,y)) = x - \frac{x}{1+2x} = \frac{2x^2}{1+2x} \geqslant p(fx,fy)$$

holds true. Hence in this case relation (5.2) is satisfied, as well as the other assumptions of Theorem 5.1 (with $m = 4$). We deduce that $f$ has a unique fixed point $z \in A_1 \cap A_2 \cap A_3 \cap A_4 = \{0\}$.

On the other hand, consider the same problem in the standard metric $d(x,y) = p^s(x,y)$ and take $x = 1$ and $y = \frac{1}{3}$. Then $d(fx,fy) = \big|\frac{1}{4} - \frac{1}{18}\big| = \frac{7}{36}$ and $\max M_f^4(x,y) = \max\{\frac{2}{3}, \frac{3}{4}, \frac{5}{18}, \frac{1}{2}(\frac{17}{18} + \frac{1}{12})\} = \frac{2}{3}$ and so

$$\max M_f^4(x,y) - \varphi(\max M_f^4(x,y)) = \tfrac{1}{2} - \varphi\left(\tfrac{1}{2}\right) = \tfrac{4}{15}.$$

Hence, $d(fx,fy) \leqslant \max M_f^4(x,y) - \varphi(\max M_f^4(x,y))$ does not hold and the existence of a fixed point of $f$ cannot be derived from [**45**, Theorem 2.1].

## 6. Cyclic generalized $f_\psi$-contractions in partial metric spaces

We will denote by $\Psi$ the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions: $(\Psi_1)$ $\psi$ is continuous; $(\Psi_2)$ $\psi(t) < t$ for all $t > 0$.

Obviously, if $\psi \in \Psi$, then $\psi(0) = 0$ and $\psi(t) \leqslant t$ for all $t \geqslant 0$.

We introduce the notion of cyclic generalized $f_\psi$-contraction as follows.

DEFINITION 6.1. Let $(X,p)$ be a partial metric space. Let $m$ be a positive integer, $A_1, A_2, \dots, A_m$ be nonempty subsets of $X$ and $Y = \bigcup_{i=1}^m A_i$. An operator $f : Y \to Y$ is a cyclic generalized $f_\psi$-contraction for some $\psi \in \Psi$, if:

(I) $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of $Y$ with respect to $f$;

(II) there exist $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$, such that for all $(x,y) \in A_i \times A_{i+1}$, $i = 1,2,\dots,m$ (with $A_{m+1} = A_1$),

$$p(fx,fy) \leqslant \alpha\, \mathbf{m}(x,y) + \beta\, \mathbf{M}(x,y),$$

where $\mathbf{m}(x,y)=\psi\left(p(y,fy)\dfrac{1+p(x,fx)}{1+p(x,y)}\right)$, and

$$\mathbf{M}(x,y)=\max\big\{\psi(p(x,y)),\psi(p(x,fx)),\psi(p(y,fy)),\psi\big(\tfrac{1}{2}\big(p(x,fy)+p(y,fx)\big)\big)\big\}.$$

The main result of this section is the following.

THEOREM 6.1. *Let $(X,p)$ be a 0-complete partial metric space, $m\in\mathbb{N}$, $A_1$, $A_2,\dots,A_m$ nonempty closed subsets of $X$ and $Y=\bigcup_{i=1}^m A_i$. Suppose $f:Y\to Y$ is a cyclic generalized $f_\psi$-contraction mapping, for some $\psi\in\Psi$. Then $f$ has a unique fixed point. Moreover, the fixed point of $f$ belongs to $\bigcap_{i=1}^m A_i$.*

PROOF. Let $x_0\in A_1$ (such a point exists since $A_1\neq\emptyset$). Define the sequence $\{x_n\}$ in $X$ by $x_{n+1}=fx_n$, $n=0,1,2,\dots$ We shall prove that

$$\lim_{n\to\infty}p(x_n,x_{n+1})=0. \tag{6.1}$$

If for some $k$, we have $p(x_{k+1},x_k)=0$, then (6.1) follows immediately. So, we can suppose that $p(x_n,x_{n+1})>0$ for all $n$. From condition (I), we observe that for all $n$, there exists $i=i(n)\in\{1,2,\dots,m\}$ such that $(x_n,x_{n+1})\in A_i\times A_{i+1}$. Then, from condition (II), we have

$$p(x_n,x_{n+1})\leqslant\alpha\,\mathbf{m}(x_{n-1},x_n)+\beta\,\mathbf{M}(x_{n-1},x_n),\quad n=1,2,\dots \tag{6.2}$$

On the other hand, we have

$$\mathbf{m}(x_{n-1},x_n)=\psi\left(p(x_n,x_{n+1})\frac{1+p(x_{n-1},x_n)}{1+p(x_{n-1},x_n)}\right)=\psi(p(x_n,x_{n+1})),$$

and

$$\begin{aligned}\mathbf{M}(x_{n-1},x_n)=\max\big\{&\psi(p(x_{n-1},x_n)),\psi(p(x_n,x_{n+1})),\\&\psi\big(\tfrac{1}{2}\big(p(x_{n-1},x_{n+1})+p(x_n,x_n)\big)\big)\big\}.\end{aligned}$$

$\star$ If $\mathbf{M}(x_{n-1},x_n)=\psi(p(x_n,x_{n+1}))$, we obtain from (6.2) and the properties of $\psi$ that

$$p(x_n,x_{n+1})\leqslant(\alpha+\beta)\psi(p(x_n,x_{n+1}))<p(x_n,x_{n+1}),\ \text{since } \alpha+\beta<1$$

which implies that such case is impossible.

$\star$ If $\mathbf{M}(x_{n-1},x_n)=\psi\big(\frac{1}{2}\big(p(x_{n-1},x_{n+1})+p(x_n,x_n)\big)\big)$, we obtain from (6.2) and the properties of $\psi$ that

$$\begin{aligned}p(x_n,x_{n+1})&\leqslant\alpha\,\psi(p(x_n,x_{n+1}))+\beta\,\psi\big(\tfrac{1}{2}\big(p(x_{n-1},x_{n+1})+p(x_n,x_n)\big)\big)\\&<\alpha\,p(x_n,x_{n+1})+\beta\,\tfrac{1}{2}\big(p(x_{n-1},x_{n+1})+p(x_n,x_n)\big).\end{aligned} \tag{6.3}$$

By ($\mathrm{p}_4$), we have $p(x_{n-1},x_{n+1})+p(x_n,x_n)\leqslant p(x_{n-1},x_n)+p(x_n,x_{n+1})$. Therefore we have

$$\tfrac{1}{2}[p(x_{n-1},x_{n+1})+p(x_n,x_n)]\leqslant\tfrac{1}{2}p(x_{n-1},x_n)+\tfrac{1}{2}p(x_n,x_{n+1}). \tag{6.4}$$

Combining (6.3) with (6.4), we obtain $p(x_n,x_{n+1})\leqslant\frac{\beta}{2-2\alpha-\beta}p(x_{n-1},x_n)$.

$\star$ If $\mathbf{M}(x_{n-1}, x_n) = \psi(p(x_{n-1}, x_n))$, we obtain from (6.2) and the properties of $\psi$ that

$$p(x_n, x_{n+1}) \leqslant \alpha\, \psi(p(x_n, x_{n+1})) + \beta\, \psi(p(x_{n-1}, x_n)) < \alpha\, p(x_n, x_{n+1}) + \beta\, p(x_{n-1}, x_n),$$

that is, $p(x_n, x_{n+1}) \leqslant \frac{\beta}{1-\alpha} p(x_{n-1}, x_n)$. Define $\lambda := \max\{\frac{\beta}{1-\alpha}, \frac{\beta}{2-2\alpha-\beta}\} < 1$. Consequently, it can be concluded that

$$p(x_n, x_{n+1}) \leqslant \lambda p(x_{n-1}, x_n) \leqslant \lambda^2 p(x_{n-2}, x_{n-1}) \leqslant \cdots \leqslant \lambda^n p(x_0, x_1).$$

Therefore, since $0 \leqslant \lambda < 1$, taking the limit as $n \to \infty$, we have $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$, which is (6.1). Therefore, conditions (3.3) are satisfied and $\{x_n\}$ is a 0-Cauchy sequence in $(X, p)$. As in the proof of Theorem 3.1 it follows that there exists $z \in Y$ such that $p(z, z) = \lim_{n\to\infty} p(z, x_n) = 0$ and that

$$z \in \bigcap_{i=1}^{m} A_i. \tag{6.5}$$

Now, we shall prove that $z$ is a fixed point of $f$. Indeed, from (6.5), since for all $n$, there exists $i(n) \in \{1, 2, \ldots, m\}$ such that $x_n \in A_{i(n)}$, applying (II) with $x = z$ and $y = x_n$, we obtain

$$p(x_{n+1}, fz) = p(fx_n, fz) \leqslant \alpha\, \mathbf{m}(x_n, z) + \beta\, \mathbf{m}(x_n, z), \tag{6.6}$$

for all $n$. On the other hand, we have

$$\mathbf{m}(x_n, z) = \psi\left(p(z, fz)\frac{1 + p(x_n, x_{n+1})}{1 + p(x_n, z)}\right),$$

and

$$\mathbf{m}(x_n, z) = \max\big\{\psi(p(z, x_n)), \psi(p(z, fz)), \psi(p(x_n, x_{n+1})),\\ \psi\big(\tfrac{1}{2}\big(p(z, x_{n+1}) + p(x_n, fz)\big)\big)\big\}.$$

Using (6.5) and the continuity of $\psi$, we obtain that

$$\lim_{n\to\infty} \mathbf{m}(x_n, z) = \max\big\{\psi(p(z, fz)), \psi(p(z, fz)/2)\big\}. \tag{6.7}$$

Passing to the limit as $n \to \infty$ in (6.6), using (6.7) and (6.5), we get

$$p(z, fz) \leqslant \alpha\psi(p(z, fz)) + \beta \max\big\{\psi(p(z, fz)), \psi(p(z, fz)/2)\big\}.$$

Suppose that $p(z, fz) > 0$. In this case, using condition $(\Psi_2)$, we have

$$\psi(p(z, fz)) < p(z, fz) \quad\text{and}\quad \max\big\{\psi(p(z, fz)), \psi(p(z, fz)/2)\big\} < p(z, fz),$$

which implies that $p(z, fz) < (\alpha + \beta)p(z, fz)$, a contradiction, since $\alpha + \beta < 1$. Then we have $p(z, fz) = 0$, that is, $z$ is a fixed point of $f$.

Finally, we prove that $z$ is the unique fixed point of $f$. Assume that $u$ is another fixed point of $f$, that is, $fu = u$. From condition (I), this implies that $u \in \bigcap_{i=1}^{m} A_i$. Then we can apply (II) for $x = z$ and $y = u$. We obtain

$$p(z, u) = p(fz, fu) \leqslant \alpha\, \mathbf{m}(z, u) + \beta\, \mathbf{m}(z, u).$$

Since $z$ and $u$ are fixed points of $f$, we can show easily that $\mathbf{m}(z,u)=0$ and $\mathbf{m}(z,u)=\psi(p(z,u))$. If $p(z,u)>0$, we get

$$p(z,u)=p(fz,fu)\leqslant \beta\mathbf{m}(z,u)=\beta\psi(p(z,u))<\beta\, p(z,u)<p(z,u),$$

a contradiction. Then we have $p(z,u)=0$, that is, $z=u$. Thus we have proved the uniqueness of the fixed point. □

Next, we derive some fixed point results from our main Theorem 6.1.

If we take $m=1$ and $A_1=X$ in Theorem 6.1, then we get immediately the following fixed point result.

Corollary 6.1. *Let $(X,d)$ be a 0-complete partial metric space and $f:X\to X$ satisfies the following condition: there exists $\psi\in\Psi$ such that*

$$\begin{aligned}p(fx,fy)&\leqslant \alpha\,\psi\left(p(y,fy)\frac{1+p(x,fx)}{1+p(x,y)}\right)\\&\quad+\beta\,\max\left\{\psi(p(x,y)),\psi(p(x,fx)),\psi(p(y,fy)),\psi\big(\tfrac12\big(p(x,fy)+p(y,fx)\big)\big)\right\},\end{aligned}$$

*for all $x,y\in X$. Then $f$ has a unique fixed point.*

Remark 6.1. Corollary 6.1 extends and generalizes many existing fixed point theorems in the literature [**8**, **22**].

Corollary 6.2. *Let $(X,p)$ be a 0-complete partial metric space, $m\in\mathbb{N}$, $A_1$, $A_2,\dots,A_m$ nonempty closed subsets of $X$, $Y=\bigcup_{i=1}^m A_i$ and $f:Y\to Y$. Suppose that there exists a nondecreasing function $\psi\in\Psi$ such that:*

(a) *$Y=\bigcup_{i=1}^m A_i$ is a cyclic representation of $Y$ with respect to $f$;*

(b) *for all $(x,y)\in A_i\times A_{i+1}$, $i=1,2,\dots,m$ (with $A_{m+1}=A_1$),*

$$\begin{aligned}p(fx,fy)&\leqslant \alpha\,\psi\left(p(y,fy)\frac{1+p(x,fx)}{1+p(x,y)}\right)\\&\quad+\beta\,\psi\big(\max\left\{p(x,y),p(x,fx),p(y,fy),\tfrac12(p(x,fy)+p(y,fx))\right\}\big).\end{aligned}\tag{6.8}$$

*Then $f$ has a unique fixed point. Moreover, the fixed point of $f$ belongs to $\bigcap_{i=1}^m A_i$.*

Proof. It follows from Theorem 6.1 by observing that if $\psi$ is nondecreasing, we have

$$\mathbf{m}(x,y)=\psi\big(\max\left\{p(x,y),p(x,fx),p(y,fy),\tfrac12(p(x,fy)+p(y,fx))\right\}\big).$$ □

Remark 6.2. It is clear that the conclusions of the previous corollary remain valid if in condition (6.8) the second term of the right-hand side is substituted by any of the following terms:

$$\beta\,\psi(p(x,y)),\quad \beta\,\psi\big(\tfrac12(p(x,fy)+p(y,fx))\big),\quad \beta\max\{\psi(p(x,fx)),\psi(p(y,fy))\},$$
$$\text{or}\quad \beta\max\{\psi(p(x,y)),\psi(p(x,fx)),\psi(p(y,fy))\}.$$

Corollary 6.3. *Let $(X,p)$ be a 0-complete partial metric space, $m\in\mathbb{N}$, $A_1$, $A_2,\dots,A_m$ nonempty closed subsets of $X$, $Y=\bigcup_{i=1}^m A_i$ and $f:Y\to Y$. Suppose that there exist five positive constants $a_j$ with $\sum_{j=1}^5 a_j<1$ such that*

(a) *$Y=\bigcup_{i=1}^m A_i$ is a cyclic representation of $Y$ with respect to $f$;*

(b) *for all* $(x,y) \in A_i \times A_{i+1}$, $i = 1,2,\dots,m$ *(with* $A_{m+1} = A_1$*),*

$$p(fx,fy) \leqslant a_1 \left( p(y,fy) \frac{1+p(x,fx)}{1+p(x,y)} \right) + a_2 p(x,y) + a_3 p(x,fx) + a_4 p(y,fy) + a_5\big(\tfrac{1}{2}(p(x,fy)+p(y,fx))\big).$$

*Then $f$ has a unique fixed point. Moreover, the fixed point of $f$ belongs to $\bigcap_{i=1}^m A_i$.*

Proof. It follows from Theorem 6.1 when $\psi(t) = (a_1+a_2+a_3+a_4+a_5)t$. □

We illustrate Theorem 6.1 by an example which is obtained by modifying the one from [**30**].

Example 6.1. Let $X = \mathbb{R}^+$ be equipped by the usual partial metric $p(x,y) = \max\{x,y\}$. Suppose $A_1 = [0,1]$, $A_2 = [0,\frac{1}{2}]$ and $Y = \bigcup_{i=1}^2 A_i$.

Consider the mapping $f : Y \to Y$ defined by $fx = \frac{x^2}{2(1+x)}$. It is clear that $\bigcup_{i=1}^2 A_i$ is a cyclic representation of $Y$ with respect to $f$.

Let us take $\psi : [0,+\infty) \to [0,+\infty)$ such that $\psi(t) = \frac{t^2}{1+t}$, $t \in [0,1]$. Then $\psi$ has the properties mentioned in Theorem 6.1. Moreover, the mapping $f$ is a cyclic contraction of type (4.1). Indeed, take arbitrary elements, say $y \leqslant x$, from $Y$. Then

$$p(fx,fy) = \max\left\{ \frac{x^2}{2(1+x)}, \frac{y^2}{2(1+y)} \right\} = \frac{x^2}{2(1+x)}.$$

On the other hand,

$$\begin{aligned}\mathbf{M}(x,y) &= \max\Big\{ \psi(p(x,y)), \psi\big(p\big(x, \tfrac{x^2}{2(1+x)}\big)\big), \psi\big(p\big(y, \tfrac{y^2}{2(1+y)}\big)\big), \\ &\qquad\qquad \psi\big(\tfrac{1}{2}\big[p\big(x, \tfrac{y^2}{2(1+y)}\big) + p\big(y, \tfrac{x^2}{2(1+x)}\big)\big]\big) \Big\} \\ &= \max\Big\{ \psi(x), \psi(x), \psi(y), \psi\big(\tfrac{1}{2}\big[x + \max\{y, \tfrac{x^2}{2(1+x)}\}\big]\big) \Big\} \\ &= \psi(x).\end{aligned}$$

(It was used that the function $\psi$ is increasing and, since $x \geqslant y$ and $x \geqslant \frac{x^2}{2(1+x)}$, that $\frac{1}{2}\big(x + \max\big\{y, \frac{x^2}{2(1+x)}\big\}\big) \leqslant x$.)

Hence in this case $p(fx,fy) \leqslant \frac{1}{2}\mathbf{M}(x,y)$ is satisfied for $\alpha = 0$. Then relation (II) holds for $\alpha = 0$ and $\beta = \frac{1}{2}$.

Therefore, all conditions of Theorem 6.1 are satisfied (with $m = 2$), and so $f$ has a fixed point (which is $z = 0 \in \bigcap_{i=1}^2 A_i$) such that $p(z,z) = 0$.

Remark 6.3. Our results extend and generalize many existing fixed point theorems in the literature [**8**, **21**, **22**].

## 7. Application to well posedness and limit shadowing of fixed point problem

The notion of well-posedness of a fixed point problem has evoked much interest to several mathematicians, for example, De Blasi and Myjak [**9**], Lahiri and Das [**24**], Popa [**36**, **37**] and others.

DEFINITION 7.1. [**9**] Let $(X, d)$ be a metric space and $f : X \to X$ be a mapping. The fixed point problem of $f$ is said to be well posed if:

(i) $f$ has a unique fixed point $x$ in $X$;

(ii) for any sequence $\{x_n\}$ of points in $X$ such that $\lim_{n\to\infty} d(fx_n, x_n) = 0$, we have $\lim_{n\to\infty} d(x_n, x) = 0$.

The limit shadowing property of fixed point problems has been discussed in the papers [**32, 35, 39**] and others.

DEFINITION 7.2. [**35**] Let $(X, d)$ be a metric space and $f : X \to X$ be a mapping. The fixed point problem of $f$ is said to have limit shadowing property in $X$ if assuming that $\{x_n\}$ in $X$ satisfies $d(x_n, fx_n) \to 0$ as $n \to \infty$, it follows that there exists $x \in X$ such that $d(x_n, f^n x) \to 0$ as $n \to \infty$.

We can give similar definitions in partial metric spaces.

Concerning the well-posedness and limit shadowing of the fixed point problem for a mapping in a partial metric space satisfying the conditions of Theorem 3.1–Theorem 6.1, we have the following results.

THEOREM 7.1. *Let $f : Y \to Y$ be a self-mapping as in Theorem 3.1. Then the fixed point problem for $f$ is well posed.*

PROOF. Owing to Theorem 3.1, we know that $f$ has a unique fixed point $z = fz \in Y$, such that $p(z, fz) = 0$. Let $\{x_n\} \subset Y$ be such that $\lim_{n\to\infty} p(x_n, fx_n) = 0$. Then

$$\begin{aligned}
p(x_n, z) &\leqslant p(x_n, fx_n) + p(fx_n, fz)\\
&\leqslant p(x_n, fx_n) + Ap(x_n, z) + Bp(x_n, fx_n) + Cp(z, fz)\\
&\quad + Dp(x_n, fz) + Ep(z, fx_n) + F\frac{p(x_n, fx_n)\cdot p(z, fz)}{1 + p(x_n, z)}\\
&\leqslant (1 + B + E)p(x_n, fx_n) + (A + D + E)p(x_n, z).
\end{aligned}$$

Passing to the limit as $n \to \infty$ in the above inequality, and using that $A+D+E < 1$, we get that $p(x_n, z) \to 0$ as $n \to \infty$ which is equivalent to saying that $x_n \to z$ as $n \to \infty$. □

THEOREM 7.2. *Let $f : Y \to Y$ be a self-mapping as in Theorem 3.1. Then $f$ has the limit shadowing property.*

PROOF. Owing to Theorem 3.1, we know that $f$ has a unique fixed point $z = fz \in Y$, such that $p(z, fz) = 0$. Let $\{x_n\} \subset Y$ be such that $\lim_{n\to\infty} p(x_n, fx_n) = 0$. Then, as in the previous proof,

$$p(x_n, z) \leqslant (1 + B + E)p(x_n, fx_n) + (A + D + E)p(x_n, z).$$

Passing to the limit as $n \to \infty$ in the above inequality, and using that $A+D+E < 1$, it follows that $p(x_n, f^n z) = p(x_n, z) \to 0$ as $n \to \infty$. □

THEOREM 7.3. *Let $X$ be a nonempty set, $(X, p)$ and $(X, \rho)$ be two partial metric spaces, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ nonempty closed subsets of $X$ and $Y = \bigcup_{i=1}^m A_i$. Suppose that:*

(1) *$Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of $Y$ with respect to $f$;*
(2) *$p(x,y) \leqslant \rho(x,y)$ for all $x,y \in Y$;*
(3) *$(Y,p)$ is a 0-complete partial metric space;*
(4) *$f : (Y,p) \to (Y,p)$ is continuous;*
(5) *$f : (Y,\rho) \to (Y,\rho)$ is Hardy–Rogers rational type cyclic contractive.*

*Then, $\{f^n x_0\}$ converges to $z$ in $(Y,p)$ for any $x_0 \in Y$ and $z$ is the unique fixed point of $f$.*

Proof. Let $x_0 \in Y$. As in Theorem 3.1, assumption (5) implies that $\{f^n x_0\}$ is a 0-Cauchy sequence in $(Y,\rho)$. Taking (2) into account, $\{f^n x_0\}$ is a 0-Cauchy sequence in $(Y,p)$ and due to (3) it converges to $z$ in $(Y,\rho)$ for any $x_0 \in Y$. Condition (4) implies the uniqueness of $z$. □

Remark 7.1. Similar consequences of Theorems 3.2–5.1 can be obtained.

Remark 7.2. The results of this paper are obtained under the assumption that the given partial metric space is 0-complete. Taking into account Lemma 2.1.(b) and Example 2.2, it follows that they also hold if the space is complete, but that our assumption is weaker.

## References


1. M. Abbas, T. Nazir, S. Romaguera, *Fixed point results for generalized cyclic contraction mappings in partial metric spaces*, Rev. Real Acad. Cienc. Exac., Fis. Nat., Ser. A, Mat. **106** (2012), 287–297.
2. T. Abdeljawad, J. O. Alzabut, A. Mukheimer, Y. Zaidan, *Banach contraction principle for cyclical mappings on partial metric spaces*, Fixed Point Theory Appl. **2012**:154 (2012), doi:10.1186/1687-1812-2012-154.
3. T. Abdeljawad, J. O. Alzabut, A. Mukheimer, Y. Zaidan, *Best proximity points for cyclic contraction mappings with boundedly compact decompositions in partial metric spaces*, J. Comput. Anal. Appl. **15** (2013), 678–685.
4. T. Abdeljawad, E. Karapinar, K. Taş, *Existence and uniqueness of a common fixed point on partial metric spaces*, Appl. Math. Lett. **24** (2011), 1900–1904.
5. R. Agarwal, M. A. Alghamdi, N. Shahzad, *Fixed point theory for cyclic generalized contractions in partial metric spaces*, Fixed Point Theory Appl. **2012**:40 (2012), doi:10.1186/1687-1812-2012-40.
6. Ya. I. Alber, S. Guerre-Delabriere, *Principles of weakly contractive maps in Hilbert spaces*, New Results in Operator Theory and its Applications (I. Gohberg and Yu. Lyubich, eds.), Oper. Theory Adv. Appl., **98**, Birkhäuser, Basel, 1997, pp. 7–22.
7. I. Altun, F. Sola, H. Simsek, *Generalized contractions on partial metric spaces*, Topology Appl. **157** (2010), 2778–2785.
8. C. Di Bari, P. Vetro, *Fixed points for weak $\varphi$-contractions on partial metric spaces*, Int. J. Engin. Contemp. Math. Sci. **1** (2011), 5–13.
9. F. S. De Blasi, J. Myjak, *Sur la porosité des contractions sans point fixed*, C. R. Acad. Sci. Paris **308** (1989), 51-54.
10. D. W. Boyd, J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458–464.
11. M. Bukatin, R. Kopperman, S. Matthews, H. Pajoohesh, *Partial metric spaces*, Am. Math. Monthly **116** (2009), 708–718.
12. C. M. Chen, *Fixed point theory for the cyclic weaker Meir-Keeler function in complete metric spaces*, Fixed Point Theory Appl. **2012**:17 (2012), doi:10.1186/1687-1812-2012-17.

13. M. H. Escardo, *Pcf extended with real numbers*, Theoretical Computer Sci. **162** (1996), 79–115.
14. R. H. Haghi, Sh. Rezapour, N. Shahzad, *Be careful on partial metric fixed point results*, Topology Appl. **160** (2013), 450–454.
15. R. Heckmann, *Approximation of metric spaces by partial metric spaces*, Appl. Categ. Structures **7** (1999), 71–83.
16. P. Hitzler, A. Seda, *Mathematical Aspects of Logic Programming Semantics*, Chapman & Hall/CRC Studies in Informatic Series, CRC Press, 2011.
17. J. Jachymski, *Equivalent conditions for generalized contractions on (ordered) metric spaces*, Nonlinear Anal. **74** (2011), 768–774.
18. G. S. Jeong, B. E. Rhoades, *Maps for which* $F(T) = F(T^n)$, Fixed Point Theory Appl. **6** (2005), 87–131.
19. Z. Kadelburg, H. K. Nashine, S. Radenović, *Fixed point results under various contractive conditions in partial metric spaces*, Rev. Real Acad. Cienc. Exac., Fis. Nat., Ser. A, Mat. doi:10.1007/s13398-012-0066-6.
20. E. Karapinar, *Fixed point theory for cyclic weak* $\phi$*-contraction*, Appl. Math. Lett. **24** (2011), 822–825.
21. E. Karapinar, I. M. Erhan, *Fixed point theorems for operators on partial metric spaces*, Appl. Math. Lett. **24** (2011), 1894–1899.
22. E. Karapinar, I. M. Erhan, A. Y. Ulus, *Fixed point theorem for cyclic maps on partial metric spaces*, Appl. Math. Inf. Sci. **6** (2012), 239–244.
23. M. S. Khan, M. Swaleh, S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Austral. Math. Soc. **30** (1984), 1–9.
24. B. K. Lahiri, P. Das, *Well-posednes and porosity of certain classes of operators*, Demonstratio Math. **38** (2005), 170–176.
25. W. A. Kirk, P. S. Srinavasan, P. Veeramani, *Fixed points for mappings satisfying cylical contractive conditions*, Fixed Point Theory **4** (2003), 79–89.
26. S. G. Matthews, *Partial metric topology*, Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci. **728** (1994), 183–197.
27. H. K. Nashine, *New fixed point theorems for mappings satisfying generalized weakly contractive condition with weaker control functions*, Ann. Polon. Math. **104** (2012), 109–119.
28. H. K. Nashine, Z. Kadelburg, *Cyclic contractions and fixed point results through control functions on partial metric spaces*, Intern. J. Anal. **2013**, Article ID 726387, doi:10.1155/2012/726387.
29. H. K. Nashine, Z. Kadelburg, S. Radenović, J. K. Kim, *Fixed point theorems under Hardy–Rogers weak contractive conditions on* 0*-complete ordered partial metric spaces*, Fixed Point Theory Appl. **2012**:180 (2012), doi:10.1186/1687-1812-2012-180.
30. H. K. Nashine, Z. Kadelburg, S. Radenović, *Common fixed point theorems for weakly isotone increasing mappings in ordered partial metric spaces*, Math. Comput. Modelling **57** (2013), 2355–2365, doi:10.1016/j.mcm.2011.12.019.
31. S. Oltra, O. Valero, *Banach's fixed point theorem for partial metric spaces*, Rend. Istit. Math. Univ. Trieste **36** (2004), 17–26.
32. M. Păcurar, I. A. Rus, *Fixed point theory for cyclic* $\phi$*-contractions*, Nonlinear Anal. **72** (2010), 1181–1187.
33. M. A. Petric, *Some results concerning cyclical contractive mappings*, General Math. **18** (2010), 213–226.
34. M. A. Petric, *Best proximity point theorems for weak cyclic Kannan contractions*, Filomat **25**:1 (2011), 145–154.
35. S. Ju. Piljugin, *Shadowing in Dynamical Systems*, Springer, 1999.
36. V. Popa, *Well-posedness of fixed point problems in orbitally complete metric spaces*, Stud. Cerc. St. Ser. Mat. Univ. **16** (2006), Supplement. Proceedings of ICMI 45, Bacau, Sept. 18–20 (2006), 209–214.

37. V. Popa, *Well-posedness of fixed point problems in compact metric spaces*, Bul. Univ. Petrol-Gaze, Ploiesti, Sec. Mat. Inform. Fiz. **60** (1) (2008), 1–4.
38. S. Radenović, Z. Kadelburg, D. Jandrlić, A. Jandrlić, *Some results on weak contraction maps*, Bull. Iranian Math. Soc. **38**(3) (2012), 625–645.
39. I. A. Rus, *The theory of a metrical fixed point theorem: Theoretical and applicative relevances*, Fixed Point Theory **9** (2008), 541–559.
40. B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **336** (1977), 257–290.
41. B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal. **47** (2001), 2683–2693.
42. S. Romaguera, *A Kirk type characterization of completeness for partial metric spaces*, Fixed Point Theory Appl. (2010), Article ID 493298, 6 pages, doi:10.1155/2010/493298
43. B. Samet, C. Vetro, F. Vetro, *From metric spaces to partial metric spaces*, Fixed Point Theory Appl. **2013**:5 (2013), doi:10.1186/1687-2012-2013-5.
44. O. Valero, *On Banach fixed point theorems for partial metric spaces*, Appl. General Topology **6** (2005), 229–240.
45. Q. Zhang, Y. Song, *Fixed point theory for generalized $\varphi$-weak contractions*, Appl. Math. Lett. **22** (2009), 75–78.

(Received 16 01 2013)

Department of Mathematics
Disha Institute of Management and Technology
Satya Vihar, Vidhansabha-Chandrakhuri Marg,
Mandir Hasaud, Raipur-492101(Chhattisgarh)
India
drhknashine@gmail.com, nashine_09@rediffmail.com

Faculty of Mathematics
University of Belgrade
Studentski trg 16, 11000 Beograd
Serbia
kadelbur@matf.bg.ac.rs

Faculty of Mechanical Engineering
University of Belgrade
Kraljice Marije 16, 11120 Beograd
Serbia
radens@beotel.net