PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 93 (107) (2013), 49–63 DOI: 10.2298/PIM1307049K

ILL-POSED ABSTRACT VOLTERRA EQUATIONS

Marko Kostić

Communicated by Stevan Pilipović

Abstract. The study of ill-posed abstract Volterra equations is a recent subject. In this paper, we investigate equations on the line, continue the research of (a, k)-regularized C-resolvent families, subordination principles, abstract semilinear Volterra integrodifferential equations, and provide several illustrative examples.

1. Introduction and preliminaries

The present paper can be viewed as a contribution to the theory of abstract Volterra equations that are not well-posed in the usual sense. The main part of our investigation is devoted to the study of equations on the line, (kC)-parabolic problems and L^p-stability of

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)\,ds,\ t \geqslant 0. \tag{1.1}$$

Several results from the theory of fractional differential equations in Banach spaces [**2**, **15**, **16**] are reconsidered and slightly improved. In [**23**] a method is developed to treat certain classes of semilinear Volterra integrodifferential equations in Banach space. We examine possibility of extension of this method within the framework of the theory of (local) C-regularized semigroups. Although the work mentioned above is partially confined to the scalar case, we analyze in the last section a class of nonscalar hyperbolic problems on the line.

We mainly use the following condition

(P1): $k(t)$ is Laplace transformable, i.e., it is locally integrable on $[0, \infty)$ and there exists $\beta \in \mathbb{R}$ so that
$$\tilde{k}(\lambda) = \mathcal{L}(k)(\lambda) := \lim_{b\to\infty} \int_0^b e^{-\lambda t}k(t)\,dt := \int_0^\infty e^{-\lambda t}k(t)\,dt$$
exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \beta$. Put $\operatorname{abs}(k) := \inf\{\operatorname{Re}\lambda : \tilde{k}(\lambda) \text{ exists}\}$.

2010 *Mathematics Subject Classification:* Primary 46D03, 47D06; Secondary 43D60, 43D62, 43D99.

Partially supported by grant 144016 of Ministry of Science and Technological Development, Republic of Serbia.

Let us recall that a function $k \in L^1_{\mathrm{loc}}([0,\tau))$ is called a kernel, if for every $\phi \in C([0,\tau))$, $\int_0^t k(t-s)\phi(s)\,ds = 0$, $t \in [0,\tau)$ implies $\phi(t) = 0$, $t \in [0,\tau)$, and that $0 \in \operatorname{supp} k$ implies that $k(t)$ is a kernel. Henceforth $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$. In the second section, E denotes a non-trivial complex Banach space, A is a closed linear operator in E and $L(E) \ni C$ is an injective operator which satisfies $CA \subseteq AC$. The norm in E is denoted by $\|\cdot\|$, $[\mathrm{R}(C)]$ stands for the Banach space $\mathrm{R}(C)$ equipped with the norm $\|x\|_{\mathrm{R}(C)} = \|C^{-1}x\|$, $x \in \mathrm{R}(C)$, and $[D(A)]$ stands for the Banach space $D(A)$ equipped with the graph norm $\|x\|_{[D(A)]} = \|x\| + \|Ax\|$, $x \in D(A)$. The C-resolvent set of A, denoted by $\rho_C(A)$, is the set which consists of all complex numbers λ satisfying that the operator $\lambda - A$ is injective and that $\mathrm{R}(C) \subseteq \mathrm{R}(\lambda - A)$; the resolvent set of A is also denoted by $\rho(A)$. We basically follow the notation used in the monograph of Prüss [**21**] and refer the reader to [**21**, Definition 10.2, p. 256] for the definition of a (strongly, uniformly) integrable family of operators. The notions of (a,k)-regularized C-resolvent families, (a,C)-regularized resolvent families, the condition (H5) as well as local (convoluted) C-semigroups and cosine functions are understood in the sense of [**12**, **13**].

2. The scalar case

Of concern are the following abstract Volterra equations on the line:

$$u(t) = \int_0^\infty a(s)Au(t-s)\,ds + \int_{-\infty}^t k(t-s)g'(s)\,ds, \tag{2.1}$$

where $g : \mathbb{R} \to E$, $a \in L^1_{\mathrm{loc}}([0,\infty))$, $a \neq 0$, $k \in C([0,\infty))$, $k \neq 0$, and

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)\,ds, \quad t \in (-\tau,\tau), \tag{2.2}$$

where $\tau \in (0,\infty]$ and $f \in C((-\tau,\tau) : E)$. Notice that equation (2.1) appears in the study of the problem of heat flow with memory [**19**].

Proposition 2.1. *Assume A is a subgenerator of a global (a,k)-regularized C-resolvent family $(S(t))_{t\geqslant 0}$, $g : \mathbb{R} \to \mathrm{R}(C)$, $C^{-1}g(\cdot)$ is differentiable for a.e. $t \in \mathbb{R}$, $C^{-1}g(t) \in D(A)$ for a.e. $t \in \mathbb{R}$,*

- (i) *the mapping $s \mapsto S(t-s)(C^{-1}g)'(s)$, $s \in (-\infty,t]$ is an element of the space $L^1((-\infty,t] : [D(A)])$ for a.e. $t \in \mathbb{R}$, and*
- (ii) *the mapping $s \mapsto k(t-s)g'(s)$, $s \in (-\infty,t]$ is an element of the space $L^1((-\infty,t] : E)$ for a.e. $t \in \mathbb{R}$.*

Put $u(t) := \int_{-\infty}^t S(t-s)(C^{-1}g)'(s)\,ds$, $t \in \mathbb{R}$. Then $C(\mathbb{R} : E) \ni u$ satisfies (2.1).

Proof. The continuity of $u(t)$ can be proved by using the dominated convergence theorem and the strong continuity of $(S(t))_{t\geqslant 0}$. The proof of (2.1) follows from the following computation:

$$
\begin{aligned}
&\int_0^\infty a(s)Au(t-s)\,ds + \int_{-\infty}^t k(t-s)\,g'(s)\,ds\\
&= \int_0^\infty a(s)A\int_{-\infty}^{t-s} S(t-s-r)(C^{-1}g)'(r)\,dr\,ds + \int_{-\infty}^t k(t-s)\,g'(s)\,ds\\
&= \int_0^\infty \int_0^{s'} a(s'-r')AS(r')(C^{-1}g)'(t-s')\,dr'ds' + \int_{-\infty}^t k(t-s)\,g'(s)\,ds\\
&= \int_0^\infty (S(s')-k(s')C)(C^{-1}g)'(t-s')\,ds + \int_{-\infty}^t k(t-s)\,g'(s)\,ds\\
&= u(t) - \int_0^\infty k(s)\,g'(t-s')\,ds' + \int_{-\infty}^t k(t-s)\,g'(s)\,ds = u(t),\ t\in\mathbb{R}. \qquad \square
\end{aligned}
$$

Denote by $AP(E)$, $AA(E)$, $AA_c(E)$ and $AAA(E)$ the spaces which consist of all almost periodic functions, almost automorphic functions, compact almost automorphic functions and asymptotically almost automorphic functions defined on $\mathbb{R}$, respectively, and assume that the function $(C^{-1}g)'(t)$ belongs to one of these spaces [**18**]. By [**3**, Theorem 4.6], the uniform integrability of $(S(t))_{t\geqslant 0}$ implies that the solution $u(t)$ of (2.1) belongs to the same space as $(C^{-1}g)'(t)$. The above assertion remains true in the nonscalar case.

Definition 2.1. Let $p\in[1,\infty]$, $a\in L^1_{\mathrm{loc}}([0,\infty))$, $a\neq 0$ and $k\in C([0,\infty))$, $k\neq 0$. The abstract Volterra equation (1.1) is said to be:

(i) *L^p-stable (CR)* if for every $g\in L^p([0,\infty):[\mathrm{R}(C)])$ there exists a unique function $u\in L^p([0,\infty):E)$ such that $a*u\in C([0,\infty):[D(A)])$ and that $u(t)=(a*g)(t)+A(a*u)(t)$ for a.e. $t\geqslant 0$.

(ii) *L^p-stable (CS)* if for every $f\in W^{1,p}_{\mathrm{loc}}([0,\infty):[\mathrm{R}(C)])$ such that $f'\in L^p([0,\infty):[\mathrm{R}(C)])$ there exists a unique function $u\in L^p([0,\infty):E)$ satisfying $a*u\in C([0,\infty):[D(A)])$ and $u(t)=f(t)+A(a*u)(t)$ for a.e. $t\geqslant 0$.

(iii) *C-strongly L^p-stable* if for every $g\in L^p([0,\infty):[\mathrm{R}(C)])$ there exists a unique function $u\in L^p([0,\infty):[D(A)])$ such that $a*u\in C([0,\infty):[D(A)])$ and that $u(t)=(a*g)(t)+(a*Au)(t)$ for a.e. $t\geqslant 0$.

(iv) *(kC)-parabolic* if (iv.1)–(iv.2) hold, where:

 (iv.1) $a(t)$ and $k(t)$ satisfy (P1) and there exist meromorphic extensions of the functions $\tilde{a}(\lambda)$ and $\tilde{k}(\lambda)$ on $\mathbb{C}_+$, denoted by $\hat{a}(\lambda)$ and $\hat{k}(\lambda)$. Let N be the subset of $\mathbb{C}_+$ which consists of all zeroes and possible poles of $\hat{a}(\lambda)$ and $\hat{k}(\lambda)$.

 (iv.2) There exists $M\geqslant 1$ such that, for every $\lambda\in\mathbb{C}_+\smallsetminus N$, $1/\hat{a}(\lambda)\in\rho_C(A)$ and $\|\hat{k}(\lambda)(I-\hat{a}(\lambda)A)^{-1}C\|\leqslant M/|\lambda|$.

 If $k(t)\equiv 1$, resp. $C=I$, then it is also said that (1.1) is C-parabolic, resp. k-parabolic.

Before proceeding further, notice that the definition of (kC)-parabolicity of (1.1) extends the corresponding one given by Prüss [**21**, Definition 3.1, p. 68]. As an illustrative example of a k-parabolic problem, we quote the backwards heat equation on $L^2[0,\pi]$ (cf. [**13**] for more details).

Remark 2.1. (i) Assume (1.1) is (kC)-parabolic and there exists an analytic mapping $F:\mathbb{C}_+\to L(E)$ such that $F(\lambda)=\hat{k}(\lambda)(I-\hat{a}(\lambda)A)^{-1}C$, $\lambda\in\mathbb{C}_+\smallsetminus N$ and $\sup_{\lambda\in\mathbb{C}_+}\|\lambda^2F'(\lambda)\|<\infty$. By [**21**, Theorem 0.4] and [**12**, Theorem 2.7(iii)–(iv)], we infer that, for every $\alpha\in(0,1]$, A is a subgenerator of an $(a,k*\frac{t^{\alpha-1}}{\Gamma(\alpha)})$-regularized C-resolvent family $(S_\alpha(t))_{t\geqslant 0}$ which satisfies $\sup_{h>0,t\geqslant 0}h^{-\alpha}\|S_\alpha(t+h)-S_\alpha(t)\|<\infty$; furthermore, if A is densely defined, then A is a subgenerator of a bounded (a,k)-regularized C-resolvent family $(S(t))_{t\geqslant 0}$ that is norm continuous in $t>0$. Hereon $\Gamma(\cdot)$ denotes the Gamma function.

(ii) Assume A is the integral generator of a bounded analytic C-regularized semigroup of angle $\alpha\in(0,\frac{\pi}{2}]$, $a(t)$ satisfies (P1) and admits a meromorphic extension $\hat{a}(\lambda)$ on $\mathbb{C}_+$. Let $\epsilon\in(0,\alpha)$ and $1/\hat{a}(\lambda)\in\Sigma_{\frac{\pi}{2}+\alpha-\epsilon}$, $\lambda\in\mathbb{C}_+\smallsetminus N$. Then (1.1) is C-parabolic.

Assume $n\in\mathbb{N}$, $a(t)$ satisfies (P1) and $\operatorname{abs}(a)=0$. Following [**21**, Definition 3.3, p. 69], $a(t)$ is said to be *n-regular* if there exists $c>0$ such that

$$|\lambda^m\hat{a}^{(m)}(\lambda)|\leqslant c|\hat{a}(\lambda)|,\quad \lambda\in\mathbb{C}_+,\quad 1\leqslant m\leqslant n.$$

Set $a^{(-1)}(t):=\int_0^t a(s)\,ds$, $t\geqslant 0$ and suppose that $a(t)$ and $b(t)$ are n-regular for some $n\in\mathbb{N}$. Then $\hat{a}(\lambda)\neq 0$, $\lambda\in\mathbb{C}_+$, $(a*b)(t)$ and $a^{(-1)}(t)$ are n-regular, and $a'(t)$ is n-regular provided that $\operatorname{abs}(a')=0$. Furthermore, $a(t)$ is n-regular iff there exists $c'>0$ such that $|(\lambda^m\hat{a}(\lambda))^{(m)}|\leqslant c'|\hat{a}(\lambda)|$, $\lambda\in\mathbb{C}_+$, $1\leqslant m\leqslant n$, and in the case $\arg\hat{a}(\lambda)\neq\pi$, $\lambda\in\mathbb{C}_+$, n-regularity of $a(t)$ is also equivalent to the existence of a constant $c''>0$ such that $|\lambda^m(\ln\hat{a}(\lambda))^{(m)}|\leqslant c''$, $\lambda\in\mathbb{C}_+$, $1\leqslant m\leqslant n$.

The following theorem is an extension of [**21**, Theorem 3.1, p. 73].

Theorem 2.1. *Assume $n\in\mathbb{N}$, $a(t)$ is n-regular,* (1.1) *is C-parabolic and the mapping $\lambda\mapsto(I-\tilde{a}(\lambda)A)^{-1}C$, $\lambda\in\mathbb{C}_+$ is continuous. Denote by D_t^ζ the Riemann–Liouville fractional derivative of order $\zeta>0$. Then, for every $\alpha\in(0,1]$, A is a subgenerator of an $\big(a,\frac{t^\alpha}{\Gamma(\alpha+1)}\big)$-regularized C^2-resolvent family $(S_\alpha(t))_{t\geqslant 0}$ which satisfies $\sup_{h>0,t\geqslant 0}h^{-\alpha}\|S_\alpha(t+h)-S_\alpha(t)\|<\infty$, $D_t^\alpha S_\alpha(t)C^{k-1}\in C^{k-1}((0,\infty):L(E))$, $1\leqslant k\leqslant n$ as well as:*

$$\big\|t^jD_t^jD_t^\alpha S_\alpha(t)C^{k-1}\big\|\leqslant M,\quad t\geqslant 0,\quad 1\leqslant k\leqslant n,\quad 0\leqslant j\leqslant k-1,\tag{2.3}$$

$$\begin{aligned}\big\|t^kD_t^{k-1}D_t^\alpha S_\alpha(t)C^{k-1}-s^kD_s^{k-1}D_s^\alpha S_\alpha(s)C^{k-1}\big\|&\leqslant M|t-s|\Big(1+\ln\frac{t}{t-s}\Big),\\ 0\leqslant s<t<\infty,\quad 1\leqslant k\leqslant n,&\end{aligned}\tag{2.4}$$

and, for every $T>0$, $\varepsilon>0$ and $k\in\{1,\dots,n\}$, there exists $M^\varepsilon_{T,k}>0$ such that

$$\begin{aligned}\big\|t^kD_t^{k-1}D_t^\alpha S_\alpha(t)C^{k-1}-s^kD_s^{k-1}D_s^\alpha S_\alpha(s)C^{k-1}\big\|&\leqslant M^\varepsilon_{T,k}(t-s)^{1-\varepsilon},\\ 0\leqslant s<t\leqslant T,\quad 1\leqslant k\leqslant n.&\end{aligned}\tag{2.5}$$

Furthermore, if A is densely defined, then A is a subgenerator of a bounded (a, C^2)-regularized resolvent family $(S(t))_{t\geqslant 0}$ which satisfies $S(t)C^{k-1} \in C^{k-1}((0,\infty) : L(E))$, $1 \leqslant k \leqslant n$ and (2.3)–(2.5) *with $D_t^{\alpha} S_\alpha(t)C^{k-1}$ replaced by $S(t)C^{k-1}$ $(1 \leqslant k \leqslant n)$ therein.*

Proof. We will prove the theorem provided $k \geqslant 2$ and A is nondensely defined. By [**13**, Proposition 2.4.6] and an elementary argumentation, it follows that the mapping $\lambda \mapsto (I-\tilde{a}(\lambda)A)^{-1}C$, $\lambda \in \mathbb{C}_+$ is analytic. Put $F(\lambda) := (I-\tilde{a}(\lambda)A)^{-1}C/\lambda$, $\lambda \in \mathbb{C}_+$. Then

$$F'(\lambda)C = -\frac{(I-\tilde{a}(\lambda)A)^{-1}C^2}{\lambda^2} + \frac{\tilde{a}'(\lambda)}{\lambda\tilde{a}(\lambda)}\big[(I-\tilde{a}(\lambda)A)^{-2}C^2 - (I-\tilde{a}(\lambda)A)^{-1}C^2\big], \quad \lambda \in \mathbb{C}_+$$

and

$$\begin{aligned} F''(\lambda)C^2 &= \frac{2}{\lambda^3}(I-\tilde{a}(\lambda)A)^{-1}C^3 - \frac{2}{\lambda^2}\frac{\tilde{a}'(\lambda)}{\tilde{a}(\lambda)}\big[(I-\tilde{a}(\lambda)A)^{-2}C^3 - (I-\tilde{a}(\lambda)A)^{-1}C^3\big] \\ &\quad + \frac{1}{\lambda}\frac{\tilde{a}''(\lambda)\tilde{a}(\lambda) - \tilde{a}'(\lambda)^2}{\tilde{a}(\lambda)^2}\big[(I-\tilde{a}(\lambda)A)^{-2}C^3 - (I-\tilde{a}(\lambda)A)^{-1}C^3\big] \\ &+ \frac{1}{\lambda}\Big(\frac{\tilde{a}'(\lambda)}{\tilde{a}(\lambda)}\Big)^2\big[2(I-\tilde{a}(\lambda)A)^{-3}C^3 - 3(I-\tilde{a}(\lambda)A)^{-2}C^2 + (I-\tilde{a}(\lambda)A)^{-1}C^3\big], \quad \lambda \in \mathbb{C}_+. \end{aligned}$$

Hence, $\sup_{\lambda\in\mathbb{C}_+}(\|\lambda F(\lambda)C\| + \|\lambda^2 F'(\lambda)C\| + \|\lambda^3 F''(\lambda)C^2\|) < \infty$. This inequality, in combination with [**21**, Proposition 0.1] and [**13**, Theorem 1.1.1.13], implies that, for every $\alpha \in (0,1]$, A is a subgenerator of an $\big(a, \frac{t^\alpha}{\Gamma(\alpha+1)}\big)$-regularized C^2-resolvent family $(S_\alpha(t))_{t\geqslant 0}$ which satisfies $\sup_{h>0,t\geqslant 0} h^{-\alpha}\|S_\alpha(t+h) - S_\alpha(t)\| < \infty$. Inductively, $\sup_{\lambda\in\mathbb{C}_+}\sum_{k=0}^{n}\|\lambda^{k+1}F^{(k)}(\lambda)C^k\| < \infty$, and one can apply [**21**, Theorem 0.4] in order to see that, for every $k \in \{1,\dots,n\}$, there exists a function $V_k \in C^{k-1}((0,\infty) : L(E))$ such that (2.3)–(2.5) hold with $D_t^\alpha S_\alpha(t)C^{k-1}$ replaced by $V_k(t)$. Using the uniqueness theorem for Laplace transform, one gets $S_\alpha(t)C^{k-1}x = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}V_k(s)x\,ds$, $x \in E$, $t \geqslant 0$, $1 \leqslant k \leqslant n$. Since $V_k \in L^1((0,T) : L(E))$ for all $T > 0$ and $k \in \{1,\dots,n\}$, [**2**, Theorem 1.5] implies $V_k(t) = D_t^\alpha S_\alpha(t)C^{k-1}$, $t \geqslant 0$, $1 \leqslant k \leqslant n$. This completes the proof of theorem. $\square$

Keeping in mind Theorem 2.1, one can simply transfer the representation formula [**21**, (3.41), p. 81] and the assertions of [**21**, Corollary 3.2–Corollary 3.3, pp. 74–75] to exponentially bounded (a, C)-regularized resolvent families. An application can be made to Petrovsky correct matrices of operators [**7**, **13**, **25**].

In what follows, we consider L^p-stability of (1.1).

Proposition 2.2. (i) *Assume A is a subgenerator of an integrable (a, a)-regularized C-resolvent family $(R(t))_{t\geqslant 0}$ and (H5) holds. Then* (1.1) *is L^p-stable (CR) for each $p \in [1,\infty]$.*

(ii) *Let* (1.1) *be L^p-stable (CR) for some $p \in [1,\infty]$ and let $a(t)$ satisfy* (P1). *Put $g_\mu(t) := e^{-\mu t}$, $t \geqslant 0$, $\mu \in \mathbb{C}_+$.*

(ii.1) *Then, for every $\mu \in \mathbb{C}_+$, A is a subgenerator of an $(a, a^{(-1)} * g_\mu)$-regularized C-resolvent family $(U_\mu(t))_{t\geqslant 0}$ and there exists $c(\mu) > 0$ such that $\|U_\mu(t)\| \leqslant c(\mu)t^{1/p'}$, where $[1,\infty] \ni p'$ satisfies $\frac{1}{p} + \frac{1}{p'} = 1$.*

(ii.2) *Let $D(A^2) \neq \{0\}$. Then $\tilde{a}(\lambda)$ admits a meromorphic extension $\hat{a}(\lambda)$ on $\mathbb{C}_+$. Denote by N the set which consists of all zeroes and possible poles of $\hat{a}(\lambda)$, and assume additionally $\overline{D(A)} = E$ or $\rho(A) \neq \emptyset$. Then $1/\hat{a}(\lambda) \in \rho_C(A)$, $\lambda \in \mathbb{C}_+ \setminus N$, and the mapping $\lambda \mapsto K(\lambda) := \hat{a}(\lambda)(I - \hat{a}(\lambda)A)^{-1}C$, $\lambda \in \mathbb{C}_+ \setminus N$ is uniformly bounded; if $\mathbb{C}_+ \ni \lambda_0$ and $\lim_{\lambda\to\lambda_0} \hat{a}(\lambda) = \infty$, then $0 \in \rho_C(A)$.*

(ii.3) *Let $D(A^2) \neq \{0\}$ and $p = 1$. Assume $\tilde{a}(\lambda)$, $K(\lambda)$ and N possess the same meanings as in* (ii.2)*, and $\overline{D(A)} = E$ or $\rho(A) \neq \emptyset$. Then $1/\hat{a}(\lambda) \in \rho_C(A)$, $\lambda \in \mathbb{C}_+ \setminus N$, and the mapping $\lambda \mapsto K(\lambda)$ admits a strongly continuous and uniformly bounded extension on $\overline{\mathbb{C}_+}$. Furthermore, the mapping $\lambda \mapsto \hat{a}(\lambda)$ admits a continuous extension on $\overline{\mathbb{C}_+}$ which takes values in $\mathbb{C} \cup \{\infty\}$; if* $\lim_{\lambda\to\lambda_0} \hat{a}(\lambda) = \infty$ *for some $\lambda_0 \in \overline{\mathbb{C}_+}$, then $0 \in \rho_C(A)$.*

(iii) *Let* (1.1) *be C-strongly L^p-stable. Then* (1.1) *is C-parabolic.*

Proof. We will prove only (ii). Fix $\mu \in \mathbb{C}_+$ and denote by $u_\mu(t;x)$ the unique function which satisfies

$$a * u_\mu(\cdot;x) \in C([0,\infty):[D(A)]), \quad u_\mu(t;x) = (a * g_\mu)(t)Cx + A(a * u_\mu(\cdot;x))(t)$$

for a.e. $t \geqslant 0$ and $u_\mu(t;x) \in L^p([0,\infty):E)$. By the closed graph theorem, it follows that there exists a constant $c > 0$ such that $\|u_\mu(\cdot;x)\|_p \leqslant c\|g\|_p = c(p \operatorname{Re}\mu)^{-1/p}\|x\|$. Define $U_\mu(t)x := \int_0^t u_\mu(s;x)\,ds$, $t \geqslant 0$, $x \in E$. Then $(U_\mu(t))_{t\geqslant 0}$ is a strongly continuous operator family and there exists $c(\mu) > 0$ such that $\|U_\mu(t)\| \leqslant c(\mu)t^{1/p'}$, $t \geqslant 0$. Using the Laplace transform, we get

$$\big(I - \tilde{a}(\lambda)A\big)\widetilde{U_\mu}(\lambda)x = \frac{\tilde{a}(\lambda)}{\lambda(\lambda+\mu)}, \quad x \in E, \quad \operatorname{Re}\lambda > \max(0, \operatorname{abs}(a)).$$

Combining with [**12**, Theorem 2.6(ii)], this implies that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \max(0, \operatorname{abs}(a))$ and $\tilde{a}(\lambda) \neq 0$, $1/\tilde{a}(\lambda) \in \rho_C(A)$ and that A is a subgenerator of an $(a, a^{(-1)} * g_\mu)$-regularized C-resolvent family $(U_\mu(t))_{t\geqslant 0}$, finishing the proof of (ii.1). Set $f_\mu(\lambda) := \lambda(\lambda+\mu)\widetilde{U_\mu}(\lambda)$, $\operatorname{Re}\lambda > 0$. In order to prove (ii.2), notice that for every $x \in D(A)$, $x^* \in E^*$, and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \max(0, \operatorname{abs}(a))$, $\tilde{a}(\lambda) \neq 0$ and $\langle x^*, f_\mu(\lambda)x\rangle \neq 0$, we have $\frac{1}{\tilde{a}(\lambda)} = \frac{\langle x^*, Cx\rangle + \langle x^*, f_\mu(\lambda)Ax\rangle}{\langle x^*, f_\mu(\lambda)x\rangle}$. Assume $\langle x^*, f_\mu(\lambda)x\rangle = 0$ for all $x \in D(A)$, $x^* \in E^*$ and $\lambda \in \mathbb{C}_+$. Then $\langle x^*, U_\mu(t)x\rangle = 0$ for all $x \in D(A)$, $x^* \in E^*$ and $t \geqslant 0$. This shows that, for every $x \in D(A^2)$, $x^* \in E^*$ and $t \geqslant 0$,

$$0 = \langle x^*, Cx\rangle\big(a^{(-1)} * g_\mu\big)(t) + \int_0^t a(t-s)\langle x^*, U_\mu(s)Ax\rangle ds,$$

which implies $\langle x^*, Cx\rangle = 0$, $x \in D(A^2)$, $x^* \in E^*$. This is a contradiction to the assumption $D(A^2) \neq \{0\}$. The existence of an element $x \in D(A)$ and a functional $x^* \in E^*$ such that $\langle x^*, f_\mu(\cdot)x\rangle \neq 0$ is clear now. Fix such x, x^* and assume $\langle x^*, Cx\rangle + \langle x^*, f_\mu(\lambda)Ax\rangle = 0$, $\lambda \in \mathbb{C}_+$. Then one obtains $\frac{1}{\tilde{a}(\lambda)}\langle x^*, f_\mu(\lambda)x\rangle =$

$\langle x^*, f_\mu(\lambda)x\rangle = 0$, $\operatorname{Re}\lambda > \max(0, \operatorname{abs}(a))$, $\tilde{a}(\lambda) \neq 0$ and $\langle x^*, f_\mu(\lambda)x\rangle = 0$, $\lambda \in \mathbb{C}_+$, which is a contradiction. Therefore, $\langle x^*, Cx\rangle + \langle x^*, f_\mu(\lambda)Ax\rangle \neq 0$, $\lambda \in \mathbb{C}_+$. Let N be the set which consists of numbers $\lambda_0 \in \mathbb{C}_+$ such that $\langle x^*, Cx\rangle + \langle x^*, f_\mu(\lambda_0)Ax\rangle = 0$ and that $\langle x^*, f_\mu(\lambda_0)x\rangle = 0$. Then $\tilde{a}(\lambda) = \frac{\langle x^*, f_\mu(\lambda)x\rangle}{\langle x^*, Cx\rangle + \langle x^*, f_\mu(\lambda)Ax\rangle}$, $\lambda \in \mathbb{C}_+ \smallsetminus N$, $\operatorname{Re}\lambda > \max(0, \operatorname{abs}(a))$, which shows that $\tilde{a}(\lambda)$ admits a meromorphic extension $\hat{a}(\lambda)$ on $\mathbb{C}_+$. Further on, $A\int_0^\infty e^{-\lambda t}U_\mu(t)y\,dt = \int_0^\infty e^{-\lambda t}U_\mu(t)Ay\,dt$, $\lambda \in \mathbb{C}_+$, $y \in D(A)$ and $Af_\mu(\lambda)y = f_\mu(\lambda)Ay$, $\lambda \in \mathbb{C}_+$, $y \in D(A)$. Since A is closed and the mapping $\lambda \mapsto Af_\mu(\lambda)y = f_\mu(\lambda)Ay$, $\lambda \in \mathbb{C}_+$ is analytic for every fixed $y \in D(A)$, we obtain:

$$Cy = \Big(\frac{1}{\hat{a}(\lambda)} - A\Big)f_\mu(\lambda)y, \ \ y \in \overline{D(A)}, \ \ \lambda \in \mathbb{C}_+ \smallsetminus N.$$

This implies $\frac{1}{\hat{a}(\lambda)} \in \rho_C(A)$ if $\overline{D(A)} = E$. Since $R(\cdot : A)$ and $f_\mu(\cdot)$ commutes, the above conclusion still holds if $\rho(A) \neq \emptyset$. The uniform boundedness of the mapping $\lambda \mapsto K(\lambda)$, $\lambda \in \mathbb{C}_+ \smallsetminus N$ follows as in [**21**]. Let $\lambda_0 \in \mathbb{C}_+$ and $\lim_{\lambda\to\lambda_0}\hat{a}(\lambda) = \infty$. Then $\lim_{n\to\infty} f_\mu(\lambda_n) = f_\mu(\lambda_0)$, $\lim_{n\to\infty} Af_\mu(\lambda_n) = \lim_{n\to\infty}\frac{1}{\hat{a}(\lambda_n)}f_\mu(\lambda_n) - Cy = -Cy$ and the closedness of A implies $-Af_\mu(\lambda_0)y = Cy$, $y \in E$ and $0 \in \rho_C(A)$, as required. The proof of (ii.2) is complete. In the case $p = 1$, the existence of a strongly continuous and uniformly bounded extension of the mapping $\lambda \mapsto K(\lambda)$ on $\overline{\mathbb{C}_+}$ can be proved as in [**21**]. Assume now $\varrho \in \mathbb{R}$, (λ_n) is a sequence in $\mathbb{C}_+ \smallsetminus N$ and $\lim_{n\to\infty}\lambda_n = i\varrho$. Let $\lim_{n\to\infty}\frac{1}{\hat{a}(\lambda_n)} = z \in \mathbb{C}$; then $z \in \rho_C(A)$ and $\lim_{n\to\infty} K(\lambda_n)C = (z - A)^{-1}C^2$ in $L(E)$. Let $\lim_{n\to\infty}\frac{1}{\hat{a}(\lambda_n)} = \infty$; then $Cy = \frac{1}{\hat{a}(\lambda_n)}f_\mu(\lambda_n)y - f_\mu(\lambda_n)Ay$, $\lim_{n\to\infty} K(\lambda_n)y = 0$, $y \in D(A)$, and $\lim_{n\to\infty} K(\lambda_n)y = 0$, $y \in E$. This enables one to see that $\hat{a}(\lambda)$ admits a continuous extension on $\overline{\mathbb{C}_+}$ which takes values in $\mathbb{C} \cup \{\infty\}$. The rest of the proof of (ii.3) simply follows. □

In almost the same way, one can prove the following proposition.

PROPOSITION 2.3. (i) *Let A be a subgenerator of an (a, C)-regularized resolvent family $(S(t))_{t\geqslant0}$ that is integrable and bounded. Then* (1.1) *is L^p-stable (CS) for each $p \in [1, \infty]$, and $(S(t))_{t\geqslant0}$ is L^1-stable (CS) iff $(S(t))_{t\geqslant0}$ is strongly integrable.*

(ii) *Let* (1.1) *be L^p-stable (CS) for some $p \in [1, \infty]$ and let $a(t)$ satisfy* (P1).

(ii.1) *Then, for every $\mu \in \mathbb{C}_+$, A is a subgenerator of an $(a, 1*g_\mu)$-regularized C-resolvent family $(V_\mu(t))_{t\geqslant0}$ and there exists $c(\mu) > 0$ such that $\|V_\mu(t)\| \leqslant c(\mu)t^{1/p'}$, where $[1, \infty] \ni p'$ satisfies $\frac{1}{p} + \frac{1}{p'} = 1$.*

(ii.2) *Let $D(A^2) \neq \{0\}$. Then $\tilde{a}(\lambda)$ admits a meromorphic extension $\hat{a}(\lambda)$ on $\mathbb{C}_+$. Denote by N the set which consists of all zeros and possible poles of $\hat{a}(\lambda)$, and assume additionally: $\overline{D(A)} = E$ or $\rho(A) \neq \emptyset$. Then $1/\hat{a}(\lambda) \in \rho_C(A)$, $\lambda \in \mathbb{C}_+ \smallsetminus N$, and the mapping $\lambda \mapsto H(\lambda) = (I - \hat{a}(\lambda)A)^{-1}C/\lambda$, $\lambda \in \mathbb{C}_+ \smallsetminus N$ is uniformly bounded. Furthermore, A is invertible provided $C = I$.*

(ii.3) *Let $D(A^2) \neq \{0\}$ and $p = 1$. Assume $\tilde{a}(\lambda)$, $H(\lambda)$ and N possess the same meanings as in* (ii.2), *and $\overline{D(A)} = E$ or $\rho(A) \neq \emptyset$. Then $1/\hat{a}(\lambda) \in \rho_C(A)$, $\lambda \in \mathbb{C}_+ \smallsetminus N$, and the mapping $\lambda \mapsto H(\lambda)$ admits a strongly continuous and uniformly bounded extension on $\overline{\mathbb{C}_+}$. The mapping $\lambda \mapsto$*

$\hat{a}(\lambda)$ *admits a continuous extension on* $\overline{\mathbb{C}_+}$ *which takes values in* $\mathbb{C}\cup\{\infty\}$*,* $\lim_{\lambda\to 0}\hat{a}(\lambda)=\infty$*, and in the case* $0\in\rho_C(A)$*, there exists* $\lim_{\lambda\to 0}\lambda\hat{a}(\lambda)$ *in* $(\mathbb{C}\cup\{\infty\})\smallsetminus\{0\}$.

It is worthwhile to mention that it is not clear how one can prove an analogue of [**21**, Theorem 10.1, p. 262] and its consequences in the case of a general (a,k)-regularized C-resolvent family. Nevertheless, in many cases, $(S(t))_{t\geqslant 0}$ is not integrable in any sense but (2.1) is solvable provided that the function $t\mapsto \|(C^{-1}g)'(t)\|_{[D(A)]}$ decays polynomially as $t\to-\infty$ (cf. Proposition 2.1, [**12**, Example 2.31(iii)], [**15**, Theorem 4.1], [**20**, pp. 251–252] and Theorem 2.4(iv) given below).

A function $u\in C((-\tau,\tau):E)$ is called a *solution* of (2.2) iff $a*u\in C((-\tau,\tau):[D(A)])$ and $u(t)=f(t)+A\int_0^t a(t-s)u(s)\,ds$, $t\in(-\tau,\tau)$.

Proposition 2.4. *[**12**]* (i) *Assume* $a\in L^1_{\mathrm{loc}}((-\tau,\tau))$*,* $k\in C((-\tau,\tau))$*,* $a\neq 0$ *and* $k\neq 0$*. Let* $k_+(t)=k(t)$*,* $a_+(t)=a(t)$*,* $t\in[0,\tau)$*,* $k_-(t)=k(-t)$ *and* $a_-(t)=a(-t)$*,* $t\in(-\tau,0]$*. If* $\pm A$ *are subgenerators of* $(a_\pm,k_\pm)$*-regularized* C*-resolvent families* $(S_\pm(t))_{t\in[0,\tau)}$*, then, for every* $x\in D(A)$*, the function* $u:(-\tau,\tau)\to E$ *given by* $u(t)=S_+(t)x$*,* $t\in[0,\tau)$ *and* $u(t)=S_-(-t)x$*,* $t\in(-\tau,0]$ *is a solution of* (2.2) *with* $f(t)=k(t)Cx$*,* $t\in(-\tau,\tau)$*. Furthermore, the solutions of* (2.2) *are unique provided that* $k_\pm(t)$ *are kernels.*

(ii) *Assume* $n_\pm\in\mathbb{N}$*,* $f\in C((-\tau,\tau):E)$*,* $a\in L^1_{\mathrm{loc}}((-\tau,\tau))$*,* $a\neq 0$*,* $f_+(t)=f(t)$*,* $a_+(t)=a(t)$*,* $t\in[0,\tau)$*,* $f_-(t)=f(-t)$*,* $a_-(t)=a(-t)$*,* $t\in(-\tau,0]$*, and* $\pm A$ *are subgenerators of* $(n_\pm-1)$*-times integrated* $(a_\pm,C_\pm)$*-regularized resolvent families. Assume, additionally,* $a_\pm\in BV_{\mathrm{loc}}([0,\tau))$ *if* $n_\pm>1$ *(that is:* $a_+\in BV_{\mathrm{loc}}([0,\tau))$ *if* $n_+>1$*, and* $a_-\in BV_{\mathrm{loc}}([0,\tau))$ *if* $n_->1$*) as well as:*

(ii.1) $C_\pm^{-1}f_\pm\in C^{(n_\pm)}([0,\tau):E)$*,* $f_\pm^{(k-1)}(0)\in D(A^{n_\pm-k})$ *and* $A^{n_\pm-k}f^{(k-1)}(0)\in \mathrm{R}(C_\pm)$*,* $1\leqslant k\leqslant n_\pm$*, if* $n_\pm>1$*, resp.*

(ii.2) $C_\pm^{-1}f_\pm\in C([0,\tau):E)\cap W^{1,1}_{\mathrm{loc}}([0,\tau):E)$ *if* $n_+=n_-=1$*.*

Then there exists a unique solution of (2.2).

Example 2.1. (i) Assume $-\infty<\alpha\leqslant\beta<\infty$, $1\leqslant p\leqslant\infty$, $0<\tau\leqslant\infty$, $n\in\mathbb{N}$, $E=L^p(\mathbb{R}^n)$ or $E=C_b(\mathbb{R}^n)$, $P(\cdot)$ is an elliptic polynomial of degree $m\in\mathbb{N}$, $\alpha\leqslant\mathrm{Re}(P(x))\leqslant\beta$, $x\in\mathbb{R}^n$ and $A=P(D)$. Then there exists $\omega>0$ such that, for every $r>n|\frac{1}{2}-\frac{1}{p}|$, $\pm A$ are the integral generators of exponentially bounded $(\omega\mp A)^{-r}$-regularized semigroups in E. Let $a\in L^1_{\mathrm{loc}}(\mathbb{R})$, $a\neq 0$, be such that the mappings $t\mapsto a_+(t)=a(t)$, $t\geqslant 0$ and $t\mapsto a_-(t)=a(-t)$, $t\geqslant 0$ are completely positive. By [**12**, Theorem 2.8(ii)], $\pm A$ are the integral generators of exponentially bounded $(a_\pm,(\omega\mp A)^{-r})$-regularized resolvent families provided $E=L^p(\mathbb{R}^n)$ $(1\leqslant p<\infty)$, resp. $(a_\pm,t)$-regularized $(\omega\mp A)^{-r}$-resolvent families provided $E=L^\infty(\mathbb{R}^n)$ $(C_b(\mathbb{R}^n))$. Let $f\in C((-\tau,\tau):E)$ and let $f_\pm(t)$ satisfy the assumption of Proposition 2.4(ii.2), resp. Proposition 2.4(ii.1), with $n_\pm=1$, resp. $n_\pm=2$. Then there exists a unique solution of (2.2); it is noteworthy that the above example can be reformulated in the case when A is the integral generator of an exponentially bounded integrated group or C-regularized group [**10**, **11**, **24**].

(ii) Assume $E = L^2[0,\pi]$, $A = -\Delta$ with the Dirichlet or Neumann boundary conditions, $\tau = \infty$, $\beta \in [\frac{1}{2}, 1)$, $\alpha > 1 + \beta$, $a(t) = \frac{|t|^{\beta-1}}{\Gamma(\beta)}$, $t \in (-\tau, \tau)$ and $f(t) = \mathcal{L}^{-1}(\mathbf{h}_{\alpha,\beta}(\lambda))(|t|)$, $t \in (-\tau,\tau)$, where $\mathbf{h}_{\alpha,\beta}(\lambda)$ is defined through [**12**, (2.64)] and $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. Then Proposition 2.4(i) implies that there exists a unique solution $u(t)$ of (2.2) and that $u_{|\mathbb{R}\smallsetminus\{0\}}$ is analytically extendible to the sector $\Sigma_{\frac{\pi}{2}(\frac{1}{\beta}-1)}$. By Proposition 2.4(i) and [**12**, Example 2.31(iii)], it follows that, for every $n \in \mathbb{N}$, there exists an exponentially bounded kernel $k_n(t)$ such that (2.2) has a unique solution $u_n(t)$ with A replaced by the polyharmonic operator Δ^{2^n} and $f(t)$ replaced by $k_n(t)$; moreover, $u_{n|\mathbb{R}\smallsetminus\{0\}}$ is analytically extendible to the sector $\Sigma_{\frac{\pi}{2}}$. We refer the reader to [**12**] for the analysis of preceding example in the case $\beta \in [1,2)$.

The next proposition clarifies Abel-ergodic properties of an (a,k)-regularized C-resolvent family.

Proposition 2.5. *Assume $a(t)$ and $k(t)$ satisfy* (P1)*, $\lim_{\lambda\to+\infty,\tilde{k}(\lambda)\neq 0} \lambda\tilde{k}(\lambda) = k(0)$, there exists $\omega \in \mathbb{R}$ such that $\int_0^\infty e^{-\omega t}|a(t)|\,dt < \infty$ and A is a subgenerator of an exponentially bounded (a,k)-regularized C-resolvent family $(S(t))_{t\geqslant 0}$. Then*

$$\lim_{\lambda\to+\infty,\tilde{k}(\lambda)\neq 0} \lambda\tilde{k}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}Cx = k(0)Cx, \quad x \in \overline{D(A)}. \tag{2.6}$$

Proof. Let $\|S(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$ for appropriate constants $M \geqslant 1$ and $\omega \geqslant 0$. Let $\omega_0 = \max(\omega, \operatorname{abs}(a), \operatorname{abs}(k))$. By [**12**, Theorem 2.6], we have that, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0$ and $\tilde{k}(\lambda) \neq 0$, the operator $I - \tilde{a}(\lambda)A$ is injective and that $\mathrm{R}(C) \subseteq \mathrm{R}(I - \tilde{a}(\lambda)A)$. Furthermore, $\tilde{k}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}Cx = \int_0^\infty e^{-\lambda t}R(t)x\,dt$, $x \in E$, $\operatorname{Re}\lambda > \omega_0$, $\tilde{k}(\lambda) \neq 0$, which implies

$$\left\|\tilde{k}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}C\right\| \leqslant \frac{M}{\operatorname{Re}\lambda - \omega}, \quad \operatorname{Re}\lambda > \omega_0, \quad \tilde{k}(\lambda) \neq 0. \tag{2.7}$$

The proof of (2.6) in the case $x \in D(A)$ follows from (2.7), the assumption $\lim_{\lambda\to+\infty,\tilde{k}(\lambda)\neq 0} \lambda\tilde{k}(\lambda) = k(0)$ and the identity

$$(I - \tilde{a}(\lambda)A)^{-1}Cx = \tilde{a}(\lambda)(I - \tilde{a}(\lambda)A)^{-1}CAx + Cx, \quad \operatorname{Re}\lambda > \omega_0, \quad \tilde{k}(\lambda) \neq 0.$$

The proof of (2.6) in the case $x \in \overline{D(A)}$ follows from the standard limit procedure. □

The following proposition can be also viewed as of some independent interest (see e.g. [**12**, Theorem 2.6(i)] and the proofs of [**16**, Proposition 2.7] and [**17**, Proposition 2.6]).

Proposition 2.6. (i) *Assume $a(t)$ and $k(t)$ satisfy (P1), $M \geqslant 1$, $\omega \geqslant 0$, $(S(t))_{t\geqslant 0}$ is an (a,k)-regularized C-resolvent family satisfying $\|S(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$, and $AC \notin L(E)$. Then, for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \max(\omega, \operatorname{abs}(a), \operatorname{abs}(k))$ and $\tilde{k}(\lambda) \neq 0$, we have $\tilde{a}(\lambda) \neq 0$ and $1/\tilde{a}(\lambda) \in \rho_C(A)$.*

(ii) *Assume $\alpha \in (0,1)$, A is a subgenerator of a global $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, k)$-regularized C-resolvent family $(S_\alpha(t))_{t\geqslant 0}$, $D(A) \neq \{0\}$, and $\lim_{t\to+\infty}|k(t)|$ does not exist in*

$[0,\infty]$ or $\lim_{t\to+\infty}|k(t)|\neq 0$. Then there exist no $M\geqslant 1$ and $\omega>0$ such that $\|S_\alpha(t)\|\leqslant Me^{-\omega t}$, $t\geqslant 0$.

Recall that Webb considered in [**23**] a class of abstract semilinear Volterra equations appearing in thermodynamics of materials with memory [**4, 5**]. An insignificant modification of the proofs of [**23**, Theorem 2.1–2.2, Corollary 2.1] implies the following theorem.

THEOREM 2.2. (i) *Assume A is a subgenerator of a (local) C-regularized semigroup $(T(t))_{t\in[0,\tau)}$ and there exists $t_0\in(0,\tau)$ such that:*

(i.1) *$C^{-1}f\in C^1([0,t_0]:E)$,*

(i.2) *$C^{-1}g\in C([0,t_0]\times D:E)$, where D is an open subset of $[D(A)]$, $C^{-1}g(t,x)$ is continuously differentiable with respect to t, and for each $x\in D$ there exists a neighborhood D_x about x and continuous functions $b:[0,t_0]\to[0,\infty)$ and $c:[0,t_0]\to[0,\infty)$ such that*

$$\|C^{-1}g(t,x_1)-C^{-1}g(t,x_2)\|\leqslant b(t)\|x_1-x_2\|_{[D(A)]},\ \ t\in[0,t_0],\ \ x_1,\ \ x_2\in D_x,$$

$$\Big\|\frac{\partial}{\partial t}C^{-1}g(t,x_1)-\frac{\partial}{\partial t}C^{-1}g(t,x_2)\Big\|\leqslant c(t)\|x_1-x_2\|_{[D(A)]},\ \ t\in[0,t_0],\ \ x_1,\ \ x_2\in D_x.$$

Then, for each $x\in C(D)$, there exist a number $t_1\in(0,t_0)$ and a unique function $u:[0,t_1]\to E$ such that $u\in C^1([0,t_1]:E)\cap C([0,t_1]:[D(A)])$,

$$u'(t)=Au(t)+\int_0^t g(t-s,u(s))\,ds+f(t),\ \ t\in[0,t_1]\ \text{and}\ u(0)=x. \tag{2.8}$$

Assume further $n\in\mathbb{N}$, $x\in C(D(A^n))$, $\tau=\infty$ as well as (i.1) *and* (i.2) *hold with $C^{-1}f$, $C^{-1}g$, $D=D_y$ ($y\in D(A)$), $[0,t_0]$, $b:[0,t_0]\to[0,\infty)$ and $c:[0,t_0]\to[0,\infty)$, replaced by $C^{-n}f$, $C^{-n}g$, $[D(A)]$, $[0,nt_0]$, $b_n:[0,nt_0]\to[0,\infty)$ and $c_n:[0,nt_0]\to[0,\infty)$, respectively. Then there exists a unique function $u_n:[0,nt_1]\to E$ such that $u_n\in C^1([0,nt_1]:E)\cap C([0,nt_1]:[D(A)])$ and that* (2.8) *holds with $u(t)$ and $[0,t_1]$ replaced by $u_n(t)$ and $[0,nt_1]$, respectively.*

(ii) *Assume $x\in D$,* (i.1)–(i.2) *hold, $M\geqslant 1$, $\omega\in\mathbb{R}$, $\|T(t)\|\leqslant Me^{\omega t}$, $t\in[0,\tau)$ and $x_1,x_2\in C(D)$. Denote by $u_1(t)$ and $u_2(t)$ the solutions of* (2.8) *with initial values x_1 and x_2, respectively, and set $\alpha(t)=\int_0^t e^{-\omega s}(b(s)+c(s))\,ds$, $t\in[0,t_1]$ and $\beta(t)=\max_{s\in[0,t]}e^{-\omega s}b(s)$, $t\in[0,t_1]$. Then the assumption $\{u_1(t),u_2(t)\}\subseteq D_x$, $t\in[0,t_1]$ implies:*

$$\|u_1(t)-u_2(t)\|\leqslant M\|C^{-1}x_1-C^{-1}x_2\|_{[D(A)]}e^{(M\alpha(t)+\beta(t)+Mb(0)+\omega)t},\ \ t\in[0,t_1].$$

Furthermore, if $D=D_x=[D(A)]$, $x\in D(A)$ and $M\alpha(t)+\beta(t)+Mb(0)+\omega\leqslant\gamma$, for some $\gamma\in\mathbb{R}$ and every $t\in[0,t_1]$, then $\|u_1(t)-u_2(t)\|\leqslant M\|C^{-1}x_1-C^{-1}x_2\|_{[D(A)]}e^{\gamma t}$, $t\in[0,t_1]$.

Let $\alpha>0$, $\beta>0$ and $\gamma\in(0,1)$. Then $\mathbf{D}_t^\alpha$ denotes the Caputo fractional derivative of order α, $E_\beta(z)$ denotes the Mittag-Leffler function which is given by $E_\beta(z):=\sum_{n=0}^\infty\frac{z^n}{\Gamma(\beta n+1)}$, $z\in\mathbb{C}$ and $\Phi_\gamma(t)$ denotes the Wright function which is given by $\Phi_\gamma(t):=\mathcal{L}^{-1}(E_\gamma(-\lambda))(t)$, $t\geqslant 0$ (for further information, see e.g. [**2, 9, 22**] and references therein).

The following theorem extends [**2**, Theorem 2.26] and can be applied to the coercive differential operators considered by Li, Li and Zheng in [**15**, Section 4].

THEOREM 2.3. *Assume that* $\alpha \in (1,2)$ *and* A *is a subgenerator of an* $\big(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, C\big)$*-regularized resolvent family* $(S_\alpha(t))_{t \geqslant 0}$ *which satisfies* $\|S_\alpha(t)\| \leqslant Me^{\omega t}$ *for appropriate constants* $M \geqslant 1$ *and* $\omega \geqslant 0$. *Let* $(B(t))_{t\geqslant 0} \subseteq L(E)$, $\mathrm{R}(B(t)) \subseteq \mathrm{R}(C)$, $t \geqslant 0$, $C^{-1}B(\cdot) \in C([0,\infty) : L(E))$ *and* $CB(t)x = B(t)Cx$, $x \in D(A)$. *Then, for every* $x \in D(A)$, *there exists a unique solution* $u(t)$ *of the problem*

$$\mathbf{D}_t^\alpha u(t,x) = (A + B(t))u(t,x), \quad t > 0,$$
$$u(0,x) = Cx, \quad u'(0,x) = 0.$$

The solution $u(t,x)$ *is given by* $u(t,x) = \sum_{n=0}^{\infty} S_{\alpha,n}(t)x$, $t \geqslant 0$, *where we define* $S_{\alpha,n}(t)$ $(t \geqslant 0)$ *recursively by*

$$S_{\alpha,0}(t) = S_\alpha(t) \text{ and } S_{\alpha,n}(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(s) C^{-1} B(s) S_{\alpha,n-1}(s)\, ds.$$

Denote $K(T) = \max_{t\in[0,T]} \|C^{-1}B(t)\|$, $T \geqslant 0$. *Then*

$$\|u(t,x)\| \leqslant Me^{\omega t} E_\alpha(MK_T t^\alpha)\|x\|, \quad t \in [0,T],$$
$$\|u(t,x) - S_\alpha(t)x\| \leqslant Me^{\omega t}(E_\alpha(MK_T t^\alpha) - 1)\|x\|, \quad t \in [0,T].$$

THEOREM 2.4. *[2, Section 3] Assume* $k_\beta(t)$ *satisfies (P1),* $0 < \alpha < \beta$, $\gamma = \frac{\alpha}{\beta}$ *and* A *is a subgenerator of an* $\big(\frac{t^{\beta-1}}{\Gamma(\beta)}, k_\beta\big)$*-regularized* C*-resolvent family* $(S_\beta(t))_{t\geqslant 0}$ *which satisfies* $\|S_\beta(t)\| = O(e^{\omega t})$, $t \geqslant 0$ *for some* $\omega \geqslant \max(0, \mathrm{abs}(k_\beta))$. *Assume additionally that* (H5) *holds and that there exists a function* $k_\alpha(t)$ *satisfying* (P1), $k_\alpha(0) = k_\beta(0)$ *and* $\tilde{k_\alpha}(\lambda) = \lambda^{\frac{\alpha}{\beta}-1}\tilde{k_\beta}(\lambda^{\frac{\alpha}{\beta}})$ *for all sufficiently large positive real numbers* λ. *Then* A *is a subgenerator of a* $\big(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, k_\alpha\big)$*-regularized* C*-resolvent family* $(S_\alpha(t))_{t\geqslant 0}$ *which satisfies* $\|S_\alpha(t)\| = O(e^{\omega^{\frac{\beta}{\alpha}} t})$, $t \geqslant 0$ *and*

$$S_\alpha(t)x = \int_0^\infty t^{-\gamma}\Phi_\gamma(st^{-\gamma}) S_\beta(s)x\, ds, \quad x \in E, \quad t > 0.$$

Furthermore:

(i) *The mapping* $t \mapsto S_\alpha(t)$, $t > 0$ *has an analytic extension to the sector* $\Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2}, \pi)}$.

(ii) *If* $\omega = 0$ *and* $\varepsilon \in (0, \min((\frac{1}{\gamma}-1)\frac{\pi}{2}, \pi))$, *then there exists* $M_{\gamma,\varepsilon} > 0$ *such that* $\|S_\alpha(z)\| \leqslant M_{\gamma,\varepsilon}$, $z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\pi)-\varepsilon}$.

(iii) *If* $\omega > 0$ *and* $\varepsilon \in (0, \min((\frac{1}{\gamma}-1)\frac{\pi}{2}, \frac{\pi}{2}))$, *then there exist* $\delta_{\gamma,\varepsilon} > 0$ *and* $M_{\gamma,\varepsilon} > 0$ *such that* $\|S_\alpha(z)\| \leqslant M_{\gamma,\varepsilon} e^{\delta_{\gamma,\varepsilon} \operatorname{Re} z}$, $z \in \Sigma_{\min((\frac{1}{\gamma}-1)\frac{\pi}{2},\frac{\pi}{2})-\varepsilon}$.

(iv) *Let* $\zeta \geqslant 0$. *Then the assumption* $\|S_\beta(t)\| = O(1+t^\zeta)$, $t \geqslant 0$, *resp.* $\|S_\beta(t)\| = O(t^\zeta)$, $t \geqslant 0$, *implies* $\|S_\alpha(t)\| = O(1+t^{\gamma\zeta})$, $t \geqslant 0$, *resp.* $\|S_\alpha(t)\| = O(t^{\gamma\zeta})$, $t \geqslant 0$.

We close this section with the assertion that [**2**, Theorem 3.12] remains true in the case of non-densely defined operators.

3. The nonscalar case

In this section, X and Y denote non-trivial complex Banach spaces such that Y is continuously embedded in X; $L(X) \ni C$ is an injective operator and $\tau \in (0,\infty]$. The norm in X, resp. Y, is denoted by $\|\cdot\|_X$, resp. $\|\cdot\|_Y$, and $[\mathrm{R}(C)]$ stands for the Banach space $\mathrm{R}(C)$ equipped with the norm $\|x\|_{\mathrm{R}(C)} = \|C^{-1}x\|_X$, $x \in \mathrm{R}(C)$. Given a closed linear operator A in X, we use the abbreviation $[D(A)]$ for the Banach space $D(A)$ equipped with the graph norm $\|x\|_{[D(A)]} = \|x\|_X + \|Ax\|_X$, $x \in D(A)$. Let $A(t)$ be a locally integrable function from $(-\tau,\tau)$ into $L(Y,X)$. In the sequel, we assume that $A(t)$ is not of scalar type, which means that there exist neither a function $a \in L^1_{\mathrm{loc}}((-\tau,\tau))$, $a \neq 0$, nor a closed linear operator A in X such that $Y = [D(A)]$ and that $A(t) = a(t)A$ for a.e. $t \in (-\tau,\tau)$.

Definition 3.1. [**14**] Let $\tau \in (0,\infty]$, $k \in C([0,\tau))$, $k \neq 0$ and $A \in L^1_{\mathrm{loc}}([0,\tau) : L(Y,X))$, $A \neq 0$. An operator family $(S(t))_{t\in[0,\tau)}$ is called an (A,k)-*regularized C-pseudoresolvent family* iff the following holds:

(S1) The mapping $t \mapsto S(t)x$, $t \in [0,\tau)$ is continuous in X for every fixed $x \in X$ and $S(0) = k(0)C$.

(S2) Put $U(t)x := \int_0^t S(s)x\,ds$, $x \in X$, $t \in [0,\tau)$. Then (S2) is equivalent to say that $U(t)Y \subseteq Y$, $U(t)_{|Y} \in L(Y)$, $t \in [0,\tau)$ and that $(U(t)_{|Y})_{t\in[0,\tau)}$ is locally Lipschitz continuous in $L(Y)$.

(S3) The resolvent equations

$$S(t)y = k(t)Cy + \int_0^t A(t-s)\,dU(s)y\,ds, \quad t \in [0,\tau), \quad y \in Y, \tag{3.1}$$

$$S(t)y = k(t)Cy + \int_0^t S(t-s)A(s)y\,ds, \quad t \in [0,\tau), \quad y \in Y, \tag{3.2}$$

hold; (3.1), resp. (3.2), is called the first resolvent equation, resp. the second resolvent equation.

An (A,k)-regularized C-pseudoresolvent family $(S(t))_{t\in[0,\tau)}$ is said to be an (A,k)-*regularized C-resolvent family* if additionally:

(S4) For every $y \in Y$, $S(\cdot)y \in L^\infty_{\mathrm{loc}}([0,\tau) : Y)$.

A family $(S(t))_{t\in[0,\tau)}$ in $L(X)$ is called a *weak (A,k)-regularized C-pseudoresolvent family* iff (S1) and (3.2) hold; in the case $\tau = \infty$, $(S(t))_{t\geqslant 0}$ is said to be exponentially bounded iff there exist $M \geqslant 1$ and $\omega \geqslant 0$ such that $\|S(t)\|_{L(X)} \leqslant Me^{\omega t}$, $t \geqslant 0$.

In what follows, a (weak) (A,k)-regularized C-(pseudo)resolvent family with $k(t) \equiv \frac{t^\alpha}{\Gamma(\alpha+1)}$ ($\alpha \geqslant 0$) is also called a (weak) α-times integrated A-regularized C-(pseudo)resolvent family; a (weak) 0-times integrated A-regularized C-(pseudo)-resolvent family is also said to be a (weak) A-regularized C-(pseudo)resolvent family, and a (weak) (A,k)-regularized C-(pseudo)resolvent family with $C = I$ is also said to be a (weak) (A,k)-regularized (pseudo)resolvent family.

Let us consider the equations

$$u(t) = \int_0^\infty A(s)u(t-s)\,ds + \int_{-\infty}^t k(t-s)\,g'(s)\,ds, \tag{3.3}$$

where $g : \mathbb{R} \to X$, $A \in L^1_{\mathrm{loc}}([0,\infty) : L(Y,X))$, $A \neq 0$, $k \in C([0,\infty))$, $k \neq 0$, and

$$u(t) = f(t) + \int_0^t A(t-s)u(s)\,ds, \quad t \in (-\tau,\tau), \tag{3.4}$$

where $\tau \in (0,\infty]$, $f \in C((-\tau,\tau) : X)$ and $A \in L^1_{\mathrm{loc}}((-\tau,\tau) : L(Y,X))$, $A \neq 0$. The following proposition can be applied to a class of nonscalar parabolic equations considered by Friedman and Shinbrot in [**8**].

Proposition 3.1. *Assume that there exists an (A,k)-regularized C-resolvent family $(S(t))_{t\geqslant 0}$, $g : \mathbb{R} \to \mathrm{R}(C)$, $C^{-1}g(\cdot)$ is differentiable for a.e. $t \in \mathbb{R}$, $C^{-1}g(t) \in Y$ for a.e. $t \in \mathbb{R}$,*

- (i) *the mapping $s \mapsto S(t-s)(C^{-1}g)'(s)$, $s \in (-\infty,t]$ is an element of the space $L^1((-\infty,t] : Y)$ for a.e. $t \in \mathbb{R}$, and*
- (ii) *the mapping $s \mapsto k(t-s)\,g'(s)$, $s \in (-\infty,t]$ is an element of the space $L^1((-\infty,t] : X)$ for a.e. $t \in \mathbb{R}$.*

Let $u(t) = \int_{-\infty}^t S(t-s)(C^{-1}g)'(s)\,ds$, $t \in \mathbb{R}$. Then $C(\mathbb{R} : X) \ni u$ satisfies (3.3).

A function $u \in C((-\tau,\tau) : X)$ is said to be:

- (i) a *strong solution* of (3.4) iff $u \in L^\infty_{\mathrm{loc}}((-\tau,\tau) : Y)$ and (3.4) holds on $(-\tau,\tau)$,
- (ii) a *mild solution* of (3.4) iff there exist a sequence (f_n) in $C((-\tau,\tau) : X)$ and a sequence (u_n) in $C([0,\tau) : X)$ such that $u_n(t)$ is a strong solution of (3.4) with $f(t)$ replaced by $f_n(t)$ and that $\lim_{n\to\infty} f_n(t) = f(t)$ and $\lim_{n\to\infty} u_n(t) = u(t)$, uniformly on compact subsets of $(-\tau,\tau)$.

Proposition 3.2. [**14**] (i) *Assume $k \in C((-\tau,\tau))$, $k \neq 0$ and $A \in L^1_{\mathrm{loc}}((-\tau,\tau) : L(Y,X))$, $A \neq 0$. Let $k_+(t) = k(t)$, $A_+(t) = A(t)$, $t \in [0,\tau)$, $k_-(t) = k(-t)$ and $A_-(t) = -A(-t)$, $t \in (-\tau,0]$. If there exist $(A_\pm,k_\pm)$-regularized C-resolvent families $(S_\pm(t))_{t\in[0,\tau)}$, then for every $x \in Y$ the function $u : (-\tau,\tau) \to X$ given by $u(t) = S_+(t)x$, $t \in [0,\tau)$ and $u(t) = S_-(-t)x$, $t \in (-\tau,0]$ is a strong solution of* (3.4) *with $f(t) = k(t)Cx$, $t \in (-\tau,\tau)$. Furthermore, strong solutions of* (3.4) *are unique provided that $k_\pm(t)$ are kernels.*

(ii) *Assume $n_\pm \in \mathbb{N}$, $f \in C((-\tau,\tau) : X)$, $A \in L^1_{\mathrm{loc}}((-\tau,\tau) : L(Y,X))$, $A \neq 0$, $f_+(t) = f(t)$, $A_+(t) = A(t)$, $t \in [0,\tau)$, $f_-(t) = f(-t)$, $A_-(t) = -A(-t)$, $t \in (-\tau,0]$ and there exist $(n_\pm - 1)$-times integrated $A_\pm$-regularized $C_\pm$-resolvent families. Let $f_\pm \in C^{(n_\pm)}([0,\tau) : X)$ and $f_\pm^{(i)}(0) = 0$, $0 \leqslant i \leqslant n_\pm - 1$. Then the following holds:*

- (ii.1) *Let $(C_\pm^{-1}f_\pm)^{(n_\pm-1)} \in AC_{\mathrm{loc}}([0,\tau) : Y)$ and $(C_\pm^{-1}f_\pm)^{(n_\pm)} \in L^1_{\mathrm{loc}}([0,\tau) : Y)$. Then there exists a unique strong solution $u(t)$ of* (3.4)*, and moreover $u \in C((-\tau,\tau) : Y)$.*
- (ii.2) *Let $(C_\pm^{-1}f_\pm)^{(n_\pm)} \in L^1_{\mathrm{loc}}([0,\tau) : X)$ and $\overline{Y}^X = X$. Then there exists a unique mild solution of* (3.4).

Example 3.1. (i) ([**14**, Example 2.1], cf. also Example 2.1(i)) Assume $-\infty < \alpha \leqslant \beta < \infty$, $1 \leqslant p \leqslant \infty$, $0 < \tau \leqslant \infty$, $n \in \mathbb{N}$, $X = L^p(\mathbb{R}^n)$ or $X = C_b(\mathbb{R}^n)$, $P(\cdot)$ is an elliptic polynomial of degree $m \in \mathbb{N}$, $\alpha \leqslant \mathrm{Re}(P(x)) \leqslant \beta$, $x \in \mathbb{R}^n$, $A = P(D)$ and $Y = [D(A)]$. Let $r > |\frac{1}{2} - \frac{1}{p}|$, $C_\pm = (\omega \mp A)^{-r}$ and let $a \in L^1_{\mathrm{loc}}(\mathbb{R})$, $a \neq 0$, be such that the mappings $t \mapsto a_+(t)$, $t \geqslant 0$ and $t \mapsto a_-(t) = a(-t)$, $t \geqslant 0$ are completely positive kernels which fulfill (P1); in the case $X = L^\infty(\mathbb{R}^n)$ or $X = C_b(\mathbb{R}^n)$, we assume $a(t) \equiv 1$. Suppose, in addition, $(B_{0,\pm}(t))_{t\in[0,\tau)} \subseteq L(Y) \cap L(X, [\mathrm{R}(C_\pm)])$, $(B_{1,\pm}(t))_{t\in[0,\tau)} \subseteq L(Y, [\mathrm{R}(C_\pm)])$,

(i.1) $C_\pm^{-1} B_{0,\pm}(\cdot)y \in BV_{\mathrm{loc}}([0,\tau) : Y)$ for all $y \in Y$, $C_\pm^{-1} B_{0,\pm}(\cdot)x \in BV_{\mathrm{loc}}([0,\tau) : X)$ for all $x \in X$,

(i.2) $C_\pm^{-1} B_{1,\pm}(\cdot)y \in BV_{\mathrm{loc}}([0,\tau) : X)$ for all $y \in Y$,

(i.3) $C_\pm B_\pm(t)y = B_\pm(t)C_\pm y$, $y \in Y$, $t \in [0,\tau)$, where $B_\pm(t)y = B_{0,\pm}(t)y + (a_\pm * B_{1,\pm})(t)y$, $y \in Y$, $t \in [0,\tau)$, and

(i.4) $C_\pm^{-1} f_\pm \in AC_{\mathrm{loc}}([0,\tau) : Y)$ and $(C_\pm^{-1} f_\pm)' \in L^1_{\mathrm{loc}}([0,\tau) : Y)$.

Set $B(t) = B_+(t)$, $t \in [0,\tau)$ and $B(t) = B_-(-t)$, $t \in (-\tau, 0)$. Then there exists a unique strong solution of (3.4) with $A(t) = a(t)P(D) + B(t)$, $t \in (-\tau,\tau)$.

(ii) [**14**] Let $1 < p < \infty$, $X = L^p(\mathbb{R})$, $Y = W^{4,p}(\mathbb{R})$,

$$A(t)f = -tf'''' - tf'' - 2if' - tf, \quad t \in \mathbb{R}, \quad f \in Y,$$

$s \in (1,2)$ and $f(t) = k_s(t) = \mathcal{L}^{-1}(e^{-\lambda^{1/s}})(|t|)$, $t \in \mathbb{R}$. Then there exist no exponentially bounded $(\pm A(\pm t), k_s)$-regularized resolvent families, and Proposition 3.2(i) implies that there exists a unique strong solution $u(t)$ of (3.4) on $\mathbb{R}$. Finally, one can simply prove that $u(t)$ is hyponalytic in the sense of [**12**, Definition 2.19].

References

1. W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Birkhäuser, 2001.
2. E. Bazhlekova, *Fractional Evolution Equations in Banach Spaces*, PhD Thesis, Eindhoven University of Technology, Eindhoven, 2001.
3. S. Calzadillas, C. Lizama, *Bounded mild solutions of perturbed Volterra equations with infinite delay*, Nonlinear Anal. **72** (2010), 3976–3983.
4. B. D. Coleman, M. E. Gurtin, *Equipresence and constitutive equations for rigid heat conductors*, Z. Angew. Math. Phys. **18** (1967), 199–208.
5. B. D. Coleman, V. J. Mizel, *Norms and semigroups in the theory of fading memory*, Arch. Rational Mech. Anal. **28** (1966), 87–123.
6. C. Cuevas, C. Lizama, *Almost automorphic solutions to integral equations on the line*, Semigroup Forum **79** (2009), 461–472.
7. R. deLaubenfels, *Existence Families, Functional Calculi and Evolution Equations*, Lect. Notes Math. **1570**, Springer, Berlin, 1994.
8. A. Friedman, M. Shinbrot, *Volterra integral equations in Banach spaces*, Trans. Amer. Math. Soc. **126** (1967), 131–179.
9. R. Gorenflo, Y. Luchko, F. Mainardi, *Analytical properties and applications of the Wright function*, Fract. Calc. Appl. Anal. **2** (1999), 383–414.
10. M. Hieber, *Integrated semigroups and differential operators on L^p spaces*, Math. Ann. **291** (1995), 1-16.
11. V. Keyantuo, *Integrated semigroups and related partial differential equations*, J. Math. Anal. Appl. **212** (1997), 135–153.
12. M. Kostić, *(a,k)-regularized C-resolvent families: regularity and local properties*, Abstr. Appl. Anal. vol. 2009, Art. ID 858242, 27 pages, 2009.
13. M. Kostić, *Generalized Semigroups and Cosine Functions*, Mathematical Institute, Belgrade, 2011.
14. M. Kostić, *Generalized well-posedness of hyperbolic Volterra equations of nonscalar type*, preprint.

15. F.-B. Li, M. Li, Q. Zheng, *Fractional evolution equations governed by coercive differential operators*, Abstr. Appl. Anal. 2009, Art. ID 438690, 14 pp. 34G10.
16. M. Li, Q. Zheng, *On spectral inclusions and approximations of α-times resolvent families*, Semigroup Forum **69** (2004), 356–368.
17. M. Li, Q. Zheng, J. Zhang, *Regularized resolvent families*, Taiwanese J. Math. **11** (2007), 117–133.
18. G. M. N'Guérékata, *Topics in Almost Automorphy*, Springer-Verlag, New York, 2005.
19. J. W. Nunziato, *On heat conduction in materials with memory*, Quart. Appl. Math. **29** (1971), 187–204.
20. H. Prado, *Stability properties for solution operators*, Semigroup Forum **77** (2008), 243–252.
21. J. Prüss, *Evolutionary Integral Equations and Applications*, Birkhäuser Verlag, Basel, Boston, Berlin, 1993.
22. B. Stanković, *On the function of E. M. Wright*, Publ. Inst. Math., Nouv. Sér. **10** (1970), 113–124.
23. G. F. Webb, *An abstract semilinear Volterra integrodifferential equation*, Proc. Am. Math. Soc. **69** (1978), 255–260.
24. T. Xiao, J. Liang, *Schrödinger-type evolution equations in $L^p(\Omega)$*, J. Math. Anal. Appl. **260** (2001), 55–69.
25. Q. Zheng, Y. Li, *Abstract parabolic systems and regularized semigroups*, Pacific J. Math. **182** (1998), 183–199.

Faculty of Technical Sciences
University of Novi Sad
21125 Novi Sad
Serbia
marco.s@verat.net

(Received 29 10 2009)