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A FIRST-ORDER CONDITIONAL PROBABILITY LOGIC WITH ITERATIONS

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Abstract. We investigate a first-order conditional probability logic with equality, which is, up to our knowledge, the first treatise of such logic. The logic, denoted LFPOIC$^=$, allows making statements such as: $CP_{\geqslant s}(\phi,\theta)$, and $CP_{\leqslant s}(\phi,\theta)$, with the intended meaning that the conditional probability of ϕ given θ is at least (at most) s. The corresponding syntax, semantic, and axiomatic system are introduced, and Extended completeness theorem is proven.

1. Syntax and semantics

The recent papers [**1, 3, 6**], discuss conditional probability extensions of classic propositional logic, while [**2**] introduces a first-order conditional probability logic in which iterations of conditional probability operators are not allowed. In this paper, we abandon that restriction and also extend logical language by adding equality, which causes changes in the corresponding syntax and semantics. Solving those issues is the main novelty presented in this paper.

Let $[0,1]_{\mathbb{Q}}$ denote the set of all rational numbers from the interval $[0,1]$. The language $\mathcal{L}$ of the LFOICP$^=$-logic consists of countable sets of variables $Var = \{x_1, x_2, \ldots\}$, relation symbols R_i^m, the relation symbol $=$ which is, of course, interpreted rigidly as equality, and function symbols F_j^n, where m and n are arities of these symbols, logical connectives $\wedge$ and $\neg$, the quantifier $\forall$, and binary conditional probability operators $CP_{\geqslant s}$ and $CP_{\leqslant t}$ for all $s \in [0,1]_{\mathbb{Q}}$, $t \in [0,1)_{\mathbb{Q}}$. Constants are function symbols whose arity is 0.

Terms and atomic formulas are defined as in the first-order classical logic with equality. The set of formulas $\mathrm{For}_{\mathrm{FOICP}^=}$ is the smallest set containing atomic formulas and closed under the following formation rules: if ϕ and θ are formulas, then $\neg\phi$, $CP_{\geqslant s}(\phi,\theta)$, $CP_{\leqslant t}(\phi,\theta)$, $\phi\wedge\theta$ and $(\forall x)\phi$ are formulas. We use the standard abbreviations for other connectives, while $P_{\geqslant s}(\phi)$ denotes $CP_{\geqslant s}(\phi,\top)$. A formula

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ψ is a sentence if no variable is free in ψ. The subset of all sentences is denoted by $\mathrm{Sent}_{\mathrm{FOICP}^=}$. We call a set $T \subset \mathrm{For}_{\mathrm{FOICP}^=}$ a theory if T contains only sentences.

Semantics to the set of $\mathrm{LFOICP}^=$-formulas is given in the possible-world style.

DEFINITION 1.1. An $\mathrm{LFOICP}^=$-model is a structure $M = \langle W, D, I, \mathrm{Prob}\rangle$ where:

- W is a nonempty set of objects called worlds,
- all worlds have a nonempty set D as a domain,
- I associates an interpretation of function and relation symbols with every world $w \in W$ such that the meanings of the terms are same in all worlds (we say that terms are rigid) and $I(w)(R_i^m)$ is a subset of D^m,
- Prob is a probability assignment which assigns to every $w \in W$ a probability space $\mathrm{Prob}(w) = \langle W(w), H(w), \mu(w)\rangle$, where:
  - $W(w)$ is a nonempty subset of W,
  - $H(w)$ is an algebra of subsets of $W(w)$,
  - $\mu(w)$ is a finitely additive probability measure on $H(w)$.

The fact that $\phi \in \mathrm{For}_{\mathrm{FOICP}^=}$ holds in a world w of some $\mathrm{LFOICP}^=$-model M for a valuation v of variables is denoted as $(M, w, v) \vDash \phi$ and the notation $[\phi]_w^v = \{u \in W(w) \mid (M, u, v) \vDash \phi\}$ is used throughout the paper.

DEFINITION 1.2. Let $M = \langle W, D, I, \mathrm{Prob}\rangle$ be an $\mathrm{LFOICP}^=$-model and v be a valuation. The satisfiability of $\phi \in \mathrm{For}_{\mathrm{FOICP}}$ in $w \in W$ for a given valuation v is defined as follows:

- if ϕ is a classical first-order atomic formula, then $(M, w, v) \vDash \phi$ if and only if $w \vDash \phi(\underline{a_1}, \dots, \underline{a_n})$, where $\underline{a_i}$, $i = 1, \dots, n$, are the names for $a_i = v(w)(x_i)$, and w is considered as a classical first-order model,
- if $\phi \equiv \neg\psi$, then $(M, w, v) \vDash \neg\psi$ if and only if $(M, w, v) \nvDash \psi$,
- if $\phi \equiv \psi \wedge \theta$, then $(M, w, v) \vDash \psi \wedge \theta$ if and only if $(M, w, v) \vDash \psi$ and $(M, w, v) \vDash \theta$,
- if $\phi \equiv (\forall x)\psi(x)$, then $(M, w, v) \vDash (\forall x)\psi(x)$ if and only if for every $d \in D(w)$ $(M, w, v) \vDash \psi(\underline{d})$, where $\underline{d}$ is a name for d,
- if $\phi \equiv CP_{\geqslant s}(\psi, \theta)$, then $(M, w, v) \vDash CP_{\geqslant s}(\psi, \theta)$ if and only if either $\mu(w)([\theta]_w^v) = 0$, or $\mu(w)([\theta]_w^v) > 0$ and $\dfrac{\mu(w)([\psi \wedge \theta]_w^v)}{\mu(w)([\theta]_w^v)} \geqslant s$,
- if $\phi \equiv CP_{\leqslant s}(\psi, \theta)$, then $(M, w, v) \vDash CP_{\leqslant s}(\psi, \theta)$ if and only if either $\mu(w)([\theta]_w^v) = 0$ and $s = 1$, or $\mu(w)([\theta]_w^v) > 0$ and $\dfrac{\mu(w)([\psi \wedge \theta]_w^v)}{\mu(w)([\theta]_w^v)} \leqslant s$.

We say that a formula ϕ holds in a world w of an $\mathrm{LFOICP}^=$-model M and denote it by $(M, w) \vDash \phi$ if for every valuation v, $(M, w, v) \vDash \phi$.

Since the satisfiability of a sentence ϕ in w does not depend on the given valuation v, and for all valuations sets $[\phi]_w^v$ coincide we denote the set of all worlds $u \in W(w)$ of an $\mathrm{LFOICP}^=$-model M where ϕ holds by $[\phi]_w$. We may omit the subscript when the meaning of $[\phi]$ is clear from the context: if it is written $\mu(w)([\phi])$, then it is connoted $[\phi] = [\phi]_w$.

By the above definition the conditional probability of ϕ given ψ is 1 when $\mu(w)([\psi]_w^v) = 0$ and we have expanded Kolmogorov's definition of the conditional probability in a rather usual way following [**6**] and [**7**].

DEFINITION 1.3. A formula $\phi \in \mathrm{For}_{\mathrm{FOICP}^=}$ is satisfiable if there exist an $\mathrm{LFOICP}^=$-model M, a world w in M, and a valuation v such that $(M, w, v) \vDash \phi$. A set T of formulas is satisfiable if there exist an $\mathrm{LFOICP}^=$-model M, some world w in M, and a valuation v such that $(M, w) \vDash \phi$, for every $\phi \in T$. A formula ϕ is valid if for every $\mathrm{LFOICP}^=$-model M, and every world w from M, $(M, w) \vDash \phi$.

We focus on the class of models satisfying the requirement that for every $\phi \in \mathrm{Sent}_{\mathrm{FOICP}^=}$ and every w from a model M, $[\phi]_w$ is a measurable set, i.e., $[\phi]_w \in H(w)$, and that class will be denoted by $\mathrm{LFOICP}^=_{\mathrm{Meas}}$. Also, we consider the class $\mathrm{LFOICP}^=_{All}$ of all $\mathrm{LFOICP}^=_{\mathrm{Meas}}$-models having property that for each $w \in W$ every subset of $W(w)$ is $\mu(w)-$ measurable.

DEFINITION 1.4. A probabilistic k-nested implication $\Phi_k(\tau, (\theta_i)_{i<\omega})$ for the formula τ based on the sequence $(\theta_i)_{i<\omega}$ of formulas is defined by recursion:

$$\Phi_0(\tau, (\theta_i)_{i<\omega}) \equiv \theta_0 \to \tau, \quad \Phi_{k+1}(\tau, (\theta_i)_{i<w}) \equiv \theta_{k+1} \to P_{\geqslant 1}(\Phi_k(\tau, (\theta_i)_{i<\omega})).$$

For example $\Phi_3(\tau, (\theta_i)_{i<\omega}) \equiv \theta_3 \to P_{\geqslant 1}(\theta_2 \to P_{\geqslant 1}(\theta_1 \to P_{\geqslant 1}(\theta_0 \to \tau)))$.

2. Axioms

The axiomatic system $\mathrm{Ax}_{\mathrm{LFOICP}}$ for LFOICP contains the following axiom schemata:

Axiom 1 all the axioms of the classical propositional logic,
Axiom 2 $\forall x(\phi \to \psi) \to (\phi \to \forall x\psi)$, where x is not a free variable in ϕ and $\phi, \psi \in \mathrm{For}_{\mathrm{FOICP}}$,
Axiom 3 $\forall x\phi(x) \to \phi(t)$, where $\phi(t)$ is obtained by substitution of all free occurrences of x in the first-order formula $\phi(x)$ by the term t which is free for x in $\phi(x)$,
Axiom 4 $\forall x(x = x)$,
Axiom 5 $\forall x\forall y(x = y \to (\phi(x, x) \leftrightarrow \phi(x, y)))$, for $\phi \in \mathrm{For}_{\mathrm{FOICP}}$.
Axiom 6 $CP_{\geqslant 0}(\phi, \theta)$,
Axiom 7 $CP_{<s}(\phi, \theta) \to CP_{\leqslant s}(\phi, \theta)$,
Axiom 8 $CP_{\leqslant s}(\phi, \theta) \to CP_{<t}(\phi, \theta)$, $t > s$,
Axiom 9 $P_{>0}(\theta) \to (CP_{\geqslant s}(\phi, \theta) \leftrightarrow CP_{\leqslant 1-s}(\neg\phi, \theta))$,
Axiom 10 $(P_{\geqslant s}(\phi) \wedge P_{\geqslant t}(\theta) \wedge P_{\geqslant 1}(\neg(\phi \wedge \theta))) \to P_{\geqslant \min\{1, s+t\}}(\phi \vee \theta)$,
Axiom 11 $(P_{\leqslant s}(\phi) \wedge P_{<t}(\theta)) \to P_{<s+t}(\phi \vee \theta)$, $s + t \leqslant 1$,
Axiom 12 $P_{=0}(\theta) \to CP_{=1}(\phi, \theta)$,
Axiom 13 $(P_{\geqslant t}(\theta) \wedge P_{\leqslant s}(\phi \wedge \theta)) \to CP_{\leqslant \min\{\frac{s}{t}, 1\}}(\phi, \theta)$, $t \neq 0$
Axiom 14 $(P_{\leqslant t}(\theta) \wedge P_{\geqslant s}(\phi \wedge \theta)) \to CP_{\geqslant \min\{\frac{s}{t}, 1\}}(\phi, \theta)$, $t \neq 0$

and inference rules:

Rule 1 modus ponens,
Rule 2 $\dfrac{\phi}{\forall x\phi}$, $\phi \in \mathrm{For}_{\mathrm{FOICP}}$,

Rule 3 $\dfrac{\phi}{P_{\geqslant 1}(\phi)}$, $\phi \in \mathrm{For}_{\mathrm{FOICP}}$,

Rule 4 $\dfrac{\Phi_k(CP_{\geqslant s-\frac{1}{n}}(\psi,\chi),(\theta_i)_{i<\omega}),\ \text{for every integer } n \geqslant \frac{1}{s}}{\Phi_k(CP_{\geqslant s}(\psi,\chi),(\theta_i)_{i<\omega})}$,

Rule 5 $\dfrac{\Phi_k(CP_{\leqslant s+\frac{1}{n}}(\psi,\chi),(\theta_i)_{i<\omega}),\ \text{for every integer } n \geqslant \frac{1}{1-s}}{\Phi_k(CP_{\leqslant s}(\psi,\chi),(\theta_i)_{i<\omega})}$.

Let us discuss the system $\mathrm{Ax}_{\mathrm{LFOICP}}$. The axioms 1–5 and the rules 1 and 2 correspond to the classical first-order reasoning, while the axioms 6–14 concern the probabilistic part of our system. Axiom 6 announces the nonnegativity and Axioms 7 and 8 the monotonicity of the conditional probability. Axiom 9 claims that $CP_{\geqslant s}(\phi,\theta)$ and $CP_{\leqslant 1-s}(\neg\phi,\theta)$ are equivalent if the condition has a positive probability. Axioms 10 and 11 correspond to the finite additivity of measures, while Axioms 12–14 describe the relationship between the conditional and absolute probability. Rule 3 is a form of modal necessitation. Rules 4 and 5 are the generalization of the infinitary rules which correspond to the Archimedean rule for real numbers, and do not occur in the previous papers.

Definition 2.1. $\phi \in \mathrm{For}_{\mathrm{FOICP}}$ is a theorem, which we denote by $\vdash \phi$, if there exists a denumerable sequence of formulas $\phi_0, \phi_1, \dots, \phi$ called the proof, such that each member of the sequence is an instance of some axiom schemata or is obtained from the previous formulas using an inference rule.

ϕ is deducible from a set of sentences T ($T \vdash \phi$) if there is an at most countable sequence of formulas $\phi_0, \phi_1, \dots, \phi$ called the proof, such that each member of the sequence is an instance of some axiom schemata, or is contained in T or is obtained from the previous formulas using an inference rule, with the exception that the inference rule 3 can be applied to the theorems only.

Definition 2.2. A theory T is consistent if there is at least one formula from $\mathrm{For}_{\mathrm{FOICP}}$ which can not be deduced from T. A theory T is maximal consistent if it is consistent and for each $\phi \in \mathrm{Sent}_{\mathrm{FOICP}}$, either $\phi \in T$ or $\neg\phi \in T$.

The set of all formulas which are deducible from T is called the deductive closure of T and denoted by $Cn(T)$. A theory T is deductively closed if $T = Cn(T)$.

3. Soundness and completeness

Some of the following results can be proved in the way analogous to ones presented in [**4, 6, 7**], so we emphasize only the main differences and new ideas.

Theorem 3.1 (Soundness). *The axiomatic system* $\mathrm{Ax}_{\mathrm{LFOICP}^=}$ *is sound with respect to the class of* $\mathrm{LFOICP}^{=}_{\mathrm{Meas}}$*-models.*

Proof. Axioms 4 and 5 are obviously valid (for the validity of the latter axiom the assumption about constant domains and rigidness of terms is essential), and it remains to prove, using the induction on k, that rule R4 produces a valid formula from a set of valid premises. In fact we are going to show that if in a world w of some model M for a given valuation v holds $\Phi_k(CP_{\geqslant s-\frac{1}{n}}(\psi,\chi),(\theta_i)_{i<\omega})$, for every

$n \geqslant \frac{1}{s}$, then $(M, w, v) \models \Phi_k(CP_{\geqslant s}(\psi, \chi), (\theta_i)_{i<\omega})$. For the induction basis $k = 0$ we point out to the above mentioned literature, and assume that it is fulfilled for $k = j$. Suppose that there are an $\mathrm{LFOICP}^{=}_{\mathrm{Meas}}$-model M_1, a world w_1, and a valuation v_1, such that $(M_1, w_1, v_1) \models \Phi_{j+1}(CP_{\geqslant s-\frac{1}{n}}(\psi, \chi), (\theta_i)_{i<\omega})$ for every $n \geqslant \frac{1}{s}$, and

$$(M_1, w_1, v_1) \not\models \theta_{j+1} \to P_{\geqslant 1}(\Phi_j(CP_{\geqslant s}(\psi, \chi), (\theta_i)_{i<\omega})).$$

We conclude $(M_1, w_1, v_1) \models \theta_{j+1} \wedge \neg P_{\geqslant 1}(\Phi_j(CP_{\geqslant s}(\psi, \chi), (\theta_i)_{i<\omega}))$, and for each $n \geqslant \frac{1}{s}$

$$(M_1, w_1, v_1) \models P_{\geqslant 1}(\Phi_j(CP_{\geqslant s-\frac{1}{n}}(\psi, \chi), (\theta_i)_{i<\omega})),$$

meaning that for each world u from some subset $S \subseteq W(w_1)$ whose $\mu(w_1)$ measure is equal to 1 holds $(M_1, u, v_1) \models \Phi_j(CP_{\geqslant s-\frac{1}{n}}(\psi, \chi), (\theta_i)_{i<\omega})$. By the induction hypothesis $(M_1, u, v_1) \models \Phi_j(CP_{\geqslant s}(\psi, \chi), (\theta_i)_{i<\omega})$ for all worlds from $S \subseteq W(w_1)$, $\mu(w_1)(S) = 1$, implying $(M_1, w_1, v_1) \models P_{\geqslant 1}(\Phi_j(CP_{\geqslant s}(\psi, \chi), (\theta_i)_{i<\omega})$, and the initial supposition leads to contradiction. $\square$

Theorem 3.2 (Deduction theorem). *If T is a theory and $\phi, \psi \in \mathrm{Sent}_{\mathrm{FOICP}^{=}}$, then $T \cup \{\phi\} \vdash \psi$ if and only if $T \vdash \phi \to \psi$.*

The proof of Deduction theorem for $\mathrm{LFOICP}^{=}$ differs from the proof of the corresponding theorem presented in [**4, 6, 7**] in the case when infinitary rules are applied. If $\sigma = \Phi_k(CP_{\geqslant s}(\psi, \chi), (\theta_i)_{i<\omega})$ is obtained from $T \cup \{\phi\}$ using rule R4, then:

1. $T \cup \{\phi\} \vdash \theta_k \to P_{\geqslant 1}(\Phi_{k-1}(CP_{\geqslant s-\frac{1}{n}}(\psi, \chi), (\theta_i)_{i<\omega}))$, for each integer $n \geqslant \frac{1}{s}$
2. $T \vdash (\phi \wedge \theta_k) \to P_{\geqslant 1}(\Phi_{k-1}(CP_{\geqslant s-\frac{1}{n}}(\psi, \chi), (\theta_i)_{i<\omega}))$, for $n \geqslant \frac{1}{s}$, by the induction hypothesis and using an instance of the classical propositional tautology $(p \to (q \to r)) \leftrightarrow ((p \wedge q) \to r)$
   For $i \neq k$ the sequence $(\overline{\theta}_i)_{i<\omega}$ coincides with $(\theta_i)_{i<\omega}$, and $\overline{\theta}_k \equiv \phi \wedge \theta_k$. Introducing that notation we obtain
3. $T \vdash (\phi \wedge \theta_k) \to P_{\geqslant 1}(\Phi_{k-1}(CP_{\geqslant s}(\psi, \chi), (\overline{\theta}_i)_{i<\omega}))$, by the application of the rule R4 on 2
4. $T \vdash \phi \to (\theta_k \to P_{\geqslant 1}(\Phi_{k-1}(CP_{\geqslant s}(\psi, \chi), (\theta_i)_{i<\omega})))$

The next corollary follows from several applications of the previous theorem, and makes more evident the necessity of imposing rigidness of terms.

Corollary 3.1. *$x = y \to P_{\geqslant 1}(x = y)$ is a theorem of* $\mathrm{LFOICP}^{=}$.

Proof. We deduce as follows:

1) $\vdash \forall x \forall y(x = y \to (P_{\geqslant 1}(x = x) \leftrightarrow P_{\geqslant 1}(x = y)))$ is an instance of A5,
2) $\vdash P_{\geqslant 1}(x = x) \to (x = y \leftrightarrow P_{\geqslant 1}(x = y))$, is obtained from 1) using A3 and Deduction theorem, and an instance of a propositional tautology $(p \to (q \to r)) \leftrightarrow (q \to (p \to r))$,
4) $\vdash P_{\geqslant 1}(x = x)$, using A4, A3, Deduction theorem and R3,
5) $\vdash x = y \to P_{\geqslant 1}(x = y)$, from 4) and 3) using Modus ponens. $\square$

Lemma 3.1. a) *For all $s, t \in [0,1]_{\mathbb{Q}}$ and $\phi, \theta \in \mathrm{For}_{\mathrm{FOICP}}$, if $s \leqslant t$, then $\vdash CP_{\geqslant t}(\phi, \theta) \to CP_{\geqslant s}(\phi, \theta)$.*

b) *For all* $s, t \in [0,1]_{\mathbb{Q}}, t \neq 0$ *and* $\phi, \theta \in \mathrm{For}_{\mathrm{FOICP}}$*, holds* $\vdash (P_{=t}(\theta) \wedge P_{=s}(\phi \wedge \theta)) \to CP_{=\min\{\frac{s}{t},1\}}(\phi, \theta)$.
c) $\vdash P_{\geqslant 1}(\phi \to \theta) \to (P_{\geqslant s}(\phi) \to P_{\geqslant s}(\theta))$ *for all* $\phi, \theta \in \mathrm{For}_{\mathrm{FOICP}}$.
d) $P_{\geqslant 1}(\phi_1), P_{\geqslant 1}(\phi_2) \vdash P_{\geqslant 1}(\phi_1 \vee \phi_2) \wedge P_{\geqslant 1}(\phi_1 \wedge \phi_2)$.

Proof. As an illustration we prove d), while the other statements are left to the reader. We deduce as follows:

1) $\vdash P_{\geqslant 1}(\phi_1 \to (\phi_1 \vee \phi_2))$, applying Rule 3 to an instance of a propositional tautology ,
2) $\vdash P_{\geqslant 1}(\phi_1 \to (\phi_1 \vee \phi_2)) \to (P_{\geqslant 1}(\phi_1) \to P_{\geqslant 1}(\phi_1 \vee \phi_2))$, by c) of this lemma,
3) $P_{\geqslant 1}(\phi_1) \vdash P_{\geqslant 1}(\phi_1 \vee \phi_2)$, from 1) and 2) using R1 and Deduction theorem,
4) $\vdash P_{\geqslant 1}(\phi_1) \leftrightarrow P_{\leqslant 0}(\neg\phi_1)$, an instance of A9,
5) $\vdash P_{<s}(\neg\phi_1) \to P_{<s}(\neg\phi_1 \wedge \phi_2)$, using similar arguments as above and contraposition,
6) $P_{\leqslant 0}(\neg\phi_1) \vdash P_{<\frac{1}{n}}(\neg\phi_1)$, for every $n > 0$, by A8,
7) $P_{\leqslant 0}(\neg\phi_1) \vdash P_{\leqslant\frac{1}{n}}(\neg\phi_1 \wedge \phi_2)$, for every $n > 0$, from 5) and 6), and by A7,
8) $P_{\leqslant 0}(\neg\phi_1) \vdash P_{\leqslant 0}(\neg\phi_1 \wedge \phi_2)$, from 7) using R5,
9) $\vdash P_{\geqslant 1}((\neg\phi_1 \wedge \phi_2) \vee (\phi_1 \wedge \phi_2)) \to (P_{>0}(\neg\phi_1 \wedge \phi_2) \vee P_{\geqslant 1}(\phi_1 \wedge \phi_2))$, by A11 and contraposition,
10) $\vdash P_{\geqslant 1}(\phi_2) \to P_{\geqslant 1}((\neg\phi_1 \wedge \phi_2) \vee (\phi_1 \wedge \phi_2))$, using the previous clause of this lemma
11) $P_{\geqslant 1}(\phi_2) \vdash P_{>0}(\neg\phi_1 \wedge \phi_2) \vee P_{\geqslant 1}(\phi_1 \wedge \phi_2)$, from 9) and 10),
12) $P_{\geqslant 1}(\phi_1) \vdash P_{\leqslant 0}(\neg\phi_1 \wedge \phi_2)$, from 4) and 8),
13) $P_{\geqslant 1}(\phi_1), P_{\geqslant 1}(\phi_2) \vdash P_{\geqslant 1}(\phi_1 \wedge \phi_2)$, from 11) and 12). □

Lemma 3.2. *Let T be a consistent theory. Then:*

a) *for every formula* $\phi \in \mathrm{For}_{\mathrm{FOICP}}$*, either* $T \cup \{\phi\}$ *or* $T \cup \{\neg\phi\}$ *is consistent;*
b) *if* $\neg\Phi_k(CP_{\geqslant s}(\psi, \chi), (\theta_i)_{i<\omega}) \in T$*, then there exists an integer* $n > \frac{1}{s}$ *such that* $T \cup \{\theta_k \to \neg\Phi_{k-1}(CP_{\geqslant s-\frac{1}{n}}(\psi, \chi), (\theta_i)_{i<\omega})\}$ *is consistent. Also, if* $\neg\Phi_k(CP_{\leqslant s}(\psi, \chi), (\theta_i)_{i<\omega}) \in T$*, then there exists an integer* $n > \frac{1}{1-s}$ *such that* $T \cup \{\theta_k \to \neg\Phi_{k-1}(CP_{\leqslant s+\frac{1}{n}}(\psi, \chi), (\theta_i)_{i<\omega})\}$ *is a consistent theory.*

Definition 3.1. A set T of formulas is saturated if for each formula of the form $\neg(\forall x)\phi(x)$ which is contained in T there exists a term t such that $\neg\phi(t) \in T$.

In order to prove the completeness theorem, the following theorem that states that every consistent theory T can be extended to a saturated maximal consistent theory T^* in some broader language is needed.

Theorem 3.3. *Let T be a consistent set of sentences in the first-order probability language $\mathcal{L}$, and C a countably infinite set of new constant symbols. Then T can be extended to a saturated maximal consistent theory T^* in the language $\mathcal{L}^* = \mathcal{L} \cup C$.*

Proof. Let $\phi_0, \phi_1, \ldots$, be an enumeration of all sentences in $\mathcal{L}^*$. We define a sequence of theories $T_i, i \in \omega$ as follows:

1) $T_0 = T$,

2) if $T_i \cup \{\phi_i\}$ is consistent, then $T_{i+1} = T_i \cup \{\phi_i\}$, otherwise $T_{i+1} = T_i \cup \{\neg\phi_i\}$,
3) if the set T_{i+1} is obtained by adding a formula of the form $\neg(\forall x)\psi(x)$ to the set T_i, then for some $c \in C$ which does not occur in any of the formulas $\phi_0, \dots, \phi_i$, we add $\neg\psi(c)$ to T_{i+1} such that T_{i+1} remains consistent,
4) if a formula of the form $\neg\Phi_k(CP_{\geqslant s}(\psi,\chi),(\theta_i)_{i<\omega})$ is added, then for some positive integer n, $\theta_k \to \neg\Phi_{k-1}(CP_{\geqslant s-\frac{1}{m}}(\psi,\chi),(\theta_i)_{i<\omega})$ is also added to T_{i+1}, so that T_{i+1} is consistent,
5) if a formula of the form $\neg\Phi_k(CP_{\leqslant s}(\psi,\chi),(\theta_i)_{i<\omega})$ is added, then for some positive integer m, $\theta_k \to \neg\Phi_{k-1}(CP_{\leqslant s+\frac{1}{m}}(\psi,\chi),(\theta_i)_{i<\omega})$ is also added to T_{i+1}, so that T_{i+1} is consistent,
6) $T^* = \bigcup_{i<\omega} T_i$.

T^* has required properties. □

The next corollary summarizes some obvious properties of saturated maximal consistent theories.

Corollary 3.2. *Let T be a saturated maximal consistent theory in $\mathcal{L}$ and $\phi, \psi \in \mathrm{Sent}_{\mathcal{L}}$. Then:*

a) *if $T \vdash \phi$, then $\phi \in T$, i.e. every saturated maximal consistent theory is deductively closed;*
b) *if $t = \sup\{r \mid P_{\geqslant r}(\phi) \in T\}$ and $t \in [0,1]_{\mathbb{Q}}$, then $P_{\geqslant t}(\phi), P_{\leqslant t}(\phi) \in T$.*

Definition 3.2. A cut theory P_T^- corresponding to a theory T in the language $\mathcal{L}$ is the set of sentences $P_T^- = \{\phi \in \mathrm{Sent}_{\mathrm{FOICP}^=} \mid P_{\geqslant 1}(\phi) \in T\}$.

Lemma 3.3. *If $P_T^- \vdash \psi$, then $T \vdash P_{\geqslant 1}(\psi)$.*

Proof. We use the transfinite induction on the length of the proof for ψ from P_T^-. If the proof is finite $\psi_1, \dots, \psi_l, \psi$ and $T \vdash P_{\geqslant 1}(\psi_i)$ for each $i = 1, \dots, l$, then:

1) $\psi_1 \wedge \dots \wedge \psi_l \vdash \psi$
2) $\vdash P_{\geqslant 1}((\psi_1 \wedge \dots \wedge \psi_l) \to \psi)$, by Rule 3
3) $\vdash P_{\geqslant 1}((\psi_1 \wedge \dots \wedge \psi_l) \to \psi) \to (P_{\geqslant 1}(\psi_1 \wedge \dots \wedge \psi_l) \to P_{\geqslant 1}(\psi))$, by Lemma 3.1c)
4) $P_{\geqslant 1}(\psi_1 \wedge \dots \wedge \psi_l) \vdash P_{\geqslant 1}(\psi)$, from 2) and 3) using R1 and Deduction theorem
5) $P_{\geqslant 1}(\psi_1), \dots, P_{\geqslant 1}(\psi_l) \vdash P_{\geqslant 1}(\psi_1 \wedge \dots \wedge \psi_l)$, by Lemma 3.1d)
6) $T \vdash P_{\geqslant 1}(\psi)$

We consider the case when the proof is infinite $\psi_1, \dots, \psi$. Suppose that some ψ_j is of the form $\Phi_k(CP_{\geqslant s}(\psi,\chi),(\theta_i)_{i<\omega})$, and is obtained by an application of infinitary rule R4 to formulas $\Phi_k(CP_{\geqslant s-\frac{1}{n}}(\psi,\chi),(\theta_i)_{i<\omega})$, $n \geqslant \frac{1}{s}$, which occur in the proof sequence before ψ_j. Thus, by the induction hypothesis, we have that $T \vdash P_{\geqslant 1}(\Phi_k(CP_{\geqslant s-\frac{1}{n}}(\psi,\chi),(\theta_i)_{i<\omega}))$ for every $n \geqslant \frac{1}{s}$, and since $(\top \to p) \leftrightarrow p$ is a tautology, using R4, we conclude $T \vdash P_{\geqslant 1}(\psi)$. □

The canonical model M for a consistent theory T is defined as follows. From the set $\mathcal{T}$ of all maximal saturated extensions in the expanded language $\mathcal{L}^*$ we pick one which is a extension of T, denote it by T_1, and set that the world w_1 is T_1. Note that $P^-_{T_1}$ is a consistent theory, since $\top$, $P_{\geqslant 1}(\top)$, $P_{\geqslant 1}(\top) \leftrightarrow P_{\leqslant 0}(\bot)$ are contained in every maximal theory, T_1 included, and $P^-_{T_1} \vdash \bot$ would imply $T_1 \vdash P_{\geqslant 1}(\bot)$ contradicting consistency of T_1. The corresponding probability space $\mathrm{Prob}(w_1)$ is determined with

$$W(w_1) = \{T^* \in \mathcal{T} \mid P^-_{T_1} \subseteq T^*\}, \qquad [\phi]_{w_1} = \{u \in W(w_1) \mid \phi \in u\},$$
$$H(w_1) = \{[\phi]_{w_1} \mid \phi \in \mathrm{Sent}_{\mathcal{L}^*}\}, \quad \mu(w_1)([\phi]_{w_1}) = \{s \mid P_{\geqslant s}(\phi) \in w_1\}.$$

For each element from $W(w_1)$ we proceed with this procedure and so on. Let $\mathcal{C}$ be the set of all constants from $\mathcal{L}^*$. The relation $\sim$ on $\mathcal{C}$ is defined by $c_i \sim c_j$ iff $T_1 \vdash c_i = c_j$, is an equivalence relation. Domain of the canonical model is $D = \mathcal{C}/\sim$ and its elements are classes of equivalence c^*. For $w \in W$, $I(w)$ is an interpretation such that:

- for every symbol of constant c_j, $I(w)(c_j) = c^*$ iff $c_j = c \in w$,
- for every function symbol F^m_i, $I(w)(F^m_i)$ is a function from D^m to D mapping $(c^*_1, \dots, c^*_m)$ to c^*_{m+1} iff $F^m_i(c_1, \dots, c_m) = c_{m+1} \in w$,
- for every relation symbol R^m_i

$$I(w)(R^m_i) = \{(c^*_1, \dots, c^*_m) \in D^m \mid R^m_i(c_1, \dots, c_m) \in w\}.$$

Corollary 1 guarantees that terms are rigidly interpreted, cause $t_1 = t_2 \in T_1$, $\vdash t_1 = t_2 \to P_{\geqslant 1}(t_1 = t_2)$ implies $t_1 = t_2 \in P^-_{T_1}$. It remains to be proved that $M = \langle W, D, I, \mathrm{Prob}\rangle$ is really an $\mathrm{LFOICP}^{=}_{\mathrm{Meas}}$-model showing that $H(w)$ is an algebra of subsets of $W(w)$, $\mu(w)$ is a finitely additive measure, and $(M, w) \vDash \phi$ iff $\phi \in w$. Here we provide the proof for one fact, namely we prove that $[\phi]_w \subseteq [\psi]_w$ implies $\mu(w)([\phi]_w) \leqslant \mu(w)([\psi]_w)$. For every $u \in W(w)$, if $\phi \in u$ then $\psi \in u$, and since u is a maximal theory, it means that $\phi \to \psi \in u$. Thus, $P^-_w \cup \{\neg(\phi \to \psi)\}$ is not a consistent theory, and according to Deduction theorem $P^-_w \vdash \phi \to \psi$. Using Lemma 3.3 we obtain $w \vdash P_{\geqslant 1}(\phi \to \psi)$, and by Lemma 3.1c) and Deduction theorem $w \vdash P_{\geqslant s}(\phi) \to P_{\geqslant s}(\psi)$. We summarize these facts in two following lemmas:

Lemma 3.4. *Let $M = \langle W, D, I, \mathrm{Prob}\rangle$ be as above, $w \in W$ and let ϕ, ψ be sentences from $\mathit{Sent}^{=}_{\mathrm{FOICP}}$. Then, the following hold:*

a) *$H(w)$ is an algebra of subsets of $W(w)$,*
b) *if $[\phi] = [\psi]$, then $\mu(w)([\phi]) = \mu(w)([\psi])$,*
c) *if $[\phi] = [\psi]$, then $P_{\geqslant s}(\phi) \in w$ iff $P_{\geqslant s}(\psi) \in w$, and $P_{\leqslant s}(\phi) \in w$ iff $P_{\leqslant s}(\psi) \in w$,*
d) *$\mu(w)$ is a finitely additive measure.*

Lemma 3.5. *$M = \langle W, D, I, \mathrm{Prob}\rangle$, defined as above, is an* $\mathrm{LFOICP}^{=}_{\mathrm{Meas}}$*-model.*

Theorem 3.4 (Extended completeness theorem for $\mathrm{LFOICP}^{=}_{\mathrm{Meas}}$). *A theory T is consistent if and only if it has an* $\mathrm{LFOICP}^{=}_{\mathrm{Meas}}$*-model.*

Proof. The direction from right to left follows from the soundness theorem. The theory T can be extended to some saturated maximal consistent theory w in the expanded language $\mathcal{L}^*$, and for the canonical model M holds $(M, w) \vDash T$. □

Theorem 3.5 (Extended completeness theorem for $\mathrm{LFOICP}^{=}_{\mathrm{All}}$). *A theory T is consistent if and only if it has an* $\mathrm{LFOICP}^{=}_{\mathrm{All}}$*-model.*

Proof. Applying the extension theorem for additive measures from [**5**] it is possible to obtain finitely additive measures on the power set of W whose restrictions are $\mu(w)$ from the weak canonical model M. □

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