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ON THE CLASS GAMMA AND RELATED CLASSES OF FUNCTIONS

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ABSTRACT. The gamma class $\Gamma_\alpha(g)$ consists of positive and measurable functions that satisfy $f(x+yg(x))/f(x) \to \exp(\alpha y)$. In most cases the auxiliary function g is Beurling varying and self-neglecting, i.e., $g(x)/x \to 0$ and $g \in \Gamma_0(g)$. Taking $h = \log f$, we find that $h \in E\Gamma_\alpha(g,1)$, where $E\Gamma_\alpha(g,a)$ is the class of positive and measurable functions that satisfy $(f(x+yg(x)) - f(x))/a(x) \to \alpha y$. In this paper we discuss local uniform convergence for functions in the classes $\Gamma_\alpha(g)$ and $E\Gamma_\alpha(g,a)$. From this, we obtain several representation theorems. We also prove some higher order relations for functions in the class $\Gamma_\alpha(g)$ and related classes. Two applications are given.

1. Introduction and definitions

Let $f(x)$ denote a measurable function defined on $\mathbb{R}$ and positive for large values of x. The class $\Gamma_\alpha(g)$ consists of the functions f for which there exists a measurable and positive function g such that

$$\lim_{x\to\infty} \frac{f(x+yg(x))}{f(x)} = e^{\alpha y}, \quad \forall y \in \mathbb{R}. \tag{1.1}$$

Notation: $f \in \Gamma_\alpha(g)$. If $\alpha = 0$, we write $f \in \Gamma_0(g)$. If $\alpha > 0$, then w.l.o.g. we assume that $\alpha = 1$ and we write $f \in \Gamma(g)$. If $\alpha < 0$, we may assume that $\alpha = -1$ and then we write $f \in \Gamma_-(g)$. Clearly $f \in \Gamma_-(g)$ if and only if $1/f \in \Gamma(g)$.

If $g(x) = c \neq 0$, a constant, then (1.1) can be replaced by a relation of the form

$$\lim_{x\to\infty} \frac{f(x+y)}{f(x)} \to e^{\beta y}, \quad \forall y \in \mathbb{R}.$$

Notation: $f \in L(\beta)$. If $f \in L(\beta)$, then we have $f(\log(x)) \in RV(\beta)$, a regularly varying function. Recall that a positive and measurable function f is regularly

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varying with index β if it satisfies

$$\lim_{x\to\infty} \frac{f(xy)}{f(x)} = y^{\beta}, \quad \forall y > 0.$$

Notation: $f \in RV(\beta)$. The class $L(\beta)$ appears in studying subexponential distribution functions, cf. Embrechts et al. (1997). The classes $RV(\beta)$ and $\Gamma_\alpha(g)$ appear in the context of extreme value theory, cf. de Haan (1970). For regular variation and applications, we refer to Bingham et al. (1987). For nondecreasing functions f, we have the following property. For a proof we refer to Geluk and de Haan (1987) or Bingham et al. (1987).

LEMMA 1.1. *Suppose that f is nondecreasing and that* (1.1) *holds. Then*
(i) *Relation* (1.1) *holds l.u. in y;*
(ii) *We have $g \in \Gamma_0(g)$, $g(x)/x \to 0$ and*

$$\frac{g(x+yg(x))}{g(x)} \to 1, \quad \forall y \in \mathbb{R}, \textit{l.u. in } y. \tag{1.2}$$

In Lemma 1.1 and throughout the paper, we use the abbreviation "l.u." for "local uniform" convergence. Also throughout the paper we take limits as $x \to \infty$. Lemma 1.1 motivates the following definitions. A measurable and positive function g is called Beurling varying if it satisfies $g(x)/x \to 0$ and $g \in \Gamma_0(g)$. Notation: $g \in B$. The function g is called self-neglecting if $g \in B$ and if (1.2) holds. Notation: $g \in SN$. It has been proved by Bloom (1976) that if $g \in B$ is continuous, then $g \in SN$. For an elegant proof, we refer to Geluk and de Haan (1987, Theorem 1.34). It is not clear whether $g \in B$ alone implies that $g \in SN$. The classes B and SN were used in connection with Tauberian theory, cf. Bingham and Goldie (1983) and also in connection with differential equations, cf. Omey (1981).

In the present paper we plan to study l.u. in (1.1) without assuming that f is a monotone function. This answers a question of Geluk and de Haan (1987, p. 41). At the same time we will study the rate of convergence in (1.1). Taking logarithms in (1.1), we obtain that $\log f(x+yg(x)) - \log f(x) \to \alpha y$. We look at this kind of relationship more generally and introduce the class $E\Gamma_\alpha(g,a)$ of (ultimately) positive and measurable functions f satisfying

$$\frac{f(x+yg(x)) - f(x)}{a(x)} \to \alpha y, \quad \forall y \in \mathbb{R}. \tag{1.3}$$

If $a(x) = o(f(x))$, (1.3) shows that $f \in \Gamma_0(g)$ with a remainder term. In the case where $g = a$, the class $E\Gamma$ appears in Tenenbaum (1980) and in Bingham and Goldie (1983) in connection with one-sided Tauberian theorems.

If $g(x) = c \neq 0$, a constant, then (1.3) implies that $h \in \Pi_\alpha(L)$, where $h(x) = f(\log(x))$ and $L(x) = a(\log(x))$. The class $h \in \Pi_\alpha(L)$ is the class of positive and measurable functions such that

$$\frac{h(xy) - h(x)}{L(x)} \to \alpha \log y, \quad \forall y > 0, \tag{1.4}$$

where $L \in RV(0)$, cf. de Haan (1974). If (1.4) holds then it holds l.u. in $y > 0$. For monotone increasing functions $f \in \Gamma_\alpha(g)$, we can define the inverse function

$h(x) = f^{-1}(x)$. If (1.1) then it follows that $h \in \Pi_\alpha(L)$, where $L(x) = g(h(x)) \in RV(0)$, cf. de Haan (1974).

EXAMPLES. (1) If $f(x) = \exp(-x/2)$ (cf. normal density) we have $f \in \Gamma_-(g)$ where $g(x) = 1/x$.

(2) Suppose that $f(x) = \int_x^\infty \exp(-y^2/2)\, dy$. (cf. the tail of a normal distribution). In this case we have that

$$\lim_{x\to\infty} \frac{f(x)}{x^{-1}\exp(-x^2/2)} = 1.$$

Using Example 1, we find that $h \in \Gamma_-(1/x)$.

(3) If $f \in RV(\alpha)$ then we have $f(xy)/f(x) \to y^\alpha$, l.u. in $y > 0$, cf. Bingham et al. (1987). For any function $g(x)$ such that $g(x)/x \to 0$, it follows that

$$\frac{f(x + yg(x))}{f(x)} = \frac{f(x(1 + yg(x)/x))}{f(x)} \to 1.$$

It follows that $f \in \Gamma_0(g)$ and the relation $f(x+yg(x))/f(x) \to 1$ holds l.u. in $y \in \mathbb{R}$. Clearly $f \in RV(\alpha)$ implies that $f \in L(0)$.

(4) If $g \in RV(\beta)$ and $g(x)/x \to 0$, then $g \in SN$.

(5) Suppose that $g(x) \to \infty$ and that the first derivative satisfies $g^{(1)}(x) \to 0$. Then $g \in SN$.

Here is the outline of the paper. In the present paper we study l.u. in (1.3) and without assuming that f is a monotone function. This answers a question of Geluk and de Haan (1987, p. 41). In the main result of Section 2.1 we prove l.u. convergence for functions satisfying (1.3). This result implies that we also have l.u. convergence in (1.1). Local uniform convergence then opens the gate to a number of representation theorems that are presented in Subsection 2.2. In Section 3 we discuss remainder terms in the above definitions (1.1) and (1.3) and obtain several rate of convergence results. In Section 4, we conclude the paper with some applications.

2. The class $E\Gamma_\alpha(g, a)$

If $f \in \Gamma(g)$ is not monotone, it is a natural question to try to find general conditions under which (1.1) still holds l.u. in y. The main result of this section is stated in terms of the class of functions $E\Gamma_\alpha(g, a)$ defined as follows. A positive and measurable function f is in the class $E\Gamma_\alpha(g, a)$ with measurable and positive auxiliary functions g and a if

$$\frac{f(x + yg(x)) - f(x)}{a(x)} \to \alpha y, \quad \forall y \in \mathbb{R}. \tag{2.1}$$

All the results that follow require that f is ultimately of one sign.

2.1. Local Uniform Convergence. In the next result we state conditions under which (2.1) implies that (2.1) holds l.u. in y. The result is due to de Haan and Omey (1990) and is based on Delange (1955) and Bingham and Goldie (1983). See also Bingham et al. (1987, Theorem 1.2.1).

THEOREM 2.1. *Suppose that $g \in SN$ and that $a \in \Gamma_0(g)$. If $f \in E\Gamma_\alpha(g, a)$, then* (2.1) *holds l.u. in y.*

PROOF. The proof is divided into several parts. First we consider the case where $\alpha = 0$, $a \in \Gamma_0(g)$ and where a satisfies

$$\frac{a(x + yg(x))}{a(x)} \to 1, \text{ l.u. in } y \in \mathbb{R}. \tag{2.2}$$

To prove l.u. convergence in (2.1), first we prove l.u. convergence in each interval of the form $[0, h]$ where $h > 0$. Choose ε such that $0 < \varepsilon < h$ and for $x > 0$ define the following sets:

$$\begin{aligned} I(x) &= \{y : x \leqslant y \leqslant x + 2hg(x)\}, \\ E(x) &= \big\{t \in I(x) : |f(t) - f(x)| > \tfrac{\varepsilon}{3} a(x)\big\}, \\ W(x) &= \big\{c \in [0, 2h] : |f(x + cg(x)) - f(x)| > \tfrac{\varepsilon}{3} a(x)\big\}. \end{aligned}$$

Since all functions involved are measurable, these sets are measurable. Moreover, $\mathcal{L}(E(x)) = g(x)\mathcal{L}(W(x))$, where $\mathcal{L}(.)$ denotes Lebesgue measure. By (2.1) we have $\mathcal{L}(W(x)) \to 0$. Thus, for $\varepsilon > 0$ we can find x_1 such that

$$\mathcal{L}(E(x)) = g(x)\mathcal{L}(W(x)) \leqslant \tfrac{\varepsilon}{3} g(x), \quad \forall x \geqslant x_1. \tag{2.3}$$

From (2.2) and $g \in SN$, we have, uniformly in $c \in [0, 2h]$ that

$$\frac{1}{2} \leqslant \frac{g(x + cg(x))}{g(x)} \leqslant 2, \quad \forall x \geqslant x_2, \tag{2.4}$$

$$\frac{1}{2} \leqslant \frac{a(x + cg(x))}{a(x)} \leqslant 2, \quad \forall x \geqslant x_2. \tag{2.5}$$

From (2.3) and (2.4), it follows that

$$\mathcal{L}(E(x + cg(x)) \leqslant \frac{2\varepsilon}{3} g(x), \quad \forall c \in [0, 2h], \quad \forall x \geqslant x_2.$$

It follows that $\mathcal{L}(E(x) \cup E(x + cg(x)) \leqslant \varepsilon g(x)$, $\forall c \in [0, 2h]$, $\forall x \geqslant x_2$. For the intervals $I(x)$, we have $\mathcal{L}(I(x) \cap I(x + cg(x))) \geqslant hg(x)$, $\forall c \in [0, h]$, $\forall x \geqslant x_3$. Combining these inequalities, we find that for $c \in [0, h]$ and $x \geqslant x_3$, the set

$$A = (I(x) \cap I(x + cg(x)) \setminus (E(x) \cup E(x + cg(x)))$$

has positive Lebesgue measure and hence is not empty. If $t \in A$, we have

$$\frac{|f(t) - f(x)|}{a(x)} \leqslant \frac{\varepsilon}{3}, \qquad \frac{|f(t) - f(x + cg(x))|}{a(x + cg(x))} \leqslant \frac{\varepsilon}{3}.$$

From this and (2.5) it follows that

$$\frac{|f(x + cg(x)) - f(x)|}{a(x)} \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \frac{a(x + cg(x))}{a(x)} \leqslant \varepsilon, \quad \forall c \in [0, h], x \geqslant x_3.$$

It follows that (2.1) holds uniformly in $[0, h]$.

Now we consider uniform convergence on intervals of the form $[-h, 0]$ where $h > 0$. Let $y = x + cg(x)$, $c \in [-h, 0]$. Since $g \in SN$, for $\varepsilon > 0$ we have that

$$1 - \varepsilon \leqslant \frac{g(y)}{g(x)} \leqslant 1 + \varepsilon, \quad \forall c \in [-h, 0], \quad x \geqslant x_1.$$

We also have $y = x + cg(x) \leqslant x \leqslant x + hg(x)$. Using $y - x = cg(x) \geqslant cg(y)/(1 - \varepsilon)$, we find that

$$y \leqslant x \leqslant y + \frac{-c}{1 - \varepsilon} g(y) \leqslant y + \frac{h}{1 - \varepsilon} g(y).$$

It follows that $y \leqslant x \leqslant y + \delta g(y)$ where $\delta > 0$. Clearly x is of the form $x = y + \theta g(y)$, where $0 \leqslant \theta \leqslant \delta$. To finish the proof in this case, we write

$$\frac{|f(x + cg(x)) - f(x)|}{a(x)} = \frac{|f(y) - f(y + \theta g(y)|}{a(y)} \frac{a(y)}{a(y + \theta g(y))},$$

and we find

$$\sup_{-h \leqslant c \leqslant 0} \frac{|f(x + cg(x)) - f(x)|}{a(x)} \leqslant \sup_{0 \leqslant \theta \leqslant \delta} \frac{|f(y) - f(y + \theta g(y)|}{a(y)} \sup_{0 \leqslant \theta \leqslant \delta} \frac{a(y)}{a(y + \theta g(y))}.$$

By the first part of the proof, it follows that

$$\sup_{-h \leqslant c \leqslant 0} \frac{|f(x + cg(x)) - f(x)|}{a(x)} \to 0.$$

Now we consider the case where $\alpha \neq 0$. First note that $a \in \Gamma_0(g)$ implies that $\log(a) \in E\Gamma_0(g, 1)$. The first part of the proof of the theorem shows that

$$\log(a(x + yg(x))) - \log(a(x)) \to 0, \text{ l.u. in } y.$$

Hence (2.2) holds. Now define the function $A(x)$ as follows:

$$A(x) = \alpha \int_{x^\circ}^{x} \frac{a(z)}{g(z)} \, dz.$$

This integral exists since g and a are locally bounded. Now observe that l.u. in y we have

$$\frac{A(x + yg(x)) - A(x)}{a(x)} = \alpha \int_0^y \frac{a(x + zg(x))}{g(x + zg(x))} \frac{\varphi(x)}{a(x)} \, dz \to \alpha y,$$

since $g \in SN$ and $a \in \Gamma_0(g)$. It follows that $B(x) = f(x) - A(x)$ satisfies $B \in E\Gamma_0(g, a)$. By the first part of the proof, we find that

$$\frac{B(x + yg(x)) - B(x)}{a(x)} \to 0$$

holds l.u. in $y \in \mathbb{R}$. Hence (2.1) also holds l.u. in $y \in \mathbb{R}$. □

Corollary 2.1. *Suppose that $g \in SN$ and that $f \in \Gamma_\alpha(g)$, $\alpha \in \mathbb{R}$. Then* (1.1) *holds l.u. in $y \in \mathbb{R}$.*

2.2. Representation theorems. Using the l.u. convergence in (2.1), we have the following representation theorem for functions in the class $E\Gamma_\alpha(g,a)$. We use the notation $W^{(1)}$ for the first derivative of a function W.

THEOREM 2.2. *Suppose that $g \in SN$, $a \in \Gamma_0(g)$ and that f is positive and measurable. We have $f \in E\Gamma_\alpha(g,a)$ if and only if $f(x)$ is of the form*

$$f(x) = C + \alpha \int_{x^\circ}^{x} \frac{a(z)}{g(z)}\,dz + W(x) + V(x),$$

where $W^{(1)}(x)g(x)/a(x) \to 0$ and $V(x)/a(x) \to 0$.

PROOF. First assume that $f \in E\Gamma_\alpha(g,a)$. Using the notation as in the proof of Theorem 2.1, we consider the functions $A(x)$ and $B(x) = f(x) - A(x)$. In Theorem 2.1 we obtained that $B \in E\Gamma_0(g,a)$. Now let $\Phi(x)$ be defined as

$$\Phi(x) = \int_a^x \frac{1}{g(t)}\,dt,$$

and let $\Psi(x) = B(\Phi^{-1}(x))$. We have $\Phi \in E\Gamma_1(g,1)$, $\Phi^{-1} \in E\Gamma_1(1, g(\Phi^{-1}))$ and we find that

$$\frac{\Psi(x+y) - \Psi(x)}{a(\Phi^{-1}(x))} \to 0, \text{ l.u. in } y \in \mathbb{R}.$$

Taking integrals, it follows that

$$\frac{\int_0^1 \Psi(x+y)\,dy - \Psi(x)}{a(\Phi^{-1}(x))} \to 0,$$

or $V(x) - \Psi(x) = o(a(\Phi^{-1}(x))$, where

$$V(x) = \int_0^1 \Psi(x+y)\,dy = \int_x^{x+1} \Psi(t)\,dt.$$

Note that $V^{(1)}(x) = \Psi(x+1) - \Psi(x)$ so that $V^{(1)}(x) = o(a(\Phi^{-1}(x))$. We conclude that $\Psi(x) = V(x) + o(1)\,a(\Phi^{-1}(x))$ and then also that $B(x) = W(x) + o(1)\,a(x)$, where $W(x) = V(\Phi(x))$ satisfies $W^{(1)}(x) = V^{(1)}(\Phi(x))/g(x) = o(1)\,a(x)/g(x)$. This gives the representation.

To prove the converse, we have

$$\begin{aligned} f(x+yg(x)) - f(x) = \alpha \int_x^{x+yg(x)} \frac{a(z)}{g(z)}\,dz + W(x+yg(x)) - W(x) \\ + V(x+yg(x)) - V(x). \end{aligned}$$

Now note that

$$W(x+yg(x)) - W(x) = g(x) \int_0^y W^{(1)}(x+zg(x))\,dz.$$

Using $W^{(1)}(x)g(x)/a(x) \to 0$, it follows that $W \in E\Gamma_0(g,a)$. For the other term, we have

$$\frac{1}{a(x)} \int_x^{x+yg(x)} \frac{a(z)}{g(z)}\,dz = \int_0^y \frac{a(x+zg(x))}{a(x)} \frac{g(x)}{g(x+zg(x))}\,dz \to y,$$

since $g \in SN$ and $a \in \Gamma_0(g)$. The result follows. $\square$

COROLLARY 2.2. *Suppose that $g \in SN$, $a \in \Gamma_0(g)$ and that f is positive and measurable. Then $f \in E\Gamma_\alpha(g,a)$ if and only $f(x)$ is of the form*

$$f(x) = C + \alpha \int_{x^\circ}^{x} \frac{L(z)}{g(z)}\, dz + V(x),$$

where $L \sim a \in \Gamma_0(g)$ and $V(x)/L(x) \to 0$.

PROOF. From Theorem 2.2, we have that

$$f(x) = C + \alpha \int_{x^\circ}^{x} \frac{a(z)}{g(z)}\, dz + W(x) + V(x),$$

where $W^{(1)}(x)g(x)/a(x) \to 0$ and $V(x)/a(x) \to 0$. It follows (with possibly a different value for C) that

$$W(x) = \int_a^x \frac{\varepsilon(z)a(z)}{g(z)}\, dz.$$

Taking $L(x) = (1 + \varepsilon(x)/\alpha)a(x)$, the result follows. □

2.3. Representation theorems for SN. In this section we prove representation theorems for the class SN. For a different proof of the first result, we refer to Geluk and de Haan (1987, Theorem 1.35).

THEOREM 2.3. *We have $g \in SN$ if and only if g is of the form*

$$g(x) = U(x)\, W(x),$$

where $U(x) \to 1$ and $W^{(1)}(x) \to 0$.

PROOF. Starting form $g \in SN$, we define $\Phi(x)$, where

$$\Phi(x) = \int_a^x \frac{1}{g(u)}\, du.$$

Since $g(x) > 0$ and $g(x)/x \to 0$, we find that $\Phi(x) \uparrow \infty$. As in the proof of Theorem 2.2, we have $\Phi^{-1} \in E\Gamma_1(1, g(\Phi^{-1}))$. Now we consider the function $\Psi(x) = g(\Phi^{-1}(x))$. Using the identity

$$\Psi(x+y) = g\Big(\Phi^{-1}(x) + \frac{\Phi^{-1}(x+y) - \Phi^{-1}(x)}{g(\Phi^{-1}(x))} g(\Phi^{-1}(x))\Big),$$

and $g \in SN$, it follows that $\Psi(x+y)/\Psi(x) \to 1$. Hence we obtain that $\Psi(\log(x)) \in RV(0)$. The representation theorem (cf. Bingham et al. (1987)) for $RV(0)$ shows that

$$\Psi(\log(x)) = c(x) \exp \int_a^x \varepsilon(z) z^{-1} dz,$$

where $c(x) \to c > 0$ and $\varepsilon(t) \to 0$. Replacing $\log(x)$ by t, and then t by $\Phi(x)$, we find that

$$g(x) = D(x) \exp \int_{x^\circ}^{x} \varepsilon^*(z) \frac{1}{g(z)}\, dz,$$

where $D(x) \to c > 0$ and $h(x\varepsilon^*(x) = \varepsilon^\circ(\Phi(x)) \to 0$. Taking $U(x) = D(x)/c$ and

$$W(x) = c \exp \int_{x^\circ}^{x} \varepsilon^*(z) \frac{1}{g(z)}\, dz,$$

it follows that $g(x)$ is of the desired form.

To prove the converse, first note that we have $g(x)/x \to 0$. Next we consider $W(x)$ and we write

$$W(x+yg(x)) - W(x) = g(x)\int_0^y W^{(1)}(x+zg(x))\,dz.$$

Using $W^{(1)}(x) \to 0$, we obtain that $W \in E\Gamma_0(g,g)$. Now we write

$$\frac{g(x+yg(x)) - g(x)}{g(x)} = I + II$$

where

$$I = U(x+yg(x))\frac{W(x+yg(x)) - W(x)}{g(x)}, \qquad II = \frac{U(x+yg(x)) - U(x)}{U(x)}.$$

It follows that $I + II \to 0$ l.u. in y and this proves the result. □

Alternatively, we can also use the following representation for $RV(0)$, cf. Omey and Segers (2010, Theorem 4.3).

Proposition 2.1. *The positive and measurable function L is in the class $RV(0)$ if and only if there exists real numbers $a > 0$ and c and measurable functions $u(x)$, $v(x)$ such that $u(x)/L(x) \to 0$, $v(x)/L(x) \to 0$ and*

$$L(t) = c + u(t) + \int_a^t v(z)z^{-1}dz.$$

Applying this result to Ψ, we find the following alternative for Theorem 2.3.

Proposition 2.2. *We have $g \in SN$ if and only if g is of the form*

$$g(x) = c + A(x) + \int_a^x \frac{B(u)}{g(u)}\,du,$$

where $A(x)/g(x) \to 0$ and $B(x)/g(x) \to 0$.

2.3.1. *Representation theorem for $\Gamma_\alpha(g)$.* In the next result we characterize the class $\Gamma_\alpha(g)$. In the case of nondecreasing f, the proof was given in Geluk and de Haan (1987, Theorem 1.28).

Corollary 2.3. *Suppose that $g \in SN$ and suppose that f is positive and measurable. We have $f \in \Gamma_\alpha(g)$ if and only if f is of the form*

$$f(x) = A(x)\,B(x)\exp\Big(\alpha\int_{x^\circ}^x \frac{1}{g(z)}\,dz\Big),$$

where $A(x) \to 1$ and $g(x)B^{(1)}(x)/B(x) \to 0$.

Proof. If $f \in \Gamma_\alpha(g)$ we have $\log(f) \in E\Gamma_\alpha(g,1)$ and Theorem 2.2 shows that

$$\log f(x) = C + \alpha\int_{x^\circ}^x \frac{1}{g(z)}\,dz + W(x) + o(1),$$

where $W^{(1)}(x)\,g(x) \to 0$. Taking $B(x) = \exp(C + W(x))$, we have $B^{(1)}(x) = B(x)\,W^{(1)}(x)$ and $g(x)\,B^{(1)}(x)/B(x) \to 0$. □

2.3.2. *Characterization inspired by de Haan.* It is well known that $f \in RV(\alpha)$ and $g \in RV(\beta)$, $\alpha, \beta \geqslant 0$, implies that $f(g(x)) \in RV(\alpha\beta)$. Inspired by de Haan (1974), we connect regular variation and the class Γ to obtain new characterizations for the classes $\Gamma_\alpha(g)$ and $E\Gamma(\varphi, a)$.

THEOREM 2.4. (i) *Let $g \in SN$. We have $f \in \Gamma_\alpha(g)$ if and only if f is of the form $f(x) = A(G(x))$, where $A \in RV(\alpha)$ and $G \in \Gamma(g)$.*

(ii) *Let $g \in SN$ and $a \in \Gamma_0(g)$. We have $f \in E\Gamma_\alpha(g, a)$ if and only if f is of the form $f(x) = A(G(x))$, where $A \in \Pi_\alpha(L)$ with $L \in RV(0)$ and $G \in \Gamma(g)$.*

PROOF. (i) Starting from $g \in SN$ we define the function $G(x)$ by

$$G(x) = \exp \int_a^x \frac{1}{g(t)}\, dt.$$

Clearly $G(x)$ is increasing and $G \in \Gamma(g)$. It follows that $G^{-1} \in \Pi_1(L)$, where $L(x) = g(G^{-1}(x)) \in RV(0)$. Taking $A(x) = f(G^{-1}(x))$, we have

$$A(xy) = f\Big(\frac{G^{-1}(xy) - G^{-1}(x)}{L(x)} g(G^{-1}(x)) + G^{-1}(x)\Big).$$

Since $f \in \Gamma_\alpha(g)$ and $G^{-1} \in \Pi_1(L)$, we obtain that $\frac{A(xy)}{A(x)} \to \exp \alpha \log y = y^\alpha$, and $A \in RV(\alpha)$. It follows that $f(x) = A(G(x))$ as required. For the converse, we write

$$f(x + yg(x)) = A\Big(\frac{G(x + yg(x))}{G(x)} G(x)\Big).$$

Since $G \in \Gamma(g)$ and $A \in RV(\alpha)$, it follows that $f \in \Gamma_\alpha(g)$.

(ii) We use the function $G(x)$ as before and let $A(x) = f(G^{-1}(x))$. We have

$$A(xy) - A(x) = f\Big(\frac{G^{-1}(xy) - G^{-1}(x)}{L(x)} g(G^{-1}(x)) + G^{-1}(x)\Big) - f(G^{-1}(x)).$$

Using $f \in E\Gamma_\alpha(g, a)$ and l.u. convergence, we obtain that

$$\frac{A(xy) - A(x)}{a(G^{-1}(x))} \to \alpha \log y,$$

and it follows that $A \in \Pi_\alpha(L)$ with $L(x) = a(G^{-1}(x))$. We conclude that $f(x) = A(G(x))$ with A and G as required.

For the converse, we write

$$f(x + yg(x)) - f(x) = A\Big(\frac{G(x + yg(x))}{G(x)} G(x)\Big) - A(G(x)),$$

and it follows that $f \in E\Gamma_\alpha(g, a)$ with $a(x) = L(G(x))$. $\square$

3. Remainder terms

3.1. Remainder term for the class $\Gamma_0(g)$ and $E\Gamma_1(g, a)$. In this section we obtain a result that describes the asymptotic behaviour of the remainder term in (2.1). In the first result we assume that f and g have derivatives.

Proposition 3.1. (i) *Suppose that f has a derivative $f^{(1)} \in \Gamma_0(g)$. Then*

$$f(x + yg(x)) - f(x) \sim g(x)f^{(1)}(x)\, y \tag{3.1}$$

this is $f \in E\Gamma_1(g, a)$ with $a(x) = g(x)f^{(1)}(x)$.

(ii) *Suppose that f has an N-th order derivative $f^{(N)} \in \Gamma_0(g)$. Then*

$$f(x + yg(x)) - f(x) - \sum_{k=1}^{N-1} g^k(x)f^{(k)}(x)\, \frac{y^k}{k!} \sim g^N(x)f^{(N)}(x)\, \frac{y^N}{N!}. \tag{3.2}$$

Proof. (i) For $y > 0$ we have

$$f(x + yg(x)) - f(x) = g(x) \int_0^y f^{(1)}(x + zg(x))\, dz$$

Using $f^{(1)} \in \Gamma_0(g)$, we obtain the result.

(ii) Using (3.1) we have

$$f(x + yg(x)) - f(x) - g(x)f^{(1)}(x)\, y = g^2(x) \int_0^y \int_0^{u_1} f^{(2)}(x + u_2 g(x))\, du_2 du_1,$$

and in general we have

$$I = g^N(x) \int_0^y \int_0^{u_1} \dots \int_0^{u_{N-1}} f^{(N)}(x + u_N g(x))\, du_N \dots du_1,$$

where I denotes the left handside of (3.2). The result follows. □

Remarks. (1) In order to have expressions that make sense, we should have $g(x)f^{(i)}(x) = o(f^{(i-1)}(x))$, $1 \leqslant i \leqslant N$. In this case (11) gives more precise approximations as N grows.

(2) If f is such that $f^{(N)} \in RV(\alpha)$, where $\alpha > 0$, then $f^{(i)} \in RV(\alpha + N - i)$, $0 \leqslant i \leqslant N$, and

$$\frac{xf^{(i)}(x)}{f^{(i-1)}(x)} \to \alpha + N - i.$$

The theorem can be applied with any function g for which $g(x)/x \to 0$.

3.2. Remainder term for $f \in \Gamma(g)$. In studying $\Gamma(g)$ we have some freedom in choosing the auxiliary function g. In studying the rate of convergence in the definition of $\Gamma(g)$, the choice of the auxiliary function g plays an important role. If $f \in \Gamma(g)$ and $g_1(x) \sim g(x)$, then we also have $f \in \Gamma(g_1)$. The rate of convergence in (1.1) depends on the choice of auxiliary function. The following example illustrates this point.

Example. Let $f(x) = x \exp(x)$. In this case we have $f \in \Gamma(g)$ with $g(x) = 1$. Note that with this choice of g we have

$$x\Big(\exp(-y)\frac{f(x + y)}{f(x)} - 1\Big) = y.$$

We also have $f \in \Gamma(g)$ with $g(x) = f(x)/f^{(1)}(x) = x/(1+x)$. With this choice of g we obtain that

$$\exp(-y)\frac{f(x+yg(x))}{f(x)} = \Big(1 + \frac{y}{1+x}\Big)\exp\Big(-\frac{y}{1+x}\Big).$$

Straightforward calculations show that

$$x^2\Big(\exp(-y)\frac{f(x+yg(x))}{f(x)} - 1\Big) \to -\frac{y^2}{2}.$$

The following proposition is a motivation for taking $g = f/f^{(1)}$.

Proposition 3.2. (i) *Suppose that f has a nondecreasing derivative $f^{(1)}$. If $f \in \Gamma(g)$, then $f^{(1)} \in \Gamma(g)$ and $g(x) \sim f(x)/f^{(1)}(x) \in SN$.*
(ii) *Let $g(x) = f(x)/f^{(1)}(x)$. If $g \in SN$, then $f \in \Gamma(g)$.*

Proof. (i) Let $y > 0$. We have

$$f(x+yg(x)) - f(x) = g(x)\int_0^y f^{(1)}(x+zg(x))\,dz.$$

Since $f^{(1)}$ is nondecreasing, we have

$$yg(x)f^{(1)}(x) \leqslant f(x+yg(x)) - f(x) \leqslant yg(x)f^{(1)}(x+yg(x)),$$

and from here also that

$$y\frac{g(x)f^{(1)}(x)}{f(x)} \leqslant \frac{f(x+yg(x))}{f(x)} - 1 \leqslant y\frac{g(x)f^{(1)}(x+yg(x))}{f(x)}.$$

Taking limits, on the one hand we get that

$$\limsup \frac{g(x)f^{(1)}(x)}{f(x)} \leqslant \frac{1}{y}(e^y - 1).$$

On the other hand we have

$$\frac{f(x+yg(x))}{f(x)} - 1 \leqslant y\frac{g(x)f^{(1)}(x+yg(x))}{f(x)}.$$

Now let $t = x + yg(x)$ to see that

$$\frac{g(t)}{g(x)}\frac{f(x)}{f(t)}\Big(\frac{f(t)}{f(x)} - 1\Big) \leqslant y\frac{g(t)f^{(1)}(t)}{f(t)}.$$

Since $f \in \Gamma(g)$, we have (cf. Lemma 1.1) that $g \in SN$ and hence that $g(x) \sim g(t)$. It follows that

$$\frac{1}{y}e^{-y}(e^y - 1) \leqslant \liminf \frac{g(t)f^{(1)}(t)}{f(t)}.$$

Since $y > 0$ was arbitrary, it follows that $g(x)f^{(1)}(x)/f(x) \to 1$. Since $g \in SN$ and $f \in \Gamma(g)$, we also get that $f^{(1)} \in \Gamma(g)$.

(ii) Integrating the equation $g(x) = f(x)/f^{(1)}(x)$, for $y > 0$ we get that

$$\log\frac{f(x+yg(x))}{f(x)} = \int_x^{x+yg(x)} \frac{1}{g(t)}\,dt = \int_0^y \frac{g(x)}{g(x+zg(x))}\,dz.$$

Using $g \in SN$, it follows that

$$\log \frac{f(x+yg(x))}{f(x)} \to y,$$

and hence that $f \in \Gamma(g)$. □

Now we are ready for the main result of this section. In Theorem 3.1 below we obtain the precise asymptotic behaviour of the functions $W(x,y)-1$ and $R(x,y)$, where

$$W(x,y) = \frac{f(x+yg(x))}{f(x)} \exp(-y), \qquad R(x,y) = W(x,y) - 1 + \frac{y^2}{2} g^{(1)}(x).$$

THEOREM 3.1. (i) *Suppose that* $f \in \Gamma(g)$ *where* $g(x) = f(x)/f^{(1)}(x)$. *Assume that* $g^{(1)} \in \Gamma_0(g)$ *and that* $g^{(1)}(x) \to 0$. *Then as* $x \to \infty$ *we have*

$$W(x,y) - 1 \sim -\frac{y^2}{2} g^{(1)}(x), \text{ l.u. in } y.$$

(ii) *Suppose that* $f \in \Gamma(g)$ *where* $g(x) = f(x)/f^{(1)}(x)$. *Assume that* $g^{(2)}$ *is ultimately of one sign and that* $g^{(2)} \in RV(\alpha - 2)$, *where* $\alpha < 1$.

(a) *If* $\alpha \neq 0$, *as* $x \to \infty$, *we have*

$$\frac{1}{(g^{(1)}(x))^2} R(x,y) \to \frac{\alpha+1}{\alpha} \frac{y^3}{6} + \frac{y^4}{8}, \text{ l.u. in } y.$$

(b) *If* $\alpha = 0$, *as* $x \to \infty$, *we have*

$$\frac{1}{g(x)\, g^{(2)}(x)} R(x,y) = -\frac{y^3}{6}, \text{ l.u. in } y.$$

PROOF. (i) We use $W(x,y)$ as defined above and we take $H(x) = \log f(x)$. We have

$$\log W(x,y) = H(x+yg(x)) - H(x) - y. \tag{3.3}$$

Taking derivatives of $H(x)$ we see that

$$H^{(1)}(x) = \frac{1}{g(x)}, \quad H^{(2)}(x) = -\frac{g^{(1)}(x)}{g^2(x)}.$$

Since $H^{(2)} \in \Gamma_0(g)$, we can use (3.2) for H and with $N = 2$. We obtain that

$$H(x+yg(x)) - H(x) - yg(x)\, H^{(1)}(x) \sim g^2(x)\, H^{(2)}(x)\, \frac{y^2}{2},$$

so that

$$H(x+yg(x)) - H(x) - y \sim -g^{(1)}(x)\, \frac{y^2}{2}. \tag{3.4}$$

Since $\log(x) \sim (x-1)$ as $x \to 1$, we have $\log W(x,y) \sim W(x,y) - 1$, and the first result follows from (3.3) and (3.4).

(ii) For $H(x) = \log f(x)$, we find that

$$H^{(3)}(x) = 2\frac{(g^{(1)}(x))^2}{g^3(x)} - \frac{g^{(2)}(x)}{g^2(x)}.$$

If $g^{(2)}(x) \in RV(\alpha-2)$, $\alpha<1$, $\alpha \neq 0$, we have $(\alpha-1)\, g^{(1)}(x) \sim x g^{(2)}(x)$ and $\alpha g(x) \sim x\, g^{(1)}(x)$. It follows that

$$\frac{g^3(x)\, H^{(3)}(x)}{(g^{(1)}(x))^2} \to 2 - \frac{(\alpha-1)}{\alpha} = \frac{\alpha+1}{\alpha}. \tag{3.5}$$

If $g^{(2)}(x) \in RV(-2)$, then $g^{(1)}(x) \sim x g^{(2)}(x)$ and $g \in \Pi(x g^{(1)}(x))$. Since $x g^{(1)}(x)/g(x) \to 0$, in this case it follows that

$$\frac{g^2(x)\, H^{(3)}(x)}{g^{(2)}(x)} \to -1. \tag{3.6}$$

Using (3.2), we obtain that

$$H(x+yg(x)) - H(x) - g(x)\, H^{(1)}(x) y - g^2(x)\, H^{(2)}(x)\, \frac{y^2}{2} \sim g^3(x)\, H^{(3)}(x)\, \frac{y^3}{3!},$$

or

$$H(x+yg(x)) - H(x) - y + g^{(1)}(x)\, \frac{y^2}{2} \sim g^3(x)\, H^{(3)}(x)\, \frac{y^3}{3!}.$$

Now we consider $R(x,y)$. Using $(x-1) - \log(x) \sim 1/2(x-1)^2(1+o(1))$ as $x \to 1$, we obtain that $W(x,y) - 1 - \log W(y) \sim \frac{1}{2}(W(x,y)-1)^2$, or equivalently that $W(x,y) - 1 - (H(x+yg(x)) - H(x) - y) \sim \frac{1}{2}(W(x,y)-1)^2$. Using $W(x,y) - 1 \sim -g'(x) y^2/2$, it follows that

$$R(x,y) = (H(x+yg(x)) - H(x) - y + g^{(1)}(x)\, \frac{y^2}{2} + (1+o(1))\frac{1}{8} y^4 (g^{(1)}(x))^2. \tag{3.7}$$

If $\alpha \neq 0$, we use (3.5) and (3.7) to obtain that

$$\frac{1}{(g^{(1)}(x))^2} R(x,y) \to \frac{\alpha+1}{\alpha}\frac{y^3}{6} + \frac{y^4}{8}.$$

If $\alpha = 0$, we use (3.6) and (3.7) to obtain that

$$\frac{1}{g(x)\, g^{(2)}(x)}\, R(x,y) \to -\frac{y^3}{3!}.$$

This proves the result. □

Remarks. 1) Since we should have $g(x)/x \to 0$, it makes sense to assume that $g^{(1)}(x) \to 0$. If $g^{(1)}(x) \to \alpha \neq 0$, then $g(x)/x \to \alpha$ and the corresponding function f is regularly varying. The case where $g^{(1)}(x) \to \alpha \neq 0$ was treated in detail by de Haan (1996, Theorem 5).

2) Dekkers and de Haan (1989, Theorems A.7–A.10) study into detail results as in Theorem 3.1(i) in the case where $f(x) = 1 - F(x)$, where $F(x)$ is a distribution function. See also de Haan and Omey (1990).

3) In the case of $\alpha < 1$, in (3.5) we used the fact that

$$\frac{g(x)\, g^{(2)}(x)}{(g^{(1)}(x))^2} \to \frac{\alpha-1}{\alpha}.$$

If

$$\frac{g(x)\, g^{(2)}(x)}{(g^{(1)}(x))^2} \to 1,$$

we have that $g^{(2)} \in \Gamma(a)$, where $a(x) \sim g(x)/g^{(1)}(x)$, cf. Geluk and de Haan (1987, Theorem 1.28).

EXAMPLES. (1) If $f(x)=\exp(x/\log(x))$, we have $g(x)=1+\log(x)+1/(\log(x)-1)$, and we readily find that $g^{(2)}(x) \sim -x^2$.

(2) If $f(x) = \exp(\log(x))^\alpha$, $0 < \alpha < 1$, we have $g(x) = (\log(x))^{1-\alpha}/\alpha$, and it follows that $g^{(2)}(x) \sim cx^{-2}(\log(x))^{-\alpha} \in RV(-2)$.

(3) Let $f(x)$ be given by

$$f(x) = \frac{1}{\int_x^\infty y^b \exp(-y)\,dy},$$

i.e. $1/f(x)$ is related to a gamma distribution. In this case we find that

$$g(x) = \frac{f(x)}{f^{(1)}(x)} = \frac{\int_x^\infty y^b \exp(-y)\,dy}{x^b \exp(-x)}.$$

Note that

$$g(x) = x\int_1^\infty t^b \exp(-tx+x)\,dt = x\int_0^\infty (1+u)^b \exp(-ux)\,du.$$

Using partial integration, we get that

$$g(x) = 1 + b\int_0^\infty (1+u)^{b-1} \exp(-ux)\,du.$$

It readily follows that $g^{(2)}(x) \sim 2bx^{-3} \in RV(-3)$.

(4) Let $f(x)$ be given by

$$f(x) = \frac{1}{\int_x^\infty \exp(-y^2)\,dy},$$

i.e. $1/f(x)$ is related to a normal distribution. We find that

$$g(x) = \frac{f(x)}{f^{(1)}(x)} = \exp(x^2)\int_x^\infty \exp(-y^2)\,dy.$$

Note that

$$2xg(x) = \int_0^\infty \Big(1+\frac{v}{x^2}\Big)^{-1/2} \exp(-v)\,dv.$$

We find that $2xg(x) \to 1$, $x^2g^{(1)}(x) \to -1/2$ and $x^3g^{(2)}(x) \to 3/2$.

4. Applications

4.1. Differential equations. There are several papers devoted to the asymptotic behaviour of positive, increasing solutions of the differential equation

$$y^{(2)}(x) = f(x)\,y(x) \tag{4.1}$$

where $f(x) \geqslant 0$ and $y^{(2)}(x)$ denotes the second derivative of y. One such result is the following, cf. Omey (1981, 1997).

Proposition 4.1. *Define* $q \geqslant 0$ *by* $q^2(x)f(x) = 1$ *and suppose that* $q^{(1)}(x) \to 0$. *Then the positive increasing solutions* (4.1) *satisfy:*

(i) $q(x)y^{(1)}(x)/y(x) \to 1$;

(ii) $y \in \Gamma(q)$, $y^{(1)} \in \Gamma(q)$ *and* $y^{(2)} \in \Gamma(q)$.

Proof. Let $K(x) = q(x)\, y^{(1)}(x)/y(x) \geqslant 0$. Clearly we have

$$q(x)\, K^{(1)}(x) = q^{(1)}(x)\, K(x) + 1 - K^2(x).$$

First suppose that $K^{(1)}(x) > 0$, $x > x$. In this case $K(x) \uparrow A$, where $A \leqslant \infty$. If $A = \infty$, we have

$$\frac{q(x)\, K^{(1)}(x)}{K(x)} = q^{(1)}(x) + \frac{1}{K(x)} - K(x) \to -\infty.$$

This contradicts the assumption that $K^{(1)}(x) > 0$. Hence $A < \infty$ and then we obtain that $q(x)\, K^{(1)}(x) \to 1 - A^2 \geqslant 0$. If $1 - A^2 > 0$, then $K^{(1)}(x) \sim (1 - A^2)/q(x)$ and since $q(x)/x \to 0$, we obtain that $xK^{(1)}(x) \to \infty$ and then $K(x) \to \infty$, a contradiction again. It follows that $A^2 = 1$ and $A = 1$. Next suppose that $K^{(1)}(x) < 0$, $x \geqslant x^\circ$. In this case $K(x) \downarrow A \geqslant 0$ and we obtain that

$$q(x)K^{(1)}(x) \to 1 - A^2.$$

Again we should have $A^2 = 1$. Finally suppose that $K^{(1)}(x_n) = 0$ for a sequence $x_n \to \infty$. In this case we have

$$K(x_n) = \frac{q^{(1)}(x_n) + \sqrt{(q^{(1)}(x_n))^2 + 4}}{2}$$

for this sequence. Since K is monotone between zero's of $K^{(1)}(x)$, we find that

$$K(x_n) \leqslant K(x) \leqslant K(x_{n+1}), x_n \leqslant x \leqslant x_{n+1}$$

(or a similar inequality with reversed inequality signs). Since $K(x_n) \to 1$, it follows also in this case that $K(x) \to 1$. This proves (i). Result (ii) now easily follows because $q \in SN$. □

To obtain a rate of convergence result here, we could use $y \in \Gamma(g)$ with $g(x) = y(x)/y^{(1)}(x)$ and the result of Theorem 3.1(i). We want to establish a rate of convergence result in terms of $q(x)$, where $q^2(x)f(x) = 1$. In the next result we use both $q(x)$ and $Q(x) = 1/q(x)$.

Theorem 4.1. *Suppose that* $y(x)$ *is a positive increasing solution of* (4.1). *Assume that* $Q^{(1)}(x) > 0$ *and* $Q^{(1)}(x)$ *is nondecreasing for large values of* x. *If* $Q^{(1)} \in \Gamma_0(q)$ *and* $q^{(1)}(x) \to 0$, *then*

$$\frac{y(x + zq(x))}{y(x)} \exp(-z) - 1 \sim q^{(1)}(x)\Big(z - \frac{z^2}{2}\Big).$$

Proof. To prove the result we introduce functions L, u and C by

$$L(x) = y(x)Q(x) - y^{(1)}(x), \quad u(x) = \exp\int_c^x Q(s)\, ds, \quad C(x) = u(x)L(x).$$

Since $q^{(1)}(x) \to 0$, we have $q \in SN$, $Q \in SN$ and $u \in \Gamma(q)$. Straightforward calculations show that

$$C^{(1)}(x) = u(x)y(x)Q^{(1)}(x). \tag{4.2}$$

Note that $C^{(1)}(x) > 0$ for x large. The conditions of the theorem ensure that $C^{(1)}$ is nondecreasing. By assumption we have $Q^{(1)} \in \Gamma_0(q)$. Since $y, u \in \Gamma(q)$, it is easy to see that

$$\frac{C^{(1)}(x + yq(x))}{C^{(1)}(x)} \to \exp(2y),$$

so that $C^{(1)} \in \Gamma(q/2)$. Now this implies (Lemma 1.1) that $C(x) \sim C^{(1)}(x)\, q(x)/2$. Using (4.2), it follows that $C(x) \sim u(x)\, y(x)\, Q^{(1)}(x)\, q(x)/2$, and then also that $L(x) \sim y(x)\, Q^{(1)}(x)\, q(x)/2$. Since $L(x) = y(x)\, Q(x) - y^{(1)}(x)$, we have that

$$1 - \frac{y^{(1)}(x)}{Q(x)\, y(x)} \sim \frac{Q^{(1)}(x)}{2}.$$

In terms of $q(x)$ we find that

$$\frac{y^{(1)}(x)}{y(x)} - \frac{1}{q(x)} = (1 + \varepsilon(x))\frac{q^{(1)}(x)}{2q(x)}.$$

where $\varepsilon(x) \to 0$. Integration gives

$$\log\Big(\frac{y(x + zq(x))}{y(x)}\Big) - z = I + II,$$

where

$$\begin{aligned}
I &= \int_0^z \Big(\frac{q(x)}{q(x + \theta q(x))} - 1\Big)\, d\theta = -\int_0^z \int_0^\theta \frac{q^{(1)}(x + zq(x))q(x)}{q(x + \theta q(x))}\, dz\, d\theta \\
II &= \int_0^z \big(1 + \varepsilon(x + \theta q(x))\big)\frac{q^{(1)}(x + \theta q(x))q(x)}{q(x + \theta q(x))}\, d\theta
\end{aligned}$$

For the first term we get $I \sim -q^{(1)}(x)\, z^2/2$. For the second term we get $II \sim q^{(1)}(x)z$. It follows that

$$\log\Big(\frac{y(x + zq(x))}{y(x)}\exp(-z)\Big) \sim q^{(1)}(x)\Big(z - \frac{z^2}{2}\Big).$$

Since $q^{(1)}(x) \to 0$, we obtain that

$$\frac{y(x + zq(x))}{y(x)}\exp(-z) - 1 \sim q^{(1)}(x)\Big(z - \frac{z^2}{2}\Big).$$

This proves the result. □

4.2. Motion of tagged particles. In his paper about the motion of tagged particles, Szatzschneider (1993) formulated the question to characterize positive and measurable functions L that satisfy the following relation:

$$\frac{L(x+y\sqrt{x})-L(x)}{\sqrt{L(x)}} \to 0.$$

We formulate the problem in a more general way and study functions f that satisfy the following: $f \in \Gamma_0(g)$, $g \in SN$ and

$$\frac{f(x+yg(x))-f(x)}{\sqrt{f(x)}} \to \alpha y.$$

To solve this problem, we define $h(x) = \sqrt{f(x)}$. Clearly we have

$$h(x+yg(x))-h(x) = \frac{f(x+yg(x))-f(x)}{\sqrt{f(x+yg(x))}+\sqrt{f(x)}}.$$

Since $f \in \Gamma_0(g)$, we find that $h(x+yg(x))-h(x) \to \frac{\alpha}{2}y$. This is $h \in E\Gamma_{\alpha/2}(g,1)$. The Representation Theorem 2.2 for $E\Gamma$ now shows that h is of the form

$$h(x) = C + \frac{\alpha}{2}\int_{x^\circ}^{x} \frac{1}{g(z)}dz + W(x) + o(1)$$

where $W^{(1)}(x)g(x) \to 0$. For $g(x) = \sqrt{x}$, we get that

$$h(x) = C + \alpha\sqrt{x} + W(x) + o(1),$$

where $W^{(1)}(x)\sqrt{x} \to 0$.

5. Concluding remarks

(1) In Theorem 3.1 we obtained a second order and a third order behaviour of functions $f \in \Gamma(g)$ assuming that $g^{(2)} \in RV(\alpha-2)$, $\alpha < 1$. We plan to study conditions to obtain higher order terms in another paper.

(2) In Theorem 2.4 we proved that $f \in E\Gamma_\alpha(g,a)$ implies that $f(x) = A(G(x))$, where $G \in \Gamma(g)$ and $A \in \Pi_\alpha(L)$. We showed that we can take G nondecreasing. Suppose that $\alpha \neq 0$. If f is increasing, we also find that A is increasing. It follows that the inverse function $f^{-1}(x)$ is given by $f^{-1}(x) = G^{-1}(A^{-1}(x))$. Since $G^{-1} \in \Pi_1(g(G^{-1}))$ and $A^{-1} \in \Gamma_\alpha(L(A^{-1}))$, it follows that $f^{-1} \in E\Gamma_\beta(a(f^{-1}), g(f^{-1}))$, where $\beta = 1/\alpha$.

(3) In Omey and Willekens (1987) and in Omey and Segers (2010) we studied regular variation of order n and the class Π of order n. Starting from (cf. Theorem 2.4) $f(x) = A(G(x))$ where $G \in \Gamma(g)$ and $A \in RV(\alpha)$ or $A \in \Pi(L)$, this provides an alternative way to obtain higher order asymptotics for functions in the classes $\Gamma(g)$ and $E\Gamma_\alpha(g,a)$.

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