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MICRO-LOCAL ANALYSIS IN SOME SPACES OF ULTRADISTRIBUTIONS

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Abstract. We extend some results from [**14**] and [**19**], concerning wave-front sets of Fourier–Lebesgue and modulation space types, to a broader class of spaces of ultradistributions. We relate these wave-front sets one to another and to the usual wave-front sets of ultradistributions.

Furthermore, we give a description of discrete wave-front sets by introducing the notion of discretely regular points, and prove that these wave-front sets coincide with corresponding wave-front sets in [**19**]. Some of these investigations are based on the properties of the Gabor frames.

1. Introduction

Wave-front sets with respect to Fourier–Lebesgue and modulation spaces were introduced in [**19**] and studied further in [**18**, **20**, **21**]. Among other properties, it was proved that wave-front sets of Fourier–Lebesgue and modulation spaces coincide, and that the usual wave-front sets with respect to smoothness (cf. [**13**, Sections 8.1–8.3]) can be obtained as wave-front sets of sequences of Fourier–Lebesgue or modulation spaces. Discrete versions of wave-front sets of Fourier–Lebesgue and modulation spaces, related to those wave-front sets in [**24**], were introduced and studied in [**14**]. In particular, it was proved that these wave-front sets agree with corresponding wave-front sets in [**19**].

In this paper we put questions from [**14**,**19**] in a broader context, where we allow the involved distributions to be Beurling or Roumieu type ultradistributions. This is done by relaxing a polynomial type conditions on the involved weight functions, into a subexponential type condition. An important benefit is that the families of Fourier–Lebesgue and modulation spaces in [**14**, **19**] are significantly enlarged, since growth/decay properties of these weights are crucial concerning growth and

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regularity limitations on the involved distributions. For example, in this extended situation, the modulation spaces might contain positive functions growing subexponentially. We refer to [**11**, **27**] for a review of these facts.

In this section we set the stage with a brief overview of basic notions. Then, in Section 2 we introduce wave-front sets of a Beurling type ultradistribution with respect to weighted Fourier–Lebesgue space and show that they satisfy appropriate micro-local properties, Theorem 2.1. Then, in subsection 2.1 we show that the most common wave-front sets of ultradistributions given in [**12**, **17**, **22**] can be described within our approach, Proposition 2.1.

An important part of our investigations is to establish identification properties between wave-front sets of Fourier–Lebesgue and modulation space types. This is the subject of Theorem 3.1 in Section 3. Although we follow the framework of [**19**], we note that several new problems appear when dealing with ultradistributions. For example, several properties of the wave-front sets depend on properties of the short-time Fourier transform in the framework of ultradistributions, cf. Proposition 3.2.

Finally, in Section 4 we introduce discrete versions of wave-front sets of ultradistributions and prove their invariance properties (cf. Theorem 4.2). The main ideas of our approach can be traced back to [**14**], see also [**24**, **25**]. In order to be self-contained, we could not avoid certain repetitions of [**14**]. However, here we provide additional explanations of the construction and introduce the notion of discretely regular points. We believe that the results in form of series, established when introducing discrete wave-front sets, might be useful for numerical analysis of micro-local properties of functions and ultradistributions. For example, we use Gabor frames for the description of discrete wave-front sets and note that the Gabor frame coefficients give information on micro-local properties of the signal in such way. (See [**7**, **8**] for numerical treatment of Gabor frame theory.)

Since compactly supported smooth functions are used in the process of microlocalization we are limited to the use of weights with almost exponential growth at infinity described by the Beurling–Domar condition. We refer to subsection 1.1 for the notions and to [**10**] for a discussion on the role of weights in time-frequency analysis.

Our investigation can therefore be considered as the starting point in the study of analytic wave-front sets and pseudodifferential operators with ultrapolynomial symbols, also known as symbol-global type operators. This will be done in a separate paper, [**15**].

1.1. Basic notions and notation. In this subsection we collect some notation and notions which will be used in the sequel.

We put $\mathbf{N} = \{0,1,2,\dots\}$, $\langle x\rangle = (1+|x|^2)^{1/2}$, for $x \in \mathbf{R}^d$, and $A \lesssim B$ to indicate $A \leqslant cB$ for a suitable constant $c > 0$. The symbol $B_1 \hookrightarrow B_2$ denotes the continuous and dense embedding of the topological vector space B_1 into B_2. The scalar product in L^2 is denoted by $(\cdot,\cdot)_{L^2} = (\cdot,\cdot)$. Translation and modulation operators are given by $T_x f(t) = f(t-x)$ and $M_\xi f(t) = e^{i\langle \xi,t\rangle} f(t)$.

1.1.1. *Weights.* In general, a weight function is a nonnegative function in L^∞_{loc}.

Definition 1.1. Let ω, v be nonnegative functions. Then

(1) v is called *submultiplicative* if $v(x+y) \leqslant v(x)v(y)$, for each $x, y \in \mathbf{R}^d$;
(2) ω is called *v-moderate* if $\omega(x+y) \lesssim v(x)\omega(y)$, for each $x, y \in \mathbf{R}^d$.

For a given submultiplicative weight v the set of all v-moderate weights will be denoted by $\mathscr{M}_v$.

If v is even and $\omega \in \mathscr{M}_v$, then $1/v \lesssim \omega \lesssim v$, $\omega \neq 0$ everywhere and $1/\omega \in \mathscr{M}_v$.

In the sequel v will always stand for an even submultiplicative function. Submultiplicativity implies that v is dominated by an exponential function, i.e.,

$$v \leqslant Ce^{k|\cdot|} \quad \text{for some} \quad C, k > 0.$$

Let $s > 1$. By $\mathscr{M}_{\{s\}}(\mathbf{R}^d)$ we denote the set of all weights which are moderate with respect to a weight v which satisfies $v \leqslant Ce^{k|\cdot|^{1/s}}$ for some positive constants C and k. The weight v satisfy the Beurling–Domar non-quasi-analyticity condition which takes the form $\sum_{n=0}^{\infty} \frac{1}{n^2} \log v(nx) < \infty$, $x \in \mathbf{R}^d$. We refer to [**10**] for more facts about such weights.

1.1.2. *Test function spaces and their duals.* Next we introduce spaces of test functions and their duals in the context of spaces of ultradistributions. These test function spaces correspond to the spaces C_0^∞, $\mathscr{S}$ and C^∞ in the distribution theory in [**12, 26**]. We start by giving the definition of Gelfand-Shilov type spaces.

Definition 1.2. Let $s > 1$ and $A > 0$. We denote by $\mathcal{S}_A^s(\mathbf{R}^d)$ the space of all functions $\varphi \in C^\infty(\mathbf{R}^d)$ such that the norm

$$\|\varphi\|_{s,A} = \sup_{\alpha,\beta \in \mathbf{N}_0^d} \sup_{x \in \mathbf{R}^d} \left(\frac{A^{|\alpha+\beta|}}{\alpha!^s \beta!^s} \langle x \rangle^{|\alpha|} |\varphi^{(\beta)}(x)| \right)$$

is finite. Then the spaces $\mathcal{S}^{(s)}(\mathbf{R}^d)$ and $\mathcal{S}^{\{s\}}(\mathbf{R}^d)$ are given by

$$\mathcal{S}^{(s)}(\mathbf{R}^d) = \bigcap_{A>0} \mathcal{S}_A^s(\mathbf{R}^d) \quad \mathcal{S}^{\{s\}}(\mathbf{R}^d) = \bigcup_{A>0} \mathcal{S}_A^s(\mathbf{R}^d).$$

The topologies for $\mathcal{S}^{(s)}(\mathbf{R}^d)$ and $\mathcal{S}^{\{s\}}(\mathbf{R}^d)$ are given by the projective and inductive limit, respectively, i.e.,

$$\mathcal{S}^{(s)}(\mathbf{R}^d) = \operatorname{proj} \lim_{A \to \infty} \mathcal{S}_A^s(\mathbf{R}^d), \quad \mathcal{S}^{\{s\}}(\mathbf{R}^d) = \operatorname{ind} \lim_{A \to 0} \mathcal{S}_A^s(\mathbf{R}^d).$$

We note that $\mathcal{S}_A(\mathbf{R}^d)$ is a Banach space, for every $A > 0$, and its dual is denoted by $(\mathcal{S}_A)'(\mathbf{R}^d)$. Then the Gelfand-Shilov type distribution spaces $(\mathcal{S}^{(s)})'(\mathbf{R}^d)$ and $(\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ are defined as

$$(\mathcal{S}^{(s)})'(\mathbf{R}^d) = \bigcup_{A>0} (\mathcal{S}_A^s)'(\mathbf{R}^d), \quad (\mathcal{S}^{\{s\}})'(\mathbf{R}^d) = \bigcap_{A>0} (\mathcal{S}_A^s)'(\mathbf{R}^d).$$

These spaces are the strong dual spaces of $\mathcal{S}^{(s)}(\mathbf{R}^d)$ and $\mathcal{S}^{\{s\}}(\mathbf{R}^d)$, and are called the spaces of tempered ultradistributions of Beurling type and Roumieu type, respectively. If $s > t$, then

$$\begin{aligned} \mathcal{S}^{(t)}(\mathbf{R}^d) &\hookrightarrow \mathcal{S}^{\{t\}}(\mathbf{R}^d) \hookrightarrow \mathcal{S}^{(s)}(\mathbf{R}^d) \hookrightarrow \mathcal{S}^{\{s\}}(\mathbf{R}^d) \\ &\hookrightarrow (\mathcal{S}^{\{s\}})'(\mathbf{R}^d) \hookrightarrow (\mathcal{S}^{(s)})'(\mathbf{R}^d) \hookrightarrow (\mathcal{S}^{\{t\}})'(\mathbf{R}^d) \hookrightarrow (\mathcal{S}^{(t)})'(\mathbf{R}^d). \end{aligned}$$

In order to perform (micro-)local analysis we use the following test function spaces on open sets, cf. [**16**].

Definition 1.3. Let X be an open set in $\mathbf{R}^d$. For a given compact set $K \subset X$, $s > 1$ and $A > 0$ we denote by $\mathcal{E}^s_{A,K}(X)$ the space of all $\varphi \in C^\infty(X)$ such that the norm

$$\|\varphi\|_{s,A,K} = \sup_{\beta \in \mathbf{N}_0^n} \sup_{x \in K} \frac{A^{|\beta|}}{\beta!^s} |\varphi^{(\beta)}(x)| \tag{1.1}$$

is finite.

The space of functions $\varphi \in C^\infty(X)$ such that (1.1) holds and $\operatorname{supp} \varphi \subseteq K$ is denoted by $\mathcal{D}^s_A(K)$.

Let $(K_n)_n$ be a sequence of compact sets such that $K_n \subset\subset K_{n+1}$ and $\bigcup K_n = X$. Then

$$\begin{aligned}
\mathcal{E}^{(s)}(X) &= \text{proj} \lim_{n\to\infty} (\text{proj} \lim_{A\to\infty} \mathcal{E}^s_{A,K_n})(X),\\
\mathcal{E}^{\{s\}}(X) &= \text{proj} \lim_{n\to\infty} (\text{ind} \lim_{A\to 0} \mathcal{E}^s_{A,K_n})(X),\\
\mathcal{D}^{(s)}(X) &= \text{ind} \lim_{n\to\infty} (\text{proj} \lim_{A\to\infty} \mathcal{D}^s_A(K_n)),\\
\mathcal{D}^{\{s\}}(X) &= \text{ind} \lim_{n\to\infty} (\text{ind} \lim_{A\to 0} \mathcal{D}^s_A(K_n)).
\end{aligned}$$

Obviously, $\mathcal{D}^{(s)}(X)$ ($\mathcal{D}^{\{s\}}(X)$ resp.) are subspaces of $\mathcal{E}^{(s)}(X)$ (of $\mathcal{E}^{\{s\}}(X)$ resp.) whose elements are compactly supported. We also remark that a usual notation for the space $\mathcal{D}^{\{s\}}(X)$ is $G^s_0(X)$ (cf. [**22**]).

Remark 1.1. Let $*$ denote (s) or $\{s\}$. Then $\mathcal{D}^*$, $\mathcal{S}^*$ and $\mathcal{E}^*$ correspond to C_0^∞, $\mathscr{S}$ and C^∞, respectively, and $\mathcal{D}^* \subseteq C_0^\infty$, $\mathcal{S}^* \subseteq \mathscr{S}$ and $\mathcal{E}^* \subseteq C^\infty$.

The spaces of linear functionals over $\mathcal{D}^{(s)}(X)$ and $\mathcal{D}^{\{s\}}(X)$, denoted by $(\mathcal{D}^{(s)})'(X)$ and $(\mathcal{D}^{\{s\}})'(X)$ respectively, are called the spaces of *ultradistributions* of Beurling and Roumieu type respectively, while the spaces of linear functionals over $\mathcal{E}^{(s)}(X)$ and $\mathcal{E}^{\{s\}}(X)$, denoted by $(\mathcal{E}^{(s)})'(X)$ and $(\mathcal{E}^{\{s\}})'(X)$, respectively are called the spaces of *ultradistributions of compact support* of Beurling and Roumieu type respectively, [**16**]. We have

$$(\mathcal{E}^{\{s\}})'(X) \subseteq (\mathcal{E}^{(s)})'(X),\ (\mathcal{E}^{(s)})'(X) \subseteq (\mathcal{E}^{(s)})'(\mathbf{R}^d) \ \text{ and } \ (\mathcal{E}^{\{s\}})'(X) \subseteq (\mathcal{E}^{\{s\}})'(\mathbf{R}^d).$$

Clearly,

$$\begin{aligned}
(\mathcal{E}^{\{s\}})'(\mathbf{R}^d) \subseteq (\mathcal{S}^{\{s\}})'(\mathbf{R}^d) \subseteq (\mathcal{D}^{\{s\}})'(\mathbf{R}^d)\\
(\mathcal{E}^{(s)})'(\mathbf{R}^d) \subseteq (\mathcal{S}^{(s)})'(\mathbf{R}^d) \subseteq (\mathcal{D}^{(s)})'(\mathbf{R}^d).
\end{aligned}$$

Any ultra-distribution with compact support can be viewed as an element of $(\mathcal{S}^{(1)})'(\mathbf{R}^d)$. More generally, by similar arguments as in the distribution theory in [**12**], it follows that $\mathcal{E}^*$ are exactly those elements in $\mathcal{S}^*$ or $\mathcal{D}^*$ with compact support.

1.1.3. *Fourier–Lebesgue spaces.* The Fourier transform $\mathscr{F}$ is a linear and continuous mapping on $\mathscr{S}'(\mathbf{R}^d)$ which takes the form

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x) e^{-i\langle x,\xi\rangle}\, dx$$

when $f \in L^1(\mathbf{R}^d)$. It is a homeomorphism on $(\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ (on $(\mathcal{S}^{(s)})'(\mathbf{R}^d)$ resp.) which restricts to a homeomorphism on $\mathcal{S}^{\{s\}}(\mathbf{R}^d)$ (on $\mathcal{S}^{(s)}(\mathbf{R}^d)$ resp.) and to a unitary operator on $L^2(\mathbf{R}^d)$.

Let $q \in [1,\infty]$, $s > 1$ and $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^d)$. The (weighted) Fourier–Lebesgue space $\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$ is the inverse Fourier image of $L^q_{(\omega)}(\mathbf{R}^d)$, i.e., $\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$ consists of all $f \in (\mathcal{S}^{(s)})'(\mathbf{R}^d)$ such that $\|f\|_{\mathscr{F}L^q_{(\omega)}} \equiv \|\widehat{f}\cdot\omega\|_{L^q}$ is finite. If $\omega = 1$, then the notation $\mathscr{F}L^q$ is used instead of $\mathscr{F}L^q_{(\omega)}$. We note that if $\omega(\xi) = \langle\xi\rangle^s$, then $\mathscr{F}L^q_{(\omega)}$ is the Fourier image of the Bessel potential space H^p_s.

REMARK 1.2. Whenever it is convenient, we permit an x dependence for the weight ω in the definition of Fourier–Lebesgue spaces. More precisely, for each $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$ we let $\mathscr{F}L^q_{(\omega)}$ be the set of all ultradistributions f such that

$$\|f\|_{\mathscr{F}L^q_{(\omega)}} \equiv \|\widehat{f}\,\omega(x,\cdot)\|_{L^q}$$

is finite. Since ω is v-moderate, it follows that different choices of x give rise to equivalent norms. Therefore the condition $\|f\|_{\mathscr{F}L^q_{(\omega)}} < \infty$ is independent of x, and it follows that $\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$ is independent of x although $\|\cdot\|_{\mathscr{F}L^q_{(\omega)}}$ might depend on x.

2. Wave-front sets of Fourier–Lebesgue type in spaces of Beurling type ultradistributions

In this section we introduce wave-front sets of Fourier–Lebesgue type in spaces of ultradistributions of Beurling type.

Let $s > 1$, $q \in [1,\infty]$, and $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone. If $f \in (\mathcal{S}^{(s)})'(\mathbf{R}^d)$ and $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$ we define

$$|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega)}} = |f|_{\mathscr{F}L^{q,\Gamma}_{(\omega),x}} \equiv \left(\int_\Gamma |\widehat{f}(\xi)\omega(x,\xi)|^q d\xi \right)^{1/q} \tag{2.1}$$

(with obvious interpretation when $q = \infty$). We note that $|\cdot|_{\mathscr{F}L^{q,\Gamma}_{(\omega),x}}$ defines a seminorm on $(\mathcal{S}^{(s)})'(\mathbf{R}^d)$ which might attain the value $+\infty$. Since ω is v-moderate it follows that different $x \in \mathbf{R}^d$ gives rise to equivalent seminorms $|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega),x}}$. Furthermore, if $\Gamma = \mathbf{R}^d \setminus 0$, $f \in \mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$ and $q < \infty$, then $|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega),x}}$ agrees with the Fourier–Lebesgue norm $\|f\|_{\mathscr{F}L^q_{(\omega),x}}$ of f.

For the sake of notational convenience we set

$$\mathcal{B} = \mathscr{F}L^q_{(\omega)} = \mathscr{F}L^q_{(\omega)}(\mathbf{R}^d), \quad \text{and} \quad |\cdot|_{\mathcal{B}(\Gamma)} = |\cdot|_{\mathscr{F}L^{q,\Gamma}_{(\omega),x}}. \tag{2.2}$$

We let $\Theta_{\mathcal B}(f)=\Theta_{\mathscr F L^q_{(\omega)}}(f)$ be the set of all $\xi \in \mathbf R^d\setminus 0$ such that $|f|_{\mathcal B(\Gamma)}<\infty$, for an open conical neighbourhood $\Gamma=\Gamma_\xi$ of ξ. We also let $\Sigma_{\mathcal B}(f)$ be the complement of $\Theta_{\mathcal B}(f)$ in $\mathbf R^d\setminus 0$. Then $\Theta_{\mathcal B}(f)$ and $\Sigma_{\mathcal B}(f)$ are open, respectively closed, subsets in $\mathbf R^d\setminus 0$, which are independent of the choice of $x\in \mathbf R^d$ in (2.1).

Definition 2.1. Let $s>1$, $q\in[1,\infty]$, $\mathcal B$ be as in (2.2), and let X be an open subset of $\mathbf R^d$. If $\omega\in\mathscr M_{\{s\}}(\mathbf R^{2d})$, then the wave-front set of $f\in(\mathcal D^{(s)})'(X)$, $\mathrm{WF}_{\mathcal B}(f)\equiv \mathrm{WF}_{\mathscr F L^q_{(\omega)}}(f)$ with respect to $\mathcal B$ consists of all pairs (x_0,ξ_0) in $X\times(\mathbf R^d\setminus 0)$ such that $\xi_0\in\Sigma_{\mathcal B}(\varphi f)$ holds for each $\varphi\in\mathcal D^{(s)}(X)$ such that $\varphi(x_0)\neq 0$.

We note that $\mathrm{WF}_{\mathcal B}(f)$ is a closed set in $\mathbf R^d\times(\mathbf R^d\setminus 0)$, since it is obvious that its complement is open. We also note that if $x\in\mathbf R^d$ is fixed and $\omega_0(\xi)=\omega(x,\xi)$, then $\mathrm{WF}_{\mathcal B}(f)=\mathrm{WF}_{\mathscr F L^q_{(\omega_0)}}(f)$, since $\Sigma_{\mathcal B}$ is independent of x.

The following theorem shows that wave-front sets with respect to $\mathscr F L^q_{(\omega)}$ satisfy appropriate micro-local properties. It also shows that such wave-front sets are decreasing with respect to the parameter q, and increasing with respect to the weight ω.

Theorem 2.1. *Let $s>1$, $q,r\in[1,\infty]$, X be an open set in $\mathbf R^d$ and $\omega,\vartheta\in\mathscr M_{\{s\}}(\mathbf R^{2d})$ be such that $r\leqslant q$, and $\omega(x,\xi)\lesssim\vartheta(x,\xi)$. Also let $\mathcal B$ be as in* (2.2) *and put $\mathcal B_0=\mathscr F L^r_{(\vartheta)}(\mathbf R^d)$. If $f\in(\mathcal D^{(s)})'(X)$ and $\varphi\in\mathcal D^{(s)}(X)$, then*

$$\mathrm{WF}_{\mathcal B}(\varphi\, f)\subseteq \mathrm{WF}_{\mathcal B_0}(f).$$

Proof. By the definitions it is sufficient to prove $\Sigma_{\mathcal B}(\varphi f)\subseteq\Sigma_{\mathcal B_0}(f)$ when $\varphi\in\mathcal D^{(s)}(X)$, $\vartheta=\omega$, and $f\in(\mathcal E^{(s)})'(\mathbf R^d)$, since the statement only involves local assertions. For the same reasons we may assume that $\omega(x,\xi)=\omega(\xi)$ is independent of x. Finally, we prove the assertion for $r\in[1,\infty)$. The case $r=\infty$ follows by similar arguments and is left to the reader.

Choose open cones Γ_1 and Γ_2 in $\mathbf R^d$ such that $\overline{\Gamma_2}\subseteq\Gamma_1$. We will use the fact that if $f\in(\mathcal E^{(s)})'(\mathbf R^d)$, then $|\widehat f(\xi)\omega(\xi)|\lesssim e^{N_0|\xi|^{1/s}}$ for some $N_0>0$ and prove that for every $N>0$, there exist $C_N>0$ such that

$$|\varphi f|_{\mathcal B(\Gamma_2)}\leqslant C_N\Big(|f|_{\mathcal B_0(\Gamma_1)}+\sup_{\xi\in\mathbf R^d}\big(|\widehat f(\xi)\omega(\xi)|e^{-N|\xi|^{1/s}}\big)\Big)\quad\text{when}\quad \overline{\Gamma}_2\subseteq\Gamma_1. \tag{2.3}$$

Since $\omega\in\mathscr M_{\{s\}}(\mathbf R^{2d})$ by letting $F(\xi)=|\widehat f(\xi)\omega(\xi)|$ and $\psi(\xi)=|\widehat\varphi(\xi)v(\xi)|$ we get

$$\begin{aligned}|\varphi f|_{\mathcal B(\Gamma_2)}&=\Big(\int_{\Gamma_2}|\mathscr F(\varphi f)(\xi)\omega(\xi)|^q\,d\xi\Big)^{1/q}\\&\lesssim\Big(\int_{\Gamma_2}\Big(\int_{\mathbf R^d}\psi(\xi-\eta)F(\eta)\,d\eta\Big)^q d\xi\Big)^{1/q}\lesssim J_1+J_2,\end{aligned}$$

where

$$J_1 = \left(\int_{\Gamma_2} \left(\int_{\Gamma_1} \psi(\xi-\eta)F(\eta)\,d\eta \right)^q d\xi \right)^{1/q},$$
$$J_2 = \left(\int_{\Gamma_2} \left(\int_{\complement\Gamma_1} \psi(\xi-\eta)F(\eta)\,d\eta \right)^q d\xi \right)^{1/q}.$$

Let q_0 be chosen such that $1/r_0+1/r=1+1/q$, and let χ_{Γ_1} be the characteristic function of Γ_1. Then Young's inequality gives

$$\begin{aligned} J_1 &\leqslant \left(\int_{\mathbf{R}^d} \left(\int_{\Gamma_1} \psi(\xi-\eta)F(\eta)\,d\eta \right)^q d\xi \right)^{1/q} \\ &= \|\psi * (\chi_{\Gamma_1}F)\|_{L^q} \leqslant \|\psi\|_{L^{r_0}} \|\chi_{\Gamma_1}F\|_{L^r} = C_\psi |f|_{\mathcal{B}_0(\Gamma_1)}, \end{aligned}$$

where $C_\psi = \|\psi\|_{L^{q_0}} < \infty$. If $\varphi \in \mathcal{D}^{(s)}(X)$, then for every $N>0$ there exist $C_N>0$ such that

$$\psi(\xi) = |\widehat{\varphi}(\xi)v(\xi)| \leqslant C_N e^{-(N+k)|\xi|^{1/s}} e^{k|\xi|^{1/s}} \leqslant C_N e^{-N|\xi|^{1/s}}. \tag{2.4}$$

In order to estimate J_2, we note that $\overline{\Gamma_2} \subseteq \Gamma_1$ implies that

$$|\xi-\eta|^{1/s} > 2c\max(|\xi|^{1/s}, |\eta|^{1/s}) \geqslant c(|\xi|^{1/s}+|\eta|^{1/s}), \qquad \xi \in \Gamma_2, eta \notin \Gamma_1 \tag{2.5}$$

holds for some constant $c>0$, since this is true when $1=|\xi| \geqslant |\eta|$. A combination of (2.4) and (2.5) implies that for every $N_1>0$ we have

$$\psi(\xi-\eta) \lesssim Ce^{-2N_1(|\xi|^{1/s}+|\eta|^{1/s})}.$$

This gives

$$\begin{aligned} J_2 &\lesssim \left(\int_{\Gamma_2} \left(\int_{\complement\Gamma_1} e^{-2N_1(|\xi|^{1/s}+|\eta|^{1/s})} F(\eta)\,d\eta \right)^r d\xi \right)^{1/r} \\ &\lesssim \left(\int_{\Gamma_2} \left(\int_{\complement\Gamma_1} e^{-2N_1(|\xi|^{1/s}+|\eta|^{1/s})} e^{N_1|\eta|^{1/s}} (e^{-N_1|\eta|^{1/s}}F(\eta))\,d\eta \right)^r d\xi \right)^{1/r} \\ &\lesssim \sup_{\eta\in\mathbf{R}^d} |e^{-N_1|\eta|^{1/s}}F(\eta))|, \end{aligned}$$

which proves (2.3) and the result follows. □

Next we modify the definitions from [**19**] concerning wave-front sets with respect to sequences of spaces. Let $\omega_j \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$ and $q_j \in [1,\infty]$ when j belongs to some index set J, and let $\mathcal{B}$ be the array of spaces, given by

$$(\mathcal{B}_j) \equiv (\mathcal{B}_j)_{j\in J}, \quad \text{where} \quad \mathcal{B}_j = \mathscr{F}L^{q_j}_{(\omega_j)} = \mathscr{F}L^{q_j}_{(\omega_j)}(\mathbf{R}^d), \quad j \in J. \tag{2.6}$$

If $s>1$, $f \in (\mathcal{D}^{(s)})'(\mathbf{R}^d)$, and $(\mathcal{B}_j)$ is given by (2.6), then we let $\Theta^{\sup}_{(\mathcal{B}_j)}(f)$ be the set of all $\xi \in \mathbf{R}^d \setminus 0$ such that for some $\Gamma = \Gamma_\xi$ and each $j \in J$ it holds $|f|_{\mathcal{B}_j(\Gamma)} < \infty$. We also let $\Theta^{\inf}_{(\mathcal{B}_j)}(f)$ be the set of all $\xi \in \mathbf{R}^d \setminus 0$ such that for some $\Gamma = \Gamma_\xi$ and some $j \in J$ it holds $|f|_{\mathcal{B}_j(\Gamma)} < \infty$. Finally we let $\Sigma^{\sup}_{(\mathcal{B}_j)}(f)$ and $\Sigma^{\inf}_{(\mathcal{B}_j)}(f)$ be the complements in $\mathbf{R}^d \setminus 0$ of $\Theta^{\sup}_{(\mathcal{B}_j)}(f)$ and $\Theta^{\inf}_{(\mathcal{B}_j)}(f)$ respectively.

Definition 2.2. Let J be an index set, $q_j \in [1,\infty]$, $\omega_j \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$ when $j \in J$, $(\mathcal{B}_j)$ be as in (2.6), and let X be an open subset of $\mathbf{R}^d$.

(1) The wave-front set of $f \in (\mathcal{D}^{(s)})'(X)$, of *sup-type* with respect to $(\mathcal{B}_j)$, $\mathrm{WF}^{\sup}_{(\mathcal{B}_j)}(f)$, consists of all pairs (x_0,ξ_0) in $X \times (\mathbf{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma^{\sup}_{(\mathcal{B}_j)}(\varphi f)$ holds for each $\varphi \in \mathcal{D}^{(s)}(X)$ such that $\varphi(x_0) \neq 0$;

(2) The wave-front set of $f \in (\mathcal{D}^{(s)})'(X)$, of *inf-type* with respect to $(\mathcal{B}_j)$, $\mathrm{WF}^{\inf}_{(\mathcal{B}_j)}(f)$ consists of all pairs (x_0,ξ_0) in $X \times (\mathbf{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma^{\inf}_{(\mathcal{B}_j)}(\varphi f)$ holds for each $\varphi \in \mathcal{D}^{(s)}(X)$ such that $\varphi(x_0) \neq 0$.

Remark 2.1. We recall that if $f \in \mathscr{D}'(\mathbf{R}^d)$, and $\omega_j(x,\xi) = \langle \xi \rangle^j$ for $j \in J = \mathbf{N}$, then it follows that $\mathrm{WF}^{\sup}_{(\mathcal{B}_j)}(f)$ in Definition 2.2 is equal to the standard wave front set $\mathrm{WF}(f)$ in Chapter VIII in [**12**].

2.1. Comparisons to other types of wave-front sets. Let $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$ be moderated with respect to a weight of polynomial growth at infinity and let $f \in \mathcal{D}'(X)$. Then $\mathrm{WF}_{\mathscr{F}L^q_{(\omega)}}(f)$ in Definition 2.1 is the same as the wave-front set introduced in [**19**, Definition 3.1]. Therefore, the information on regularity in the background of wave-front sets of Fourier–Lebesgue type in Definition 2.1 might be compared to the information obtained from the classical wave-front sets, cf. Example 4.9 in [**19**].

Next we compare the wave-front sets introduced in Definition 2.1 to the wave-front sets in spaces of ultradistributions given in [**12**, **17**, **22**].

Let $s > 1$ and let X be an open subset of $\mathbf{R}^d$. The ultradistribution $f \in (\mathcal{D}^{(s)})'(X)$ ($f \in (\mathcal{D}^{\{s\}})'(X)$) is (s)-micro-regular ($\{s\}$-micro-regular) at (x_0,ξ_0) if there exists $\varphi \in \mathcal{D}^{(s)}(X)$ ($\varphi \in \mathcal{D}^{\{s\}}(X)$) such that $\varphi(x) = 1$ in a neighborhood of x_0 and an open cone Γ which contains ξ_0 such that

$$|\mathscr{F}(\varphi f)(\xi)| \lesssim e^{-N|\xi|^{1/s}}, \quad \xi \in \Gamma, \tag{2.7}$$

for each $N > 0$ (for some $N > 0$). The (s)-wave-front set ($\{s\}$-wave-front set) of f, $\mathrm{WF}_{(s)}(f)$ ($\mathrm{WF}_{\{s\}}(f)$) is defined as the complement in $X \times \mathbf{R}^d \setminus 0$ of the set of all (x_0,ξ_0) where f is (s)-micro-regular ($\{s\}$-micro-regular), cf. [**22**, Definition 1.7.1].

The $\{s\}$-wave-front set $\mathrm{WF}_{\{s\}}(f)$ can be found in [**17**] and agrees with certain wave-front set $\mathrm{WF}_L(f)$ introduced in [**12**, Chapter 8.4].

Remark 2.2. Let $s > 1$, $X \subseteq \mathbf{R}^d$ be open, $f \in (\mathcal{D}^{\{s\}})'(X)$, $\varphi \in \mathcal{E}^{\{s\}}(X)$ and $\varphi_0 \in \mathcal{D}^{(s)}(X)$ be such that $\varphi(x) = 1$ in a neighborhood $\operatorname{supp}\varphi_0$. Also let Γ_0, Γ be open cones such that $\overline{\Gamma_0} \subseteq \Gamma$. If (2.7) holds for some $N > 0$, then it follows by straightforward computations, using similar arguments as in the proof of Theorem 2.1 that (2.7) is still true for some $N > 0$ after φ has been replaced by φ_0. Hence it follows that the following conditions are equivalent:

(1) $(x_0,\xi_0) \notin \mathrm{WF}_{\{s\}}(f)$;

(2) for some $\varphi \in \mathcal{D}^{\{s\}}(X)$, such that $\varphi(x) = 1$ in a neighborhood of x_0, a conical neighborhood Γ of ξ and for some $N > 0$, it follows that (2.7) holds;

(3) for some $\varphi \in \mathcal{D}^{(s)}(X)$, such that $\varphi(x) = 1$ in a neighborhood of x_0, a conical neighborhood Γ of ξ and for some $N > 0$, it follows that (2.7) holds.

Consequently we may always choose φ in $\mathcal{D}^{(s)}(X)$ in the definition of $\mathrm{WF}_{\{s\}}(f)$, when $f \in (\mathcal{D}^{\{s\}})'(X)$.

Proposition 2.1. *Let $s > 1$, and let $\mathcal{B}_j$ be the same as in* (2.6) *with $q_j \in [1, \infty]$ and $\omega_j(\xi) \equiv e^{j|\xi|^{1/s}}$. Then the following is true:*

(1) *if $f \in (\mathcal{D}^{\{s\}})'(\mathbf{R}^d)$, then*
$$\mathrm{WF}^{\inf}_{(\mathcal{B}_j)}(f) = \bigcap_{j>0} \mathrm{WF}_{\mathcal{B}_j}(f) = \mathrm{WF}_{\{s\}}(f) \subseteq \mathrm{WF}_{(s)}(f);$$

(2) *if $f \in (\mathcal{D}^{(s)})'(\mathbf{R}^d)$, then*
$$\mathrm{WF}_{(s)}(f) = \bigcup_{j>0} \mathrm{WF}_{\mathcal{B}_j}(f) \subseteq \mathrm{WF}^{\sup}_{(\mathcal{B}_j)}(f).$$

Proof. Let $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be a cone, $\varepsilon > 0$ and let $r, q \in [1, \infty]$ be such that $r \leqslant q$. Then Hölder's inequality implies that $|f|_{\mathscr{F}L^r_{(\omega_j)}(\Gamma)} \leqslant C|f|_{\mathscr{F}L^q_{(\omega_{j+\varepsilon})}(\Gamma)}$, for a constant $C > 0$ which only depends on $\varepsilon > 0$ and d. A combination of this fact and (2.3) then shows that if $C_j = \mathscr{F}L^\infty_{(\omega_j)}(\mathbf{R}^d)$, then
$$\bigcap_{j>0} \mathrm{WF}_{\mathcal{B}_j}(f) = \bigcap_{j>0} \mathrm{WF}_{\mathcal{C}_j}(f), \qquad \bigcup_{j>0} \mathrm{WF}_{\mathcal{B}_j}(f) = \bigcup_{j>0} \mathrm{WF}_{\mathcal{C}_j}(f)$$
$$\mathrm{WF}^{\inf}_{(\mathcal{B}_j)}(f) = \mathrm{WF}^{\inf}_{(\mathcal{C}_j)}(f), \qquad \mathrm{WF}^{\sup}_{(\mathcal{B}_j)}(f) = \mathrm{WF}^{\sup}_{(\mathcal{C}_j)}(f).$$
Hence we may assume that $q_j = \infty$, for every j. The result is now a straightforward consequence of the definitions. The proof is complete. □

3. Wave-front sets with respect to modulation spaces

In this section we define wave-front sets with respect to modulation spaces, and show that they coincide with wave-front sets of Fourier–Lebesgue types.

3.1. Modulation spaces. In this subsection we consider properties of modulation spaces which will be used in microlocal analysis of ultradistributions.

Let $s > 1$ and let $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d)$ be fixed. Then the short-time Fourier transform (STFT) of $f \in \mathcal{S}^{(s)}(\mathbf{R}^d)$ with respect to the window ϕ is given by
$$V_\phi f(x, \xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(y)\, \overline{\phi(y - x)}\, e^{-i\langle \xi, y\rangle} dy.$$

The map $(f, \phi) \mapsto V_\phi f$ from $\mathcal{S}^{(s)}(\mathbf{R}^d) \times \mathcal{S}^{(s)}(\mathbf{R}^d)$ to $\mathcal{S}^{(s)}(\mathbf{R}^{2d})$ extends uniquely to a continuous mapping from $(\mathcal{S}^{(s)})'(\mathbf{R}^d) \times (\mathcal{S}^{(s)})'(\mathbf{R}^d)$ to $(\mathcal{S}^{(s)})'(\mathbf{R}^{2d})$ by duality.

Moreover, if $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$ fixed and $f \in (\mathcal{S}^{(s)})'(\mathbf{R}^d)$, then
$$f \in \mathcal{S}^{(s)}(\mathbf{R}^d) \iff V_\phi f \in \mathcal{S}^{(s)}(\mathbf{R}^{2d}). \tag{3.1}$$

We refer to [**11**, **27**] for the proofs, as well as more details on STFT in the context of Gelfand–Shilov spaces.

Now we recall the definition of modulation spaces. Let $s>1$, $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, $p,q\in [1,\infty]$, and the window $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d)\setminus 0$ be fixed. Then the *modulation space* $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is the set of all ultra-distributions $f\in (\mathcal{S}^{(s)})'(\mathbf{R}^d)$ such that

$$\|f\|_{M^{p,q}_{(\omega)}} = \|f\|_{M^{p,q,\phi}_{(\omega)}} \equiv \|V_\phi f\,\omega\|_{L^{p,q}_1} < \infty .$$

Here $\|\cdot\|_{L^{p,q}_1}$ is the norm given by

$$\|F\|_{L^{p,q}_1} \equiv \left(\int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |F(x,\xi)|^p\,dx \right)^{q/p} d\xi \right)^{1/q},$$

when $F\in L^1_{\mathrm{loc}}(\mathbf{R}^{2d})$ (with obvious interpretation when $p=\infty$ or $q=\infty$). Furthermore, the modulation space $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ consists of all $f\in (\mathcal{S}^{(s)})'(\mathbf{R}^d)$ such that

$$\|f\|_{W^{p,q}_{(\omega)}} = \|f\|_{W^{p,q,\phi}_{(\omega)}} \equiv \|V_\phi f\,\omega\|_{L^{p,q}_2} < \infty ,$$

where $\|\cdot\|_{L^{p,q}_2}$ is the norm given by

$$\|F\|_{L^{p,q}_2} \equiv \left(\int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |F(x,\xi)|^q\,d\xi \right)^{p/q} dx \right)^{1/p},$$

when $F\in L^1_{\mathrm{loc}}(\mathbf{R}^{2d})$.

If $\omega=1$, then the notations $M^{p,q}$ and $W^{p,q}$ are used instead of $M^{p,q}_{(\omega)}$ and $W^{p,q}_{(\omega)}$ respectively. Moreover we set $M^p_{(\omega)} = W^p_{(\omega)} = M^{p,p}_{(\omega)}$ and $M^p = W^p = M^{p,p}$. We note that $M^{p,q}$ are modulation spaces of classical forms, while $W^{p,q}$ are classical forms of Wiener amalgam spaces. (See [**3**] concerning the terminology.)

If $s>1$, $p,q\in[1,\infty]$ and $\omega\in\mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, then one can show that the spaces $\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$, $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ are locally the same, in the sense that

$$\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)\cap(\mathcal{E}^{(s)})'(\mathbf{R}^d) = M^{p,q}_{(\omega)}(\mathbf{R}^d)\cap(\mathcal{E}^{(s)})'(\mathbf{R}^d) = W^{p,q}_{(\omega)}(\mathbf{R}^d)\cap(\mathcal{E}^{(s)})'(\mathbf{R}^d).$$

This follows by similar arguments as in [**23**] (and replacing the space of polynomially moderated weights $\mathscr{P}(\mathbf{R}^{2d})$ with $\mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$). Later on we extend these properties in the context of wave-front sets and recover the equalities above.

The proof of the next proposition concerning topological questions of modulation spaces, and properties of the adjoint of the short-time Fourier transform V_ϕ^*F, can be found in [**1**]. Here we recall that $\langle V_\phi^*F,f\rangle \equiv \langle F, V_\phi f\rangle$, $f\in\mathcal{S}^{(s)}(\mathbf{R}^d)$, when $s>1$, $\omega\in\mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, $\phi\in\mathcal{S}^{(s)}\setminus 0$ and $F(x,\xi)\in L^{p,q}_{(\omega)}(\mathbf{R}^{2d})$.

Proposition 3.1. *Let $s>1$, $\omega\in\mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, $p,q\in[1,\infty]$, and $\phi,\phi_1\in\mathcal{S}^{(s)}(\mathbf{R}^d)$, with $(\phi,\phi_1)_{L^2}\neq 0$. Then the following is true:*

(1) *the operator V_ϕ^* from $\mathcal{S}^{(s)}(\mathbf{R}^{2d})$ to $\mathcal{S}^{(s)}(\mathbf{R}^d)$ extends uniquely to a continuous operator from $L^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ to $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, and*

$$\|V_\phi^*F\|_{M^{p,q}_{(\omega)}} \leqslant C\|V_{\phi_1}\phi\|_{L^1_{(v)}}\|F\|_{L^{p,q}_{(\omega)}};$$

(2) *$M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is a Banach space whose definition is independent on the choice of window $\phi\in\mathcal{S}^{(s)}\setminus 0$;*

(3) *the set of windows can be extended from* $\mathcal{S}^{(s)}(\mathbf{R}^d)\setminus 0$ *to* $M^1_{(v)}(\mathbf{R}^d)\setminus 0$.

3.2. Wave-front sets with respect to modulation spaces. Next we define wave-front sets with respect to modulation spaces and show that they agree with corresponding wave-front sets of Fourier–Lebesgue types. More precisely, we prove that [**19**, Theorem 6.1] holds if the weights of polynomial growth are replaced by more general submultiplicative weights.

Let $s>1$, $\phi\in\mathcal{S}^{(s)}(\mathbf{R}^d)\setminus 0$, $\omega\in\mathscr{M}_{\{s\}}$, $\Gamma\subseteq\mathbf{R}^d\setminus 0$ be an open cone and let $p,q\in[1,\infty]$. For any $f\in(\mathcal{S}^{(s)})'(\mathbf{R}^d)$ we set

$$|f|_{\mathcal{B}(\Gamma)}=|f|_{\mathcal{B}(\phi,\Gamma)}\equiv\left(\int_\Gamma\left(\int_{\mathbf{R}^d}|V_\phi f(x,\xi)\omega(x,\xi)|^p\,dx\right)^{q/p}d\xi\right)^{1/q}\qquad\text{when}\quad\mathcal{B}=M^{p,q}_{(\omega)}=M^{p,q}_{(\omega)}(\mathbf{R}^d).\tag{3.2}$$

We note that $|f|_{\mathcal{B}(\Gamma)}=\|f\|_{M^{p,q}_{(\omega)}}$ when $\Gamma=\mathbf{R}^d\setminus 0$ and $\phi\in\mathcal{S}^{(s)}(\mathbf{R}^d)$, and that $|f|_{\mathcal{B}(\phi,\Gamma)}$ might attain $+\infty$.

We also set

$$|f|_{\mathcal{B}(\Gamma)}=|f|_{\mathcal{B}(\phi,\Gamma)}\equiv\left(\int_{\mathbf{R}^d}\left(\int_\Gamma|V_\phi f(x,\xi)\omega(x,\xi)|^q\,d\xi\right)^{p/q}dx\right)^{1/p}\qquad\text{when}\quad\mathcal{B}=W^{p,q}_{(\omega)}=W^{p,q}_{(\omega)}(\mathbf{R}^d)\tag{3.3}$$

and note that similar properties hold for this semi-norm compared to (3.2).

Let $\omega\in\mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, $p,q\in[1,\infty]$, $f\in(\mathcal{S}^{(s)})'(\mathbf{R}^d)$, and let $\mathcal{B}=M^{p,q}_{(\omega)}$ or $\mathcal{B}=W^{p,q}_{(\omega)}$. Then $\Theta_{\mathcal{B}}(f)$, $\Sigma_{\mathcal{B}}(f)$ and the wave-front set $\mathrm{WF}_{\mathcal{B}}(f)$ of f with respect to the modulation space $\mathcal{B}$ are defined in the same way as in Section 2, after replacing the semi-norms of Fourier–Lebesgue types in (2.1) with the semi-norms in (3.2) or (3.3) respectively.

We need the following proposition when proving that the wave-front sets of Fourier–Lebesgue and modulation space types are the same. The first part is an extension of [**1**, Proposition 4.2].

Proposition 3.2. *Let $s>1$. Then the following is true:*

(1) *if $f\in(\mathcal{E}^{(s)})'(\mathbf{R}^d)$ and $\phi\in\mathcal{S}^{(s)}(\mathbf{R}^d)$, then*

$$|V_\phi f(x,\xi)|\lesssim e^{-h|x|^{1/s}}e^{\varepsilon|\xi|^{1/s}},\quad\text{for some } h>0 \text{ and } \varepsilon>0;\tag{3.4}$$

(2) *if $f\in(\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$ and $\phi\in\mathcal{S}^{(s)}(\mathbf{R}^d)$, then* (3.4) *holds for every $h>0$ and $\varepsilon>0$;*

(3) *if $f\in(\mathcal{D}^{(s)})'(\mathbf{R}^d)$ and $\phi\in\mathcal{D}^{(s)}(\mathbf{R}^d)\setminus 0$, then $f\in(\mathcal{E}^{(s)})'(\mathbf{R}^d)$, if and only if* $\operatorname{supp}V_\phi f\subseteq K\times\mathbf{R}^d$ *for some compact set K, and then*

$$|V_\phi f(x,\xi)|\lesssim e^{\varepsilon|\xi|^{1/s}},\quad\text{for some } \varepsilon>0.\tag{3.5}$$

(4) *if $f\in(\mathcal{D}^{(s)})'(\mathbf{R}^d)$ and $\phi\in\mathcal{D}^{(s)}(\mathbf{R}^d)\setminus 0$, then $f\in(\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$, if and only if* $\operatorname{supp}V_\phi f\subseteq K\times\mathbf{R}^d$ *for some compact set K and* (3.5) *holds for every $\varepsilon>0$.*

Proof. We only prove (1) and (3). The other statements follow by similar arguments and are left for the reader.

In order to prove (1) we assume that $f \in (\mathcal{E}^{(s)})'(\mathbf{R}^d)$ and $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d)$. Also let $\psi \in \mathcal{D}^{(s)}(\mathbf{R}^d)$ be such that $\psi = 1$ in $\operatorname{supp} f$. Then for some $\varepsilon, h > 0$ we have

$$|V_\psi \phi(x,\xi)| \lesssim e^{-h|x|^{1/s} - 2\varepsilon|\xi|^{1/s}}, \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{\varepsilon|\xi|^{1/s}}.$$

By straightforward calculations, it follows that

$$\begin{aligned} |V_\phi f(x,\xi)| &= |(V_\phi(\psi f))(x,\xi)| \lesssim (|V_\psi \phi(x,\cdot)| * |\widehat{f}|)(\xi) \\ &= \int |V_\psi \phi(x,\xi-\eta)||\widehat{f}(\eta)|\, d\eta \lesssim \int e^{-h|x|^{1/s} - 2\varepsilon|\xi-\eta|^{1/s}} e^{\varepsilon|\eta|^{1/s}} d\eta \\ &\leqslant e^{-h|x|^{1/s}} \int e^{-2\varepsilon|\eta|^{1/s} + 2\varepsilon|\xi|^{1/s} + \varepsilon|\eta|^{1/s}} d\eta \lesssim e^{-h|x|^{1/s} + 2\varepsilon|\xi|^{1/s}}, \end{aligned}$$

and (1) follows.

Next we prove (3). First assume that $\phi \in \mathcal{D}^{(s)}(\mathbf{R}^d) \setminus 0$ and $f \in (\mathcal{E}^{(s)})'(\mathbf{R}^d)$. Since both ϕ and f have compact support, it follows that $\operatorname{supp}(V_\phi f) \subseteq K \times \mathbf{R}^d$. Furthermore, $|V_\phi f(x,\xi)| \lesssim e^{\varepsilon(|x|^{1/s} + |\xi|^{1/s})}$, for some $\varepsilon > 0$, in view of [**1**]. Since $V_\phi f(x,\xi)$ has compact support in the x-variable, it follows that

$$|V_\phi f(x,\xi)| \lesssim e^{\varepsilon|\xi|^{1/s}}.$$

In order to prove the reverse direction we assume that $\operatorname{supp} V_\phi f \subseteq K \times \mathbf{R}^d$, for a compact set K. Assume that $\operatorname{supp} \phi \subseteq K$ and choose $\varphi \in \mathcal{D}^{(s)}(\mathbf{R}^d)$ such that $\operatorname{supp} \varphi \cap 2K = \emptyset$. Then $(f,\varphi) = (\|\phi\|_{L^2})^{-2}(V_\phi f, V_\phi \varphi) = 0$, which implies that f has compact support. Here the first equality is Moyal's identity (cf. [**9**]). This implies that f has compact support and the condition $f \in (\mathcal{D}^{(s)})'(\mathbf{R}^d)$ now gives $f \in (\mathcal{E}^{(s)})'(\mathbf{R}^d)$. $\square$

Theorem 3.1. *Let $s > 1$, $p,q \in [1,\infty]$, $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, $\mathcal{B} = \mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$, and let $\mathcal{C} = M^{p,q}_{(\omega)}(\mathbf{R}^d)$ or $\mathcal{C} = W^{p,q}_{(\omega)}(\mathbf{R}^d)$. If $f \in (\mathcal{D}^{(s)})'(\mathbf{R}^d)$, then*

$$\operatorname{WF}_{\mathcal{B}}(f) = \operatorname{WF}_{\mathcal{C}}(f). \tag{3.6}$$

In particular, $\operatorname{WF}_{\mathcal{C}}(f)$ *is independent of p and $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$ in* (3.2) *and* (3.3).

In the proof of Theorem 3.1, the main part concerns proving that the wave-front sets of modulation types are independent of the choice of window $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$.

Proof. We only consider the case $\mathcal{C} = M^{p,q}_{(\omega)}$. The case $\mathcal{C} = W^{p,q}_{(\omega)}$ follows by similar arguments and is left for the reader. We may assume that $f \in (\mathcal{E}^{(s)})'(\mathbf{R}^d)$ and that $\omega(x,\xi) = \omega(\xi)$ since the statements only concern local assertions.

In order to prove that $\operatorname{WF}_{\mathcal{C}}(f)$ is independent of $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$, we assume that $\phi, \phi_1 \in \mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$ and let $|\cdot|_{\mathcal{C}_1(\Gamma)}$ be the semi-norm in (3.2) after ϕ has been replaced by ϕ_1. Let Γ_1 and Γ_2 be open cones in $\mathbf{R}^d$ such that $\overline{\Gamma_2} \subseteq \Gamma_1$. The asserted independence of ϕ follows if we prove that

$$|f|_{\mathcal{C}(\Gamma_2)} \leqslant C(|f|_{\mathcal{C}_1(\Gamma_1)} + 1), \tag{3.7}$$

for some positive constant C. Let

$$\Omega_1 = \{(x,\xi)\,;\,\xi\in\Gamma_1\}\subseteq \mathbf{R}^{2d}\quad\text{and}\quad \Omega_2=\complement\Omega_1\subseteq\mathbf{R}^{2d},$$

with characteristic functions χ_1 and χ_2 respectively, and set

$$F_k(x,\xi)=|V_{\phi_1}f(x,\xi)|\,\omega(\xi)\,\chi_k(x,\xi),\quad k=1,2,$$

and $G=|V_\phi\phi_1(x,\xi)v(\xi)|$. Since ω is v-moderate, it follows from [**9**, Lemma 11.3.3] that

$$|V_\phi f(x,\xi)\,\omega(x,\xi)|\lesssim \big((F_1+F_2)*G\big)(x,\xi),$$

which implies that $|f|_{\mathcal{C}(\Gamma_2)}\lesssim J_1+J_2$, where

$$J_k=\left(\int_{\Gamma_2}\left(\int_{\mathbf{R}^d}|(F_k*G)(x,\xi)|^p\,dx\right)^{q/p}d\xi\right)^{1/q},\quad k=1,2.$$

By Young's inequality $J_1\leqslant\|F_1*G\|_{L_1^{p,q}}\leqslant\|G\|_{L^1}\|F_1\|_{L_1^{p,q}}=C|f|_{\mathcal{C}_1(\Gamma_1)}$, where $C=\|G\|_{L^1}=\|V_\phi\phi_1(x,\xi)v(\xi)\|_{L^1}<\infty$, in view of (3.1).

Next we consider J_2. For $\xi\in\Gamma_2$ fixed and integrating over $\eta\in\complement\Gamma_1$, it follows from (2.5), Propositon 3.2 and (3.1) that for some $N,k,\varepsilon>0$ and every $h>0$ we have

$$\begin{aligned}
|(F_2*G)(x,\xi)| &\lesssim \iint_{\mathbf{R}^{2d}} e^{-N|y|^{1/s}}e^{\varepsilon|\eta|^{1/s}}e^{-h(|x-y|^{1/s}+|\xi-\eta|^{1/s})}v(\xi-\eta)\,dy\,d\eta\\
&\lesssim \iint_{\mathbf{R}^{2d}} e^{-N|y|^{1/s}}e^{\varepsilon|\eta|^{1/s}}e^{-h|x-y|^{1/s}-hc(|\xi|^{1/s}+|\eta|^{1/s})}e^{k(|\xi|^{1/s}+|\eta|^{1/s})}\,dy\,d\eta\\
&\lesssim e^{-N_1|x|^{1/s}}e^{(k-hc)|\xi|^{1/s}}\iint_{\mathbf{R}^{2d}} e^{-N_1|y|^{1/s}}e^{(k+\varepsilon-hc)|\eta|^{1/s}}\,dy\,d\eta,\\
&\lesssim e^{-N_1|x|^{1/s}}e^{(k-hc)|\xi|^{1/s}}<\infty,
\end{aligned}$$

for some $N_1>0$, provided h is chosen large enough. Therefore

$$\begin{aligned}
J_2&=\left(\int_{\Gamma_2}\left(\int_{\mathbf{R}^d}|(F_2*G)(x,\xi)|^p\,dx\right)^{q/p}d\xi\right)^{1/q}\\
&\lesssim\left(\int_{\Gamma_2}\left(\int_{\mathbf{R}^d}\left(e^{-N_1|x|^{1/s}}e^{(k-hc)|\xi|^{1/s}}\right)^p dx\right)^{q/p}d\xi\right)^{1/q}<\infty.
\end{aligned}$$

This proves that (3.7), and hence $\mathrm{WF}_{\mathcal{C}}(f)$ is independent of $\phi\in\mathcal{S}^{(s)}(\mathbf{R}^d)\setminus 0$.

In order to prove (3.6) we assume from now on that ϕ in (3.2) is real-valued and has compact support. Let $p_0\in[1,\infty]$ be such that $p_0\leqslant p$ and set $\mathcal{C}_0=M^{p_0,q}_{(\omega)}$. The result follows if we prove

$$\Theta_{\mathcal{C}_0}(f)\subseteq\Theta_{\mathcal{B}}(f)\subseteq\Theta_{\mathcal{C}}(f)\quad\text{when } p_0=1,\ p=\infty,\tag{3.8}$$

$$\Theta_{\mathcal{C}}(f)\subseteq\Theta_{\mathcal{C}_0}(f).\tag{3.9}$$

The proof of the first inclusion in (3.8) follows from the estimates

$$
\begin{aligned}
|f|_{\mathcal{B}(\Gamma)} &\lesssim \Big(\int_\Gamma |\widehat{f}(\xi)\omega(\xi)|^q d\xi \Big)^{1/q} \\
&\lesssim \Big(\int_\Gamma |\mathscr{F}\Big(f \int_{\mathbf{R}^d} \phi(\cdot - x)\, dx \Big)(\xi)\omega(\xi)|^q d\xi \Big)^{1/q} \\
&\lesssim \Big(\int_\Gamma \Big(\int_{\mathbf{R}^d} |\mathscr{F}(f\phi(\cdot - x))(\xi)\omega(\xi)|\, dx \Big)^q d\xi \Big)^{1/q} \\
&= \Big(\int_\Gamma \Big(\int_{\mathbf{R}^d} |V_\phi f(x,\xi)\omega(\xi)|\, dx \Big)^q d\xi \Big)^{1/q} = C|f|_{\mathcal{C}_0(\Gamma)},
\end{aligned}
$$

for a positive constant C.

Next we prove the second inclusion in (3.8). We have

$$
\begin{aligned}
|f|_{\mathcal{C}(\Gamma_2)} &= \Big(\int_{\Gamma_2} \sup_{x\in\mathbf{R}^d} |V_\phi f(x,\xi)\omega(x,\xi)|^q d\xi \Big)^{1/q} \\
&\lesssim \Big(\int_{\Gamma_2} \sup_{x\in\mathbf{R}^d} |(|\widehat{f}| * |\mathscr{F}(\phi(\cdot - x))|)(\xi)\omega(\xi)|^q d\xi \Big)^{1/q} \\
&\lesssim \Big(\int_{\Gamma_2} |(|\widehat{f}| * |\widehat{\phi}|)(\xi)\omega(\xi)|^q d\xi \Big)^{1/q} \\
&\lesssim \Big(\int_{\Gamma_2} \big((|\widehat{f}\cdot\omega| * |\widehat{\phi}\cdot v|)(\xi)\big)^q d\xi \Big)^{1/q},
\end{aligned}
$$

where $\phi \in \mathcal{D}^{(s)}(X)$ is chosen such that $\phi = 1$ in $\operatorname{supp} f$. The second inclusion in (3.8) now follows by straightforward computations, using similar arguments as in the proof of (2.3). The details are left for the reader.

It remains to prove (3.9). Let $K \subseteq \mathbf{R}^d$ be a compact set chosen such that $V_\phi f(x,\xi) = 0$ outside $K\times\mathbf{R}^d$, and let $p_1 \in [1,\infty]$ be chosen such that $1/p_1 + 1/p_0 = 1 + 1/p$. By Hölder's inequality we get

$$
\begin{aligned}
|f|_{\mathcal{C}_0(\Gamma)} &= \Big(\int_\Gamma \Big(\int_{\mathbf{R}^d} |V_\phi f(x,\xi)\omega(x,\xi)|^{p_0} dx \Big)^{q/p_0} d\xi \Big)^{1/q} \\
&\leqslant C_K \Big(\int_\Gamma \Big(\int_{\mathbf{R}^d} |V_\phi f(x,\xi)\omega(x,\xi)|^{p} dx \Big)^{q/p} d\xi \Big)^{1/q} = C_K|f|_{\mathcal{C}(\Gamma)}.
\end{aligned}
$$

This gives (3.9), and the proof is complete. □

Remark 3.1. Let $s > 1$, $p, q \in [1,\infty]$, and $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$. If $f \in (\mathcal{E}^{(s)})'(\mathbf{R}^d)$, then it follows from the definition of the wave-front sets that then

$$
f \in \mathcal{B} \iff \mathrm{WF}_{\mathcal{B}}(f) = \emptyset,
$$

when $\mathcal{B}$ is equal to $\mathscr{F}L^q_{(\omega)}$, $M^{p,q}_{(\omega)}$ or $W^{p,q}_{(\omega)}$. In particular

$$
\mathscr{F}L^q_{(\omega)} \cap (\mathcal{E}^{(s)})'(\mathbf{R}^d) = M^{p,q}_{(\omega)} \cap (\mathcal{E}^{(s)})'(\mathbf{R}^d) = W^{p,q}_{(\omega)} \cap (\mathcal{E}^{(s)})'(\mathbf{R}^d),
$$

by Theorem 3.1.

In particular, we recover Corollary 6.2 in [**19**], Theorem 2.1 and Remark 4.6 in [**23**].

Remark 3.2. In some situations we may relax the condition on the window function $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d)$ into $\phi \in \mathcal{S}^{\{s\}}(\mathbf{R}^d)$. In fact, let $s > 1$, $\phi \in \mathcal{S}^{\{s\}}(\mathbf{R}^d) \setminus 0$ and let $f \in (\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$. Then $V_\phi f$ makes sense as an element in $(\mathcal{S}^{\{s\}})'(\mathbf{R}^{2d}) \cap C^\infty(\mathbf{R}^{2d})$. Furthermore, if $\mathcal{B} = M^{p,q}_{(\omega)}(\mathbf{R}^d)$, then analogous versions of the sets $\Theta_{\mathcal{B}}(f)$, $\Sigma_{\mathcal{B}}(f)$ and $\mathrm{WF}_{\mathcal{B}}(f)$ can be defined by replacing the condition $\varphi \in \mathcal{D}^{(s)}(\mathbf{R}^d)$ with $\varphi \in \mathcal{D}^{\{s\}}(\mathbf{R}^d)$. The investigations in this section then show that Theorem 3.1 still holds after the assumptions on f and ϕ were changed in this way.

4. Discrete versions of wave-front sets

The main goal of this section is to introduce discrete wave-front sets with respect to Fourier–Lebesgue and modulation spaces, and to relate them with the corresponding wave-front sets of continuous types, given in the previous sections. To that aim, in the first part we introduce discrete analogues of Fourier–Lebesgue norms and relate them to corresponding continuous ones, given in (2.1). Finally we define discrete versions of wave-front sets and prove that they agree one to another and to the corresponding continuous ones.

4.1. Discrete semi-norms in Fourier–Lebesgue spaces. In this subsection we introduce discrete analogues of the semi-norms in (2.1), and show that these semi-norms are finite if and only if the corresponding nondiscrete semi-norms are finite. The techniques used here are similar to those in [**14**].

Assume that $q \in [1, \infty]$, $s > 1$, $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^d)$, $\mathcal{B} = \mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$, and $H \subseteq \mathbf{R}^d$ is a discrete set. Then we set

$$|f|^{(D)}_{\mathcal{B}(H)} \equiv \left(\sum_{\xi_l \in H} |\widehat{f}(\xi_l)\omega(\xi_l)|^q \right)^{1/q}, \qquad \widehat{f} \in C(\mathbf{R}^d) \cap (\mathcal{S}^{(s)})'(\mathbf{R}^d)$$

with obvious modifications when $q = \infty$. As in the continuous case, we may allow weight functions in $\mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, i.e., $\omega = \omega(x, \xi)$. However, again we note that the condition $|f|^{(D)}_{\mathcal{B}(H)} < \infty$ is independent of $x \in \mathbf{R}^d$. For the proof of the main result of this part, we need two lemmas.

We recall that by a lattice Λ we mean the set

$$\Lambda = \{a_1 e_1 + \dots + a_d e_d;\ a_1, \dots, a_d \in \mathbf{Z}\},$$

where $e_1, \dots, e_d$ is a basis in $\mathbf{R}^d$.

The following Lemma was proved for distributions, cf. [**14, 24, 25**].

Lemma 4.1. *Let $s > 1$, $f \in (\mathcal{E}^{(s)})'(\mathbf{R}^d)$, Γ and Γ_0 be open cones in $\mathbf{R}^d \setminus 0$ such that $\overline{\Gamma_0} \subseteq \Gamma$, $q \in [1, \infty]$, and let $\Lambda \subseteq \mathbf{R}^d$ be a lattice. If $|f|_{\mathcal{B}(\Gamma)}$ is finite, then $|f|^{(D)}_{\mathcal{B}(\Gamma_0 \cap \Lambda)}$ is finite.*

PROOF. We only prove the result for $q < \infty$, leaving the small modifications in the case $q = \infty$ for the reader. Assume that $|f|_{\mathcal{B}(\Gamma)} < \infty$, and let $H = \Gamma_0 \cap \Lambda$. Also let $\varphi \in \mathcal{D}^{(s)}(\mathbf{R}^d)$ be such that $\varphi = 1$ in $\operatorname{supp} f$. Then

$$
\begin{aligned}
(|f|^{(D)}_{\mathcal{B}(\Gamma_0\cap\Lambda)})^q &= \sum_{\xi_l \in H} |\mathscr{F}(\varphi f)(\xi_l)\omega(\xi_l)|^q \\
&= (2\pi)^{-qd/2} \sum_{\xi_l\in H} \left| \int \widehat{\varphi}(\xi_l - \eta)\widehat{f}(\eta)\omega(\xi_l)\, d\eta \right|^q \lesssim (S_1 + S_2),
\end{aligned}
$$

where

$$
S_1 = \sum_{\xi_l\in H} \left(\int_\Gamma \psi(\xi_l - \eta)F(\eta)\, d\eta \right)^q, \quad S_2 = \sum_{\xi_l\in H} \left(\int_{\complement\Gamma} \psi(\xi_l - \eta)F(\eta)\, d\eta \right)^q .
$$

Here we set $F(\xi) = |\widehat{f}(\xi)\omega(\xi)|$ and $\psi(\xi) = |\widehat{\varphi}(\xi)v(\xi)|$ as in the proof of Theorem 2.1.

We need to estimate S_1 and S_2. By Hölder's inequality we get

$$
\begin{aligned}
S_1 &= \sum_{\xi_l\in H} \left(\int_\Gamma \psi(\xi_l - \eta)F(\eta)\, d\eta \right)^q = \sum_{\xi_l\in H} \left(\int_\Gamma \psi(\xi_l - \eta)^{1/q'}(\psi(\xi_l - \eta)^{1/q}F(\eta))\, d\eta \right)^q \\
&\leqslant \|\psi\|_{L^1}^{q/q'} \sum_{\xi_l\in H} \int_\Gamma \psi(\xi_l - \eta)F(\eta)^q d\eta \leqslant C' \int_\Gamma F(\eta)^q d\eta = C'|f|^q_{\mathcal{B}(\Gamma)},
\end{aligned}
$$

where

$$
C' = \|\psi\|_{L^1}^{q/q'} \sup_{\eta\in\mathbf{R}^d} \sum_{\xi_l\in H} \psi(\xi_l - \eta)
$$

is finite by (2.4). This proves that S_1 is finite.

It remains to prove that S_2 is finite. We observe that

$$
|\xi_l - \eta|^{1/s} \geqslant 2c \max(|\xi_l|^{1/s}, |\eta|^{1/s}) \geqslant c(|\xi_l|^{1/s} + |\eta|^{1/s})
$$

when $\xi_l \in H$ and $\eta \in \complement\Gamma$, for some $c > 0$. Since $f \in (\mathcal{E}^{(s)})'(\mathbf{R}^d)$, it follows that $|F| \lesssim e^{N_0|\cdot|^{1/s}}$ for a positive constant N_0. Furthermore, since $\varphi \in \mathcal{D}^{(s)}(\mathbf{R}^d)$, it follows that for every $N \geqslant 0$ we have $\psi \lesssim e^{-N|\cdot|^{1/s}}$. This gives

$$
\begin{aligned}
S_2 &\lesssim \sum_{\xi_l\in H} \left(\int_{\complement\Gamma} e^{-N|\xi_l-\eta|^{1/s}} e^{N_0|\eta|^{1/s}} d\eta \right)^q \\
&\lesssim \sum_{\xi_l\in H} e^{-qNc|\xi_l|^{1/s}} \left(\int e^{-(Nc-N_0)|\eta|^{1/s}} d\eta \right)^q ,
\end{aligned}
$$

where we have used the fact that ω is v-moderate. The result now follows, since the right-hand side is finite when $N > N_0/c$. The proof is complete. □

Next we prove a converse of Lemma 4.1, in the case when the lattice Λ is dense enough. Let $e_1, \dots, e_d$ in $\mathbf{R}^d$ be a basis for Λ, i.e., for some $x_0 \in \Lambda$ we have

$$
\Lambda = \{x_0 + t_1e_1 + \cdots + t_de_d;\, t_1, \dots, t_d \in \mathbf{Z}\}.
$$

A parallelepiped D, spanned by $e_1, \dots, e_d$ for Λ and with corners in Λ, is called a Λ-parallelepiped. This means that for some $x_0 \in \Lambda$ and for some basis $e_1, \dots, e_d$ for Λ we have $D = \{x_0 + t_1 e_1 + \dots + t_d e_d;\ t_1, \dots, t_d \in [0,1]\}$.

We let $\mathcal{A}(\Lambda)$ be the set of all Λ-parallelepipeds. For future references we note that if $D_1, D_2 \in \mathcal{A}(\Lambda)$, then their volumes $|D_1|$ and $|D_2|$ agree, and for convenience we let $\|\Lambda\|$ denote the common value, i.e., $\|\Lambda\| = |D_1| = |D_2|$.

Let Λ_1 and Λ_2 be lattices in $\mathbf{R}^d$ with bases $e_1, \dots, e_d$ and $\varepsilon_1, \dots, \varepsilon_d$ respectively. Then the pair (Λ_1, Λ_2) is called *admissible lattice pair*, if for some $0 < c \leqslant 2\pi$ we have $\langle e_j, \varepsilon_j \rangle = c$ and $\langle e_j, \varepsilon_k \rangle = 0$ when $j \neq k$. If in addition $c < 2\pi$, then (Λ_1, Λ_2) is called a *strongly admissible lattice pair*. If instead $c = 2\pi$, then the pair (Λ_1, Λ_2) is called a *weakly admissible lattice pair*.

Here we note that if the lattice pair (Λ_1, Λ_2) is weakly admissible, then every choice of ϕ in (7.2.2) in [**12**] gives rise to a Fourier series expansion of a D_1-periodic distribution f, where $D_1 \in \mathcal{A}(\Lambda_1)$. Hence, there is an ambiguity concerning the choice of ϕ for expressing f in a Fourier series. On the other hand, if (Λ_1, Λ_2) is strongly admissible and the restriction of f to the open set D_1 has compact support, then the Fourier coefficients are defined in a canonical way. (See also (4.2) below.)

LEMMA 4.2. *Let $s > 1$, (Λ_1, Λ_2) be an admissible lattice pair, $D_1 \in \mathcal{A}(\Lambda_1)$, and let $f \in (\mathcal{E}^{(s)})'(\mathbf{R}^d)$ be such that an open neighbourhood of its support is contained in D_1. Also let Γ and Γ_0 be open cones in $\mathbf{R}^d$ such that $\overline{\Gamma_0} \subseteq \Gamma$. If $|f|^{(D)}_{\mathcal{B}(\Gamma \cap \Lambda_2)}$ is finite, then $|f|_{\mathcal{B}(\Gamma_0)}$ is finite.*

PROOF. Since D_1 contains an open neighbourhood of the support of f, we may modify Λ_1 (and therefore D_1) such that the lattice pair (Λ_1, Λ_2) is strongly admissible, and such that the hypothesis still holds. From now on we therefore assume that (Λ_1, Λ_2) is strongly admissible.

We use similar arguments as in the proof of Lemma 4.1. Again we prove the result only for $q < \infty$. The small modifications to the case $q = \infty$ are left for the reader.

Assume that $|f|^{(D)}_{\mathcal{B}(\Gamma \cap \Lambda_2)} < \infty$, and let $\varphi \in \mathcal{D}^{(s)}(D_1^\circ)$ be equal to one in the support of f, where D_1° denotes the interior of the set D_1. By expanding $f = \varphi f$ into a Fourier series on D_1 we get $\widehat{f}(\xi) = C \sum_{\xi_l \in \Lambda_2} \widehat{\varphi}(\xi - \xi_l) \widehat{f}(\xi_l)$, where the positive constant C only depends on Λ_2. (Cf. e.g. (4.1) below.) We have

$$(|f|_{\mathcal{B}(\Gamma_0)})^q = \int_{\Gamma_0} |\widehat{f}(\xi)\omega(\xi)|^q d\xi = C^q \int_{\Gamma_0} \Big| \sum_{\xi_l \in \Lambda_2} \widehat{\varphi}(\xi - \xi_l) \widehat{f}(\xi_l) \omega(\xi) \Big|^q d\xi \lesssim S_1 + S_2,$$

where

$$S_1 = \int_{\Gamma_0} \Big| \sum_{\xi_l \in H_1} \widehat{\varphi}(\xi - \xi_l) \widehat{f}(\xi_l) \omega(\xi) \Big|^q d\xi, \quad S_2 = \int_{\Gamma_0} \Big| \sum_{\xi_l \in H_2} \widehat{\varphi}(\xi - \xi_l) \widehat{f}(\xi_l) \omega(\xi) \Big|^q d\xi,$$

$H_1 = \Gamma \cap \Lambda_2$ and $H_2 = \complement \Gamma \cap \Lambda_2$.

We have to estimate S_1 and S_2. Let ω be moderate with respect to the weight $v(\cdot) = e^{k|\cdot|^{1/s}}$. By Minkowski's inequality we get

$$
\begin{aligned}
S_1 &\leqslant C \int_{\Gamma_0} \Big(\sum_{\xi_l \in H_1} |\widehat{\varphi}(\xi-\xi_l) v(\xi-\xi_l)| |\widehat{f}(\xi_l)\omega(\xi_l)| \Big)^q d\xi \\
&\leqslant C' \int_{\Gamma_0} \Big(\sum_{\xi_l \in H_1} |\widehat{\varphi}(\xi-\xi_l) v(\xi-\xi_l)| |\widehat{f}(\xi_l)\omega(\xi_l)|^q \Big) d\xi \leqslant C'' \sum_{\xi_l \in H_1} |\widehat{f}(\xi_l)\omega(\xi_l)|^q,
\end{aligned}
$$

where $C' = C \sup_\xi \|\widehat{\varphi}(\xi-\xi_l)v(\xi-\xi_l)\|^{q/q'}_{l^1(\Lambda_2)} < \infty$, and $C'' = C'\|\varphi\|_{\mathscr{F}L^1_{(v)}} < \infty$. This proves that S_1 is finite when $|f|^{(D)}_{\mathcal{B}(\Gamma\cap\Lambda_2)} < \infty$.

It remains to prove that S_2 is finite. We recall that

$$
|\xi-\xi_l|^{1/s} \geqslant 2c\max(|\xi|^{1/s}, |\xi_l|^{1/s}) \geqslant c(|\xi|^{1/s} + |\xi_l|^{1/s}) \text{ when } \xi \in \Gamma_0 \text{ and } \xi_l \in H_2,
$$

and use the same arguments as in the proof of Lemma 4.1 to obtain

$$
\begin{aligned}
S_2 &\lesssim \int_{\Gamma_0} \Big(\sum_{\xi_l \in H_2} e^{-N|\xi-\xi_l|^{1/s}} e^{N_0|\xi_l|^{1/s}} \Big)^q d\xi \\
&\lesssim \int_{\Gamma_0} e^{-qNc|\xi|^{1/s}} \Big(\sum_{\xi_l \in H_2} e^{-(Nc-N_0)|\xi_l|^{1/s}} \Big)^q d\xi.
\end{aligned}
$$

The result now follows, since the right-hand side is finite when $N > N_0/c$. The proof is complete. □

Theorem 4.1. *Let $s > 1$, (Λ_1, Λ_2) be an admissible lattice pair, $D_1 \in \mathcal{A}(\Lambda_1)$, and let $f \in (\mathcal{E}^{(s)})'(\mathbf{R}^d)$ be such that an open neighbourhood of its support is contained in D_1. Also let Γ and Γ_0 be open cones in $\mathbf{R}^d$ such that $\overline{\Gamma_0} \subseteq \Gamma$. If $|f|^{(D)}_{\mathcal{B}(\Gamma\cap\Lambda_2)}$ is finite, then $|\varphi f|^{(D)}_{\mathcal{B}(\Gamma_0\cap\Lambda_2)}$ is finite for every $\varphi \in \mathcal{D}^{(s)}(X)$.*

For the proof we recall that $|\varphi f|_{\mathcal{B}(\Gamma_0)}$ is finite when $f \in (\mathcal{E}^{(s)})'(\mathbf{R}^d)$, $\varphi \in \mathcal{D}^{(s)}(X)$, and $|f|_{\mathcal{B}(\Gamma)}$ is finite. This follows from the proof of Theorem 2.1.

Proof. Let Γ_1, Γ_2 be open cones such that $\overline{\Gamma_j} \subseteq \Gamma_{j+1}$ for $j = 0, 1$, $\overline{\Gamma_2} \subseteq \Gamma$, and assume that $|f|^{(D)}_{\mathcal{B}(\Gamma\cap\Lambda_2)} < \infty$. Then Lemma 4.2 shows that $|f|_{\mathcal{B}(\Gamma_2)}$ is finite. Hence, Theorem 2.1 implies that $|\varphi f|_{\mathcal{B}(\Gamma_1)} < \infty$. This gives $|\varphi f|^{(D)}_{\mathcal{B}(\Gamma_0\cap\Lambda_2)} < \infty$, in view of Lemma 4.1. The proof is complete. □

4.2. Gabor pairs. In this subsection we recall in Definition 4.1 the notion of Gabor pairs, which are later on used in the definition of discrete version of wave-front sets with respect to modulation spaces. We refer to [**14**] for an explanation that conditions in Definition 4.1 are quite general.

By Definition 4.1 it follows that our analysis can be applied to the most general classes of non-quasianalytic ultradistributions, and it also points out the role of Beurling–Domar weights in definitions of $\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$ and $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, cf. [**2**,**10**,**11**]. On the other hand, a larger class of quasianalytic ultradistributions can not be treated by the technique given here, since the corresponding test function spaces do not contain smooth functions of compact support.

Assume that $e_1,\dots,e_d$ is a basis for the lattice Λ_1, and that (Λ_1,Λ_2) is a weakly admissible lattice pair. If $f\in L^2_{\mathrm{loc}}$ is periodic with respect to Λ_1, and D is the parallelepiped, spanned by $\{e_1,\dots,e_d\}$, then we may make Fourier expansion of f as

$$f(x)=\sum_{\xi_l\in\Gamma_2} c_l e^{i\langle x,\xi_l\rangle},\qquad x\in\mathbf{R}^d \tag{4.1}$$

(with convergence in L^2_{loc}), where the coefficients c_l are given by

$$c_l=\int_\Delta f(y)e^{-i\langle y,\xi_l\rangle}\,dy. \tag{4.2}$$

Here $y=y_1e_1+\cdots+y_de_d$, $dy=dy_1\cdots dy_d$, and $\Delta=[0,1]^d$. For nonperiodic functions and distributions we instead make Gabor expansions. Because of the support properties of the involved Gabor atoms and their duals, we are usually forced to change the assumption on the involved lattice pairs. More precisely, instead of assuming that (Λ_1,Λ_2) should be a weakly admissible lattice pair, we assume from now on that (Λ_1,Λ_2) is a strongly admissible lattice pair, with $\Lambda_1=\{x_j\}_{j\in J}$ and $\Lambda_2=\{\xi_l\}_{l\in J}$. Also let $s>1$ and

$$\phi,\psi\in\mathcal{D}^{(s)}(\mathbf{R}^d),\quad \phi_{j,l}(x)=\phi(x-x_j)e^{i\langle x,\xi_l\rangle},\quad \psi_{j,l}(x)=\psi(x-x_j)e^{i\langle x,\xi_l\rangle} \tag{4.3}$$

be such that $\{\phi_{j,l}\}_{j,l\in J}$ and $\{\psi_{j,l}\}_{j,l\in J}$ are dual Gabor frames (see [**6**, **9**] for the definition and basic properties of Gabor frames and their duals). If $f\in(\mathcal{S}^{(s)})'(\mathbf{R}^d)$ then

$$f=\sum_{j,l\in J} c_{j,l}\phi_{j,l}, \tag{4.4}$$

where

$$c_{j,l}=C_{\phi,\psi}\cdot(f,\psi_{j,l}) \tag{4.5}$$

and the constant $C_{\phi,\psi}$ depends on the frames only. Here $(\cdot,\cdot)$ denotes the unique extension of the L^2-form on $\mathcal{S}^{(s)}(\mathbf{R}^d)\times\mathcal{S}^{(s)}(\mathbf{R}^d)$ into $(\mathcal{S}^{(s)})'(\mathbf{R}^d)\times(\mathcal{S}^{(s)})'(\mathbf{R}^d)$.

Note that the convergence is in $(\mathcal{S}^{(s)})'(\mathbf{R}^d)$ due to Proposition 3.2.

Definition 4.1. Assume that $\varepsilon\in(0,1]$, $\{x_j\}_{j\in J}=\Lambda_1\subseteq\mathbf{R}^d$ and $\{\xi_l\}_{l\in J}=\Lambda_2\subseteq\mathbf{R}^d$ are lattices and let $\Lambda_1(\varepsilon)=\varepsilon\Lambda_1$. Also let $\phi,\psi\in C_0^\infty(\mathbf{R}^d)$ be nonnegative, and set

$$\phi^\varepsilon=\phi(\cdot/\varepsilon),\quad \phi^\varepsilon_{j,l}=\phi^\varepsilon(\cdot-\varepsilon x_j)e^{i\langle\cdot,\xi_l\rangle},$$
$$\psi^\varepsilon=\psi(\cdot/\varepsilon),\quad \psi^\varepsilon_{j,l}=\psi^\varepsilon(\cdot-\varepsilon x_j)e^{i\langle\cdot,\xi_l\rangle}$$

when $\varepsilon x_j\in\Lambda_1(\varepsilon)$ (i.e., $x_j\in\Lambda_1$) and $\xi_l\in\Lambda_2$. Then the pair

$$(\{\phi_{j,l}\}_{j,l\in J},\{\psi_{j,l}\}_{j,l\in J}) \tag{4.6}$$

is called a Gabor pair with respect to the lattices Λ_1 and Λ_2 if for each $\varepsilon\in(0,1]$, the sets $\{\phi^\varepsilon_{j,l}\}_{j,l\in J}$ and $\{\psi^\varepsilon_{j,l}\}_{j,l\in J}$ are dual Gabor frames.

By Definition 4.1 and Chapters 5–13 in [**9**] it follows that if $f \in (\mathcal{S}^{(s)})'(\mathbf{R}^d)$ and if $(\{\phi_{j,l}\}_{j,l\in J}, \{\psi_{j,l}\}_{j,l\in J})$ is a Gabor pair, then

$$f = \sum_{j,l\in J} c_{j,l}(\varepsilon)\phi^{\varepsilon}_{j,l} \tag{4.4$'$}$$

in $(\mathcal{S}^{(s)})'(\mathbf{R}^d)$, for every $\varepsilon \in (0,1]$, where

$$c_{j,l}(\varepsilon) = (f, \psi^{\varepsilon}_{j,l}). \tag{4.5$'$}$$

We remark that if the pair in (4.6) is a Gabor pair, then it follows from the investigations in [**9**] that the lattice pair (Λ_1, Λ_2) in Definition 4.1 is strongly admissible.

The following proposition explains that any pair of dual Gabor frames satisfying a mild additional condition, generates a Gabor pair. The proof can be found in [**14**].

Proposition 4.1. *Let $\phi, \psi \in C_0^\infty(\mathbf{R}^d)$ be nonnegative functions and let $\phi_{j,l}$ and $\psi_{j,l}$ be given by* (4.3). *Also, let Λ_1 and Λ_2 be the same as in Definition* 4.1. *If $\{\phi_{j,l}\}_{j,l\in J}$ and $\{\psi_{j,l}\}_{j,l\in J}$ are dual Gabor frames such that*

$$\sum_{x_j\in\Lambda_1} \phi(\cdot - x_j)\psi(\cdot - x_j) = \|\Lambda_1\|^{-1}, \tag{4.7}$$

holds, then (4.6) *is a Gabor pair.*

Remark 4.1. If $\phi = \psi$, then (4.7) describes the tight frame property of the corresponding Gabor frame, cf. [**9**, Theorem 6.4.1].

Remark 4.2. Let $p, q \in [1,\infty]$, $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, and $f \in (\mathcal{E}^{(s)})'(\mathbf{R}^d)$. If $(\{\phi_{j,l}\}_{j,l\in J}, \{\psi_{j,l}\}_{j,l\in J})$ is a Gabor pair such that (4.4) and (4.5) hold, then it follows that $f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ if and only if

$$\|f\|_{[\varepsilon]} \equiv \left(\sum_{l\in J} \left(\sum_{j\in J} |c_{j,l}(\varepsilon)\omega(\varepsilon x_j, \xi_j)|^p \right)^{q/p} \right)^{1/q}$$

is finite for every $\varepsilon \in (0,1]$. Furthermore, for every $\varepsilon \in (0,1]$, the norm $f \mapsto \|f\|_{[\varepsilon]}$ is equivalent to the modulation space norm (1.3) (cf. [**2**, **4**, **5**, **9**].)

4.3. Discrete versions of wave-front sets with respect to Fourier–Lebesgue and modulation spaces. We start with two definitions.

Definition 4.2. Let $s > 1$, $q \in [1,\infty]$, $f \in (\mathcal{S}^{(s)})'(\mathbf{R}^d)$, X be an open subset of $\mathbf{R}^d$, $(x_0, \xi_0) \in X \times (\mathbf{R}^d \setminus 0)$, $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^d)$ and $\mathcal{B} = \mathscr{F}L^q_{(\omega)}$. The point (x_0, ξ_0) is called *discretely regular* with respect to $\mathcal{B}$ if

$$|\varphi f|^{(D)}_{\mathcal{B}(\Gamma\cap\Lambda_2)} < \infty,$$

for some choice of strongly admissible lattice pair (Λ_1, Λ_2) such that $x_0 \notin \Lambda_1$, an open conical neighborhood Γ of ξ_0, and some choice of $\varphi \in \mathcal{D}^{(s)}(X)$ such that $\varphi(x_0) \neq 0$.

For the definition of discrete wave-front sets of modulation spaces, we consider Gabor pairs $(\{\phi_{j,l}\}_{j,l\in J}, \{\psi_{j,l}\}_{j,l\in J})$, and let $J_{x_0}(\varepsilon) = J_{x_0}(\varepsilon,\phi,\psi) = J_{x_0}(\varepsilon,\phi,\psi,\Lambda_1)$ be the set of all $j\in J$ such that $x_0 \in \operatorname{supp}\phi^{\varepsilon}_{j,l}$ or $x_0\in\operatorname{supp}\psi^{\varepsilon}_{j,l}$.

Definition 4.3. Let $s>1$, $p,q\in[1,\infty]$, $f\in(\mathcal S^{(s)})'(\mathbf R^d)$, X be an open subset of $\mathbf R^d$, $(x_0,\xi_0)\in X\times(\mathbf R^d\smallsetminus 0)$, $\omega\in\mathscr M_{\{s\}}(\mathbf R^{2d})$ and $\mathcal C = M^{p,q}_{(\omega)}(\mathbf R^d)$. Also let $\Lambda_1,\Lambda_2\subseteq\mathbf R^d$ be lattices such that $x_0\notin\Lambda_1$, $\phi,\psi\in\mathcal D^{(s)}(\mathbf R^d)$ be such that $(\{\phi_{j,l}\}_{j,l\in J}, \{\psi_{j,l}\}_{j,l\in J})$ is a Gabor pair (with respect to Λ_1 and Λ_2), and let $c_{j,l}(\varepsilon)$ be the same as in (4.5)$'$. The point (x_0,ξ_0) is called *discretely regular* with respect to $\mathcal C$ if

$$\Bigg(\sum_{\xi_l\in\Gamma\cap\Lambda_2}\Bigg(\sum_{j\in J_{x_0}(\varepsilon)}|c_{j,l}(\varepsilon)\omega(\xi_l)|^p\Bigg)^{q/p}\Bigg)^{1/q}<\infty,$$

for some $\varepsilon\in(0,1]$, an open conical neighborhood Γ of ξ_0, and for some $\varphi\in\mathcal D^{(s)}(X)$ such that $\varphi(x_0)\neq 0$.

The discrete wave-front set of f with respect to $\mathcal C$ ($\mathcal B$), denoted $\mathrm{DF}_{\mathcal C}(f)$ ($\mathrm{DF}_{\mathcal B}(f)$), consists of all $(x_0,\xi_0)\in X\times(\mathbf R^d\smallsetminus 0)$, where f is not discretely regular with respect to $\mathcal C$ ($\mathcal B$).

Roughly speaking, $(x_0,\xi_0)\in\mathrm{DF}_{\mathcal C}(f)$ means that f is not locally in $\mathcal C$, in the direction ξ_0. The following result shows that our wave-front sets coincide.

Theorem 4.2. *Let $s>1$, $X\subseteq\mathbf R^d$ be open and let $f\in(\mathcal D^{(s)})'(X)$. Then*

$$\mathrm{WF}_{\mathcal B}(f)=\mathrm{WF}_{\mathcal C}(f)=\mathrm{DF}_{\mathcal B}(f)=\mathrm{DF}_{\mathcal C}(f). \tag{4.8}$$

Proof. By Theorem 3.1 and Lemmas 4.1 and 4.2, it follows that the first two equalities in (4.8) hold. The result therefore follows if we prove that $\mathrm{DF}_{\mathcal B}(f)=\mathrm{DF}_{\mathcal C}(f)$.

First assume that $(x_0,\xi_0)\notin\mathrm{DF}_{\mathcal B}(f)$, and choose $\varphi\in\mathcal D^{(s)}(X)$, an open neighbourhood $X_0\subset\overline{X_0}\subset X$ of x_0 and conical neighbourhoods Γ,Γ_0 of ξ_0 such that

- $\overline{\Gamma_0}\subseteq\Gamma$, $\quad\varphi(x)=1\quad$ when $\quad x\in X_0$,
- $|\varphi f|^{(D)}_{\mathcal B(H)}<\infty$, $\quad$ when $\quad H=\Lambda_2\cap\Gamma$.

Now let $(\{\phi_{j,l}\}_{j,l\in J}, \{\psi_{j,l}\}_{j,l\in J})$ be a Gabor pair and choose $\varepsilon\in(0,1]$ such that $\operatorname{supp}\phi^{\varepsilon}_{j,l}$ and $\operatorname{supp}\psi^{\varepsilon}_{j,l}$ are contained in X_0 when $x_0\in\operatorname{supp}\phi^{\varepsilon}_{j,l}$ and $x_0\in\operatorname{supp}\psi^{\varepsilon}_{j,l}$. Since

$$c_{j,l}(\varepsilon)=C(f,\psi^{\varepsilon}_{j,l})_{L^2(\mathbf R^d)}=\mathscr F(f\,\psi(\cdot/\varepsilon-x_j))(\xi_l),$$

it follows from these support properties that if $H_0=\Lambda_2\cap\Gamma_0$, then

$$\Bigg(\sum_{\xi_l\in H_0}|\mathscr F(f\,\psi(\cdot/\varepsilon-x_j))(\xi_l)\omega(\xi_l)|^q\Bigg)^{1/q}=|f\,\psi(\cdot/\varepsilon-x_j)|^{(D)}_{\mathcal B(H_0)} \tag{4.9}$$
$$=|f\,\varphi\psi(\cdot/\varepsilon-x_j)|^{(D)}_{\mathcal B(H_0)},$$

when $j \in J_{x_0}(\varepsilon)$. Hence, by combining Theorem 4.1 with the facts that $J_{x_0}(\varepsilon)$ is finite and $|\varphi\, f|^{(D)}_{\mathcal{B}(H)} < \infty$, it follows that the expressions in (4.9) are finite and

$$\Big(\sum_{\xi_l \in H_0} \Big(\sum_{j \in J_{x_0}(\varepsilon)} |\mathscr{F}(f\, \psi(\cdot/\varepsilon - x_j))(\xi_l)\omega(\xi_l)|^p \Big)^{q/p} \Big)^{1/q} < \infty .$$

This implies that $(x_0,\xi_0) \notin \mathrm{DF}_{\mathcal{C}}(f)$, and we have proved that $\mathrm{DF}_{\mathcal{C}}(f) \subseteq \mathrm{DF}_{\mathcal{B}}(f)$.

In order to prove the reverse inclusion we assume that $(x_0,\xi_0) \notin \mathrm{DF}_{\mathcal{C}}(f)$, and we choose $\varepsilon \in (0,1]$, Gabor pair $(\{\phi_{j,l}\}_{j,l\in J}, \{\psi_{j,l}\}_{j,l\in J})$ and conical neighbourhoods Γ, Γ_0 of ξ_0 such that $\overline{\Gamma_0} \subseteq \Gamma$ and

$$\Big(\sum_{\xi_l \in H} \Big(\sum_{j \in J_{x_0}(\varepsilon)} |\mathscr{F}(f\, \psi(\cdot/\varepsilon - x_j))(\xi_l)\omega(\xi_l)|^p \Big)^{q/p} \Big)^{1/q} < \infty , \tag{4.10}$$

when $H = \Lambda_2 \cap \Gamma$. Also choose $\varphi, \kappa \in \mathcal{D}^{(s)}(X)$ such that $\varphi(x_0) \neq 0$ and

$$\kappa(x) \sum_{j \in J_{x_0}(\varepsilon)} \psi(x/\varepsilon - x_j) = 1, \quad \text{when} \quad x \in \operatorname{supp} \varphi .$$

Since $J_{x_0}(\varepsilon)$ is finite, Hölder's inequality gives

$$\begin{aligned} |\varphi\, f|^{(D)}_{\mathcal{B}(H_0)} &= \Big| \sum_{j \in J_{x_0}(\varepsilon)} (\varphi\kappa)\,(f\, \psi(\cdot/\varepsilon - x_j)) \Big|^{(D)}_{\mathcal{B}(H_0)} \\ &\leqslant \Big(\sum_{\xi_l \in H_0} \Big(\sum_{j \in J_{x_0}(\varepsilon)} |\mathscr{F}((\varphi\kappa) f\, \psi(\cdot/\varepsilon - x_j))(\xi_l)\omega(\xi_l)| \Big)^{q} \Big)^{1/q} \\ &\lesssim \Big(\sum_{\xi_l \in H_0} \Big(\sum_{j \in J_{x_0}(\varepsilon)} |\mathscr{F}((\varphi\kappa) f\, \psi(\cdot/\varepsilon - x_j))(\xi_l)\omega(\xi_l)|^p \Big)^{q/p} \Big)^{1/q} , \end{aligned}$$

where $H_0 = \Lambda_2 \cap \Gamma_0$. By Theorem 4.1 and (4.10) it now follows that the right-hand side in the last estimates is finite. Hence, $|\varphi\, f|^{(D)}_{\mathcal{B}(H_0)} < \infty$, which shows that $(x_0,\xi_0) \notin \mathrm{DF}_{\mathcal{B}}(f)$, and we have proved that $\mathrm{DF}_{\mathcal{B}}(f) \subseteq \mathrm{DF}_{\mathcal{C}}(f)$. The proof is complete. □

We may define discrete versions, $\mathrm{DF}^{\inf}_{(B_j)}(f)$ and $\mathrm{DF}^{\inf}_{(B_j)}(f)$, of the wave-front sets $\mathrm{WF}^{\inf}_{(B_j)}(f)$ and $\mathrm{WF}^{\inf}_{(B_j)}(f)$ of sequence types, as it is done in [**14**]. Then it follows that Theorem 4.2 can be extended to involve such wave-front sets. Hence Proposition 2.1 is still true, after $\mathrm{WF}^{\inf}_{(B_j)}(f)$ and $\mathrm{WF}^{\inf}_{(B_j)}(f)$ are replaced by $\mathrm{DF}^{\inf}_{(B_j)}(f)$ and $\mathrm{DF}^{\inf}_{(B_j)}(f)$, respectively. In particular we obtain the following discrete interpretation of the wave-front set $\mathrm{WF}_{\{s\}}(f)$.

Corollary 4.1. *Let $q \in [1,\infty]$, $s > 1$, and let $\omega_k(\xi) \equiv e^{k|\xi|^{1/s}}$, $\xi \in \mathbf{R}^d$, $k > 0$. If $f \in (\mathcal{D}^{(s)})'(\mathbf{R}^d)$, then*

$$\bigcap_{k>0} \mathrm{DF}_{\mathscr{F}L^q_{(\omega_k)}}(f) = \mathrm{WF}_{\{s\}}(f).$$

We remark that a discrete analogue of $\mathrm{WF}_{\{s\}}(f)$ also can be introduced in a similar way as in [**24**, **25**]. Let us denote this set by $\mathrm{WF}_{s,T}(f)$, and refer to it as *toroidal* s-wave-front set. It can be proved that

$$\mathrm{WF}_{s,T}(f) = \mathbf{T}^d \times \mathbf{Z}^d \cap \mathrm{WF}_{\{s\}}(f),$$

where $\mathbf{T}^d$ is the torus in $\mathbf{R}^d$.

A significant difference between the toroidal wave-front sets and our discrete wave-front sets lies in the fact that $\mathrm{WF}_{s,T}(f)$ only informs about the *rational* directions for the propagation of singularities of f at a certain point, while $\mathrm{DF}(f) = \mathrm{WF}(f)$ takes care of *all* directional for f to that point, we refer to [**14**] for an example.

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