PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 91(105) (2012), 63–81 DOI: 10.2298/PIM1205063J

ON THE PRINCIPLE OF STATIONARY ISOENERGETIC ACTION

Božidar Jovanović

Abstract. We present several variants of the Maupertuis principle, both on the exact and the nonexact symplectic manifolds.

1. Introduction

1.1. The principle of least action, or the principle of stationary action, says that the trajectories of a mechanical system can be obtained as extremals of a certain action functional. It is one of the basic tools in physics being applied both in classical and quantum setting.

Consider a Lagrangian system (Q, L), where Q is a configuration space and $L(q, \dot{q}, t)$ is a Lagrangian, $L : TQ \times \mathbb{R} \to \mathbb{R}$. Let $q = (q_1, \dots, q_n)$ be local coordinates on Q. The motion of the system is described by the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}, \quad i = 1, \dots, n. \tag{1.1}$$

The solutions of the Euler–Lagrange equations are exactly the critical points of the action integral

$$S_L(\gamma) = \int_a^b L(q, \dot{q}, t)\, dt$$

in a class of curves $\gamma : [a, b] \to Q$ with fixed endpoints $\gamma(a) = q_0$, $\gamma(b) = q_1$ (the *Hamiltonian principle of least action* (1834), e.g., see [**28**]).

The Legendre transformation $\mathbb{F}L : TQ \to T^*Q$ is defined by

$$\mathbb{F}L(q, \xi, t) \cdot \eta = \frac{d}{ds}\Big|_{s=0} L(q, \xi + s\eta, t) \iff p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, \dots, n, \tag{1.2}$$

where $\xi, \eta \in T_qQ$ and $(q_1, \dots, q_n, p_1, \dots, p_n)$ are canonical coordinates of the cotangent bundle T^*Q. In order to have a Hamiltonian description of the dynamics (see the section below), we suppose that the Legendre transformation (1.2) is a diffeomorphism. The corresponding Lagrangian L is called *hyperregular* [**21**].

2010 *Mathematics Subject Classification:* Primary 37J05, 37J55; Secondary 70H25, 70H30.
Dedicated to the memory of Academician Anton Bilimović (1879–1970).

If the Lagrangian L does not depend on time then the equations (1.1) possess the energy first integral

$$E(q,\dot q) = \mathbb{F}L(q,\dot q)\cdot \dot q - L(q,\dot q) = \sum_i \frac{\partial L}{\partial \dot q_i}\dot q_i - L. \tag{1.3}$$

In that case we have

THEOREM 1.1 (the Maupertuis principle). *Suppose that h is a regular value of E. Among all curves $q=\gamma(\tau)$ connecting two points q_0 and q_1 and parametrized so that the energy has a fixed value $E=h$, the trajectory of the equations of dynamics* (1.1) *is an extremal of the reduced action*

$$S(\gamma) = \int_a^b \mathbb{F}L\left(q(\tau),\frac{dq}{d\tau}\right)\cdot\frac{dq}{d\tau}\,d\tau = \int_a^b \frac{\partial L}{\partial \dot q}(\tau)\cdot\frac{dq}{d\tau}\,d\tau, \quad q_0=\gamma(a),\, q_1=\gamma(b) \tag{1.4}$$

It is important to note that the interval $[a,b]$, parametrizing the curve $q=\gamma(\tau)$, is not fixed and it can be different for different curves being compared, while the energy must be the same.

Contrary to the Hamiltonian principle, the *Maupertuis principle*, or *principle of stationary isoenergetic action* determines the shape of a trajectory but not the time. In order to determine the time, we have to use the energy constant.

Historically, a variant of Theorem 1.2 was the first variational approach to mechanics. It is attributed to Maupertuis (1744), Euler (1744) and Jacobi (1842), who gave an important geometric interpretation of the principle (see [**28**]).

1.2. The classical proofs of the Maupertuis principle can be found in [**28, 36, 2**]. In Serbian, see the second volume of Bilimović's course in Theoretical mechanics [**4**], or Dragović and Milinković's monograph [**10**].

Weinstein [**34**] and Novikov [**25**] formulated multi-valued variational principles that provided the study of the existence of periodic orbits on non exact symplectic manifolds. We feel a need to present these results, along with the classical ones, in a unified way.

In the first part of the paper, we derive the principle of stationary isoenergetic action, both on the exact (Section 2) and the nonexact symplectic manifolds (Section 3). The variants of the Maupertuis principle presented in Section 3 are our small contribution to the subject. They slightly differ from the existing variational principles formulated either for closed trajectories, or formulated without imposing the constraint given by the energy.

In the second part of the paper we point out a contact interpretation of the Maupertuis principle (Sections 4, 5). There, it is illustrated how some of the well known properties of the system of harmonic oscillators, the Kepler problem (Moser's regularization) and the Neumann system (relationship with a geodesic flow on an ellipsoid), have natural descriptions within a framework of the contact geometry. We believe that one should expect other interesting relations between the contact structures and integrable systems as well.

It is a great pleasure to dedicate this paper to Anton Bilimović, since his work has fundamentally influenced the development of Serbian theoretical mechanics.

2. Principle of stationary isoenergetic action in a phase space

2.1. Hamiltonian equations. Let $L(q,\dot q,t)$ be a hyperregular Lagrangian. We can pass from velocities $\dot q_i$ to the momenta p_j by using the standard Legendre transformation (1.2). In the coordinates (q,p) of the cotangent bundle T^*Q, the equations of motion (1.1) read:

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i},\qquad \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i},\qquad i=1,\dots,n, \tag{2.1}$$

where the Hamiltonian function $H(q,p,t)$ is the *Legendre transformation* of L

$$H(q,p,t)=E(q,\dot q,t)|_{\dot q=\mathbb{F}L^{-1}(q,p,t)}=\mathbb{F}L(q,\dot q,t)\cdot\dot q-L(q,\dot q,t)|_{\dot q=\mathbb{F}L^{-1}(q,p,t)}.$$

Let $p\,dq=\sum_i p_i dq_i$ be the *canonical 1-form* and

$$\omega=d(p\,dq)=dp\wedge dq=\sum_{i=1}^n dp_i\wedge dq_i$$

the *canonical symplectic* form of the cotangent bundle T^*Q. The system of equations (2.1) is Hamiltonian, that is the vector field

$$X_H(q,p)=(\partial H/\partial p_1,\dots,\partial H/\partial p_n,-\partial H/\partial q_1,\dots,-\partial H/\partial q_n)$$

can be defined by

$$i_{X_H}\omega(\,\cdot\,)=\omega(X_H,\,\cdot\,)=-dH(\,\cdot\,). \tag{2.2}$$

2.2. Characteristic line bundles. More generally, consider a $2n$-dimensional symplectic manifold P with a closed, nondegenerate 2-form ω. Let $H:P\times\mathbb{R}\to\mathbb{R}$ be a smooth, in general time dependent, function. Consider the corresponding Hamiltonian equation

$$\dot x=X_H, \tag{2.3}$$

where the Hamiltonain vector field $X_H(x,t)$ is defined by (2.2).

If the Hamiltonian H does not depend on time, it is the first integral of the system. Let M be a regular connected component of the invariant variety $H=h$, which means $dH|_M\neq 0$.

Since $dH(\xi)=0$, $\xi\in T_xM$, from (2.2) we see that X_H generates the symplectic orthogonal of T_xM for all $x\in M$ — the *characteristic line bundle* $\mathcal{L}_M$ of M. It is the kernel of the form ω restricted to M:

$$\mathcal{L}_M=\{\xi\in T_xM\,|\,\omega(\xi,T_xM)=0,\ x\in M\}.$$

Note that $\mathcal{L}_M$ is determined only by M and not by H. If F is another Hamiltonian defining M, $M\subset F^{-1}(c)$, $dF|_M\neq 0$, then the restrictions of the Hamiltonian vector fields X_H and X_F to M are proportional.

A *variation* of a curve $\gamma:[a,b]\to M$ is a mapping: $\Gamma:[a,b]\times[0,\epsilon]\to M$, such that $\gamma(t)=\Gamma(t,0)$, $t\in[a,b]$. Denote $\gamma_s(t)=\Gamma(t,s)$ and $\delta\gamma(t)=\frac{d}{ds}|_{s=0}\gamma_s(t)\in T_{\gamma(t)}M$.

From Cartan's formula we get (e.g., see Griffits [**12**]):

Lemma 2.1. *Let (M, α) be a manifold endowed with a 1-form α, $\gamma : [a, b] \to M$ be an immersed curve and Γ be a variation of γ. The Lie derivative of the form $\Gamma^*\alpha$ in the direction of $\partial/\partial s$ at the points $[a, b] \times \{0\}$ is equal to*

$$L_{\partial/\partial s}\Gamma^*\alpha|_{(t,0)} = \gamma^*(i_{\delta\gamma(t)}d\alpha) + d\gamma^*(\alpha(\delta\gamma(t))).$$

Theorem 2.1. *Assume that the symplectic form ω is exact: $\omega = d\alpha$. Let M be a regular component of the invariant hypersurface $H^{-1}(h)$. The integral curves $\gamma : [a, b] \to M$ of the characteristic line bundle $\mathcal{L}_M$ are extremals of the (reduced) action functional $A(\gamma) = \int_\gamma \alpha = \int_a^b \alpha(\dot\gamma)\, dt$ in the class of variations $\gamma_s(t)$ such that $\alpha(\delta\gamma(a)) = \alpha(\delta\gamma(b)) = 0$.*

The proof is a direct consequence of Lemma 2.1. We have

$$\frac{d}{ds}\Big(\int_{\gamma_s} \alpha\Big)\Big|_{s=0} = \int_a^b \omega\big(\delta\gamma(t), \dot\gamma(t)\big)dt + \alpha(\delta\gamma(b)) - \alpha(\delta\gamma(a)). \tag{2.4}$$

The expression above is equal to zero for all variations $\gamma_s(t)$ if and only if $\dot\gamma$ is in the kernel of the form $\omega = d\alpha$ restricted to M. That is, $\gamma(t)$ is an integral curve of the line bundle $\mathcal{L}_M$.

2.3. Applying Theorem 2.1 to the symplectic space $(T^*Q, dp \wedge dq)$ we obtain Poincaré's formulation of the Maupertuis principle in a phase space [**27**].

Theorem 2.2. *If the Hamiltonian function $H = H(q, p)$ does not depend on time, then the phase trajectories of the canonical equations* (2.1) *lying on the regular connected component M of the surface $\{H(q, p) = h\}$ are extremals of the reduced action*

$$A(\gamma) = \int_\gamma p\, dq \tag{2.5}$$

in the class of curves γ lying on M and connecting the subspaces $T^*_{q_0}Q$ and $T^*_{q_1}Q$.

Note that Theorem 1.1 follows from Theorem 2.2 (e.g., see Arnold [**2**]). Suppose that the Hamiltonian system (2.1) is a Legendre transformation of the Lagrangian system (1.1). The main observation is that if $\underline{\gamma}(\tau)$ is a configuration space curve parametrized such that $E(\underline{\gamma}, d\underline{\gamma}/d\tau) = h$, then the lifted curve $\gamma = \mathbb{F}L(\underline{\gamma}, d\underline{\gamma}/d\tau)$ lyes on M and the reduced actions (1.4) and (2.5) for $\underline{\gamma}$ and γ are equal: $S(\underline{\gamma}) = A(\gamma)$ (see Fig. 1).

2.4. Jacobi's metric. Consider a natural mechanical system on Q defined by the Lagrangian function:

$$L(q, \dot q) = T + B - V = \frac{1}{2}\sum_{ij} K_{ij}\dot q_i \dot q_j + \sum_i B_i \dot q_i - V(q). \tag{2.6}$$

Here $ds^2 = \sum_{ij} K_{ij} dq_i dq_j$ is a Riemannian metric on Q, $V(q)$ is a potential function and $\theta = \sum_i B_i dq_i$ is a 1-form defining a gyroscopic (or magnetic) field $\sigma = d\theta$ (see Section 3).

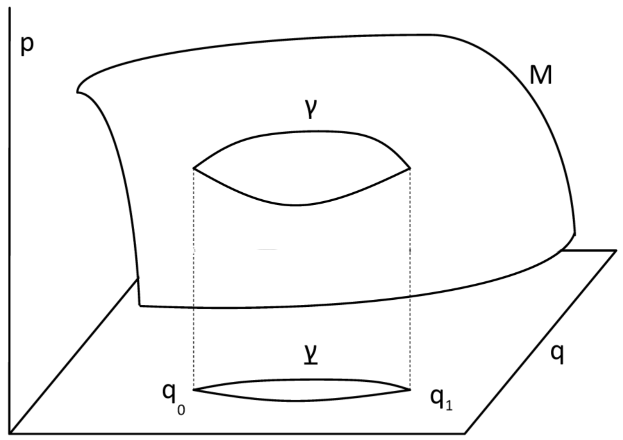

FIGURE 1.

The energy of the system (1.3) is the sum of the kinetic and the potential energy

$$E(q, \dot{q}) = T + V = \frac{1}{2} \sum_{ij} K_{ij} \dot{q}_i \dot{q}_j + V(q).$$

In the region of the configuration space Q_h where $V(q) < h$, we can define the *Jacobi metric*

$$ds_J^2 = 2(h - V(q))\, ds^2 = 2(h - V(q)) \sum_{ij} K_{ij} dq_i dq_j. \tag{2.7}$$

The following version of the Maupertuis principle for Lagrangians of the form (2.6) is well known (e.g., see Kozlov [**19**]).

THEOREM 2.3. *Among all curves $q = \gamma(\tau)$ connecting the points $q_0, q_1 \in Q_h$ and parametrized so that the energy has a fixed value $E = h$, the trajectory of the equations of dynamics* (1.1) *with Lagrangian* (2.6) *is an extremal of the integral*

$$S(\gamma) = \int_\gamma ds_J + \theta. \tag{2.8}$$

In particular, if there are no gyroscopic forces, the trajectories of the system within Q_h, up to reparametrization, are geodesic lines of the Jacobi metric ds_J^2.

Indeed, in order to guarantee a fixed value of the energy

$$E = T + V = \frac{1}{2} \sum_{ij} K_{ij} \frac{dq_i}{d\tau} \frac{dq_j}{d\tau} + V(q) = \frac{1}{2} \left(\frac{ds}{d\tau} \right)^2 + V(q) = h,$$

the parameter τ of the curve $q = \gamma(\tau)$ must be proportional to the length $d\tau = ds/\sqrt{2(h-V)}$. Therefore

$$\int_a^b \frac{\partial L}{\partial \dot q}(\tau)\cdot \frac{dq}{d\tau}\,d\tau = \int_a^b \Big(\sum_{ij} K_{ij}\frac{dq_i}{d\tau}\frac{dq_j}{d\tau} + \sum_i B_i \frac{dq_i}{d\tau}\Big)d\tau$$
$$= \int_a^b \Big(2(h - V(q)) + \sum_i B_i \frac{dq_i}{d\tau}\Big)d\tau = \int_\gamma ds_J + \theta.$$

Remark 2.1. The variational principle stated in Theorem 2.3 is used in the study of periodic trajectories of natural mechanical systems with exact magnetic fields (see [**31**] and references therein). Note also that the Maupertuis principle for a configuration space Q being a Banach space can be found in [**21, 30**].

2.5. The Hamiltonian principle of least action. Consider a *Poincaré–Cartan* 1-form $p\,dq - H dt$ on the extended phase space $T^*Q \times \mathbb{R}(q,p,t)$, where $H : T^*Q\times\mathbb{R} \to \mathbb{R}$ is a Hamiltonian function. The phase trajectories of the canonical equations (2.1) are extremals of the action

$$A_H(\gamma) = \int_\gamma pdq - Hdt \tag{2.9}$$

in the class of curves $\gamma(t) = (q(t), p(t), t)$ connecting the subspaces $T^*_{q_0}Q \times \{t_0\}$ and $T^*_{q_1}Q \times \{t_1\}$ (Poincaré's modification of the Hamiltonian principle of least action [**27**]). Namely, a vector $(\xi, 1)$, $\xi \in T_{(q,p)}(T^*Q)$ belongs to $\ker d(pdq - Hdt)$ at (q, p, t) if and only if $\xi = X_H(q, p, t)$ (see [**2, 21**]).

Obviously, we can replace $(T^*Q, dp \wedge dq)$ by an arbitrary exact symplectic manifold $(P, \omega = d\alpha)$. In particular, if we consider the action $A_H(\gamma) = \int_\gamma \alpha - Hdt$ on the free loop space $\Omega(P) = C^\infty(S^1, P)$, $S^1 = \mathbb{R}/\mathbb{Z}$ of P and H is 1-periodic in t-variable, then the critical points of A_H are 1-periodic orbits of the equation (2.3).

For a given time-independent Hamiltonian $H : P \to \mathbb{R}$ with a regular level set $H^{-1}(h)$, the periodic orbits having all positive periods and energy h can be obtained by the use of modified action:

$$A_{H,h}(\gamma, \lambda) = \int_0^1 \alpha(\dot\gamma)dt - \lambda\int_0^1 (H(\gamma(t)) - h)dt, \tag{2.10}$$

defined on the space $\Omega(P) \times \mathbb{R}^+$ (see [**29, 34**]). The critical points (γ, λ) of $A_{H,h}$ correspond to λ-periodic orbits $x(t) = \gamma(t/\lambda)$ that lie on the energy hypersurface $H^{-1}(h)$. Moreover, Weinstein defined actions A_H and $A_{H,h}$ when the symplectic form is not exact as well [**35**].

The Lagrangian analogue of the functional (2.10) is

$$S_{L,h}(\gamma, \lambda) = \int_0^1 \lambda L(\gamma, \dot\gamma/\lambda)dt + \lambda h, \qquad \gamma \in \Omega(Q),\ \lambda > 0$$

(see [**9**]). The pair (γ, λ) is a critical point of $S_{L,h}$ if and only if $q(t) = \gamma(t/\lambda)$ is a λ-periodic solution of the Euler–Lagrange equation (1.1) with energy h.

Variational principles related to the action (2.9), which arise by a reduction process are given in [**8**].

3. The Maupertuis principle on nonexact symplectic manifolds

3.1. Magnetic flows. Consider a natural mechanical system given by Lagrangian function (2.6). After the Legendre transformation, it takes form (2.1) with the Hamiltonian function

$$H(q,p)=\frac{1}{2}\langle p-\theta,p-\theta\rangle+V(q)=\frac{1}{2}\sum_{ij}K^{ij}(p_i-B_i)(p_j-B_j)+V(q), \tag{3.1}$$

where K^{ij} is the inverse of the metric tensor K_{ij}.

The transformation $T_\theta : (q,p)\mapsto(q,p-\theta)$ is a symplectomorphism between $(T^*Q, dp\wedge dq)$ and a "twisted" cotangent bundle $(T^*Q, dp\wedge dq+\pi^*\sigma)$, where $\pi : T^*Q\to Q$ is the natural projection and $\sigma=d\theta$.

In the new coordinates, also denoted by (q,p), Hamiltonian (3.1) takes the usual form, the sum of the kinetic and the potential energy:

$$H(q,p)=\frac{1}{2}\langle p,p\rangle+V(q)=\frac{1}{2}\sum_{ij}K^{ij}p_ip_j+V(q),$$

while the equations of motion take the "noncanonical" form:

$$\frac{dq^i}{dt}=\frac{\partial H}{\partial p_i},\qquad \frac{dp_i}{dt}=-\frac{\partial H}{\partial q^i}+\sum_{j=1}^n F_{ij}\frac{\partial H}{\partial p_j}, \tag{3.2}$$

where $\sigma=\sum_{1\leqslant i<j\leqslant n}F_{ij}(q)dq_i\wedge dq_j$. The equations are Hamiltonian with respect to the symplectic form $\omega=dp\wedge dq+\pi^*\sigma$.

One can consider system (3.2) associated to a nonexact 2-form σ as well (for example, the motion of a particle in a magnetic monopole field [**21**]). In this case, Lagrangian (2.6) is defined only locally. Nevertheless, it is very interesting that the Hamiltonian (Weinstein [**34**] and Tuynman [**33**]) and the Maupertuis principles (Novikov [**26**]) of least action can be still defined.

3.2. Multivalued reduced action. Let (P,ω) be a non exact symplectic manifold and let $M=H^{-1}(h)$ be a regular isoenergetic hypersurface. The main observation concerning the Maupertuis principle can be stated as follows (see [**26, 19**] for the reduced action (2.8)).

Let $U\subset P$ be a region where ω is exact and let $\omega=d\alpha_1=d\alpha_2$. Consider a variation $\gamma_s(t)=\Gamma(t,s)$, $t\in[0,1], s\in[0,\epsilon]$ with fixed endpoints of a curve γ lying in $M\cap U$. Then

$$\int_{\gamma_\epsilon}\alpha_1-\int_{\gamma_0}\alpha_1=-\int_{[0,1]\times[0,\epsilon]}\Gamma^*d\alpha_1=-\int_{[0,1]\times[0,\epsilon]}\Gamma^*d\alpha_2=\int_{\gamma_\epsilon}\alpha_2-\int_{\gamma_0}\alpha_2.$$

Therefore

$$\int_{\gamma_\epsilon}\alpha_1-\int_{\gamma_0}\alpha_1=\int_{\gamma_\epsilon}\alpha_2-\int_{\gamma_0}\alpha_2$$

and, although $\int_\gamma\alpha_i$ depends on the form α_i, the derivative

$$\frac{d}{ds}|_{s=0}\int_0^1\alpha(\dot\gamma_s)dt=\int_0^1\omega(\delta\gamma(t),\dot\gamma(t))dt$$

does not depend on α_i, $i = 1, 2$. One can define an appropriate multi-valued functional on a space of paths with fixed endpoints, such that an extremal (if exist) is exactly the integral curve of the characteristic foliation on M. However, as in the case of the symplectic homology (see [**13**]), the situation simplifies in the aspherical case which is considered below.

3.3. Aspherical symplectic manifolds. The symplecic manifold (P, ω) is *aspherical* if ω vanishes on $\pi_2(P)$. Of course, if ω is exact or $\pi_2(P) = 0$, then (P, ω) is aspherical.

Consider the equation (2.3), where H does not depend on time. Let M be a regular component of $H^{-1}(h)$ and $c : [0, 1] \to P$ be an immersed curve with endpoints $x_0 = c(0) \in M$ and $x_1 = c(1) \in M$. Define $\Omega^h_c(x_0, x_1)$ as the space of regular paths that are homotopic to c in P:

$$\Omega^h_c(x_0, x_1) = \{\gamma : [0, 1] \to M \mid \gamma(0) = x_0,\ \gamma(1) = x_1, \dot\gamma(t) \neq 0, t \in [0, 1],\ \gamma \sim_P c\}.$$

The space of all regular paths connecting x_0 and x_1 and laying in M is the union $\Omega^h(x_0, x_1) = \bigcup_c \Omega^h_c(x_0, x_1)$, where we take representatives c for all nonhomotopic paths (in P) connecting x_0 and x_1.

If we suppose that (P, ω) is simplectically aspherical then we can define a single-valued *reduced action*:

$$A : \Omega^h(x_0, x_1) \to \mathbb{R}, \qquad A(\gamma)|_{\Omega^h_c(x_0,x_1)} = \int_D f^*_\gamma \omega, \tag{3.3}$$

where $D = \{z \mid z \in \mathbb{C},\ |z| \leqslant 1\}$ is the unit disk, $f_\gamma : D \to P$ is an arbitrary mapping that is smooth for $|z| < 1$, continuous on D and $\gamma(t) = f_\gamma(\exp(\sqrt{-1}\pi t))$, $c(t) = f_\gamma\big(\exp\big(\sqrt{-1}\pi(2 - t)\big)\big)$, $t \in [0, 1]$. That is, $f(D)$ is a surface with the boundary $\partial D = \gamma \cdot c^{-1}$.

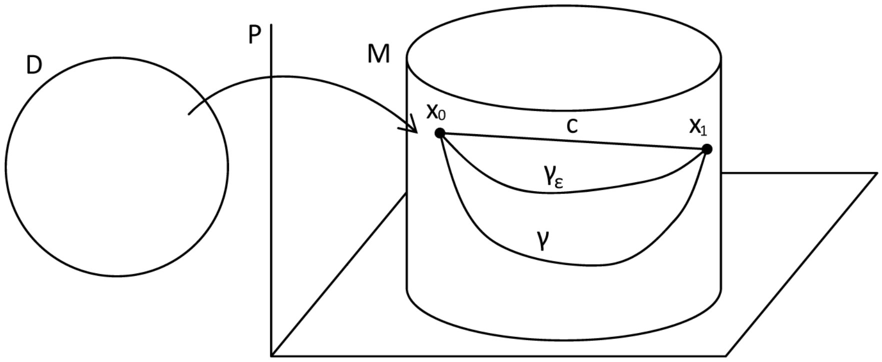

FIGURE 2.

Since $\gamma \sim_P c$ we can always find a mapping f with required properties. From $\omega|_{\pi_2(P)} = 0$, the value $A(\gamma)$ does not depend on the choice of f.

THEOREM 3.1. *The integral curves* $\gamma : [0,1] \to M$ *of the characteristic line bundle* $\mathcal{L}_M$ *that connect* x_0 *and* x_1 *are extremals of the reduced action* (3.3).

PROOF. Consider a variation $\gamma_s(t) = \Gamma(t,s)$, $t \in [0,1]$, $s \in [0,\epsilon]$ of γ lying in M. By using $\omega|_{\pi_2(P)} = 0$, we get

$$A(\gamma_\epsilon) - A(\gamma) = \int_D (f^*_{\gamma_\epsilon}\omega - f^*_\gamma\omega) = -\int_{[0,1]\times[0,s]} \Gamma^*\omega = \int_0^1\int_0^\epsilon \omega\Big(\frac{\partial\Gamma}{\partial s}, \frac{\partial\Gamma}{\partial t}\Big) dt\, ds.$$

Thus, as above,

$$\frac{d}{ds}\Big|_{s=0} A(\gamma_s) = \int_0^1 \omega\big(\delta\gamma(t), \dot\gamma(t)\big)\, dt,$$

is zero for all variations $\gamma_s(t)$ if and only if the velocity vector field $\dot\gamma(t)$ is a section of $\ker\omega|_M$. $\square$

3.4. A torus valued reduced action. Tuynaman proposed a torus-valued action, such that multi-valued Poincaré action (2.9) can be seen as a composition of a multi-valued function on a torus and a torus-valued action [**33**]. In this subsection we follow Tuynman's construction [**33**] in order to formulate the principle of stationary isoenergetic action.

Consider a manifold P with a symplectic 2-form $\omega = \sum_{a=1}^n \mu_a\beta^a$, where β^a are 2-forms, representing integrals cohomology classes. We take the decomposition with minimal n. Then the parameters μ_a are independent over $\mathbb{Q}$, in particular $\mu = \mu_1 + \cdots + \mu_n \neq 0$. To ω we associate the 1-form $\lambda = \sum_{a=1}^n \mu_a dy^a$ on a torus $\mathbb{T}^n = \{(\exp(\sqrt{-1}y^1), \dots, \exp(\sqrt{-1}y^n)\}$. It can be consider as a differential of a multi-valued function Λ on $\mathbb{T}^n$: $\lambda = d\Lambda$. Also, for $a = 1, \dots, n$, let us define principal S^1-bundles

$$\begin{array}{ccc} S^1 & \longrightarrow & Y_a \\ & & \big\downarrow{\scriptstyle \rho_a} \\ & & P \end{array}$$

having the connections θ^a with the curvature forms β^a (see Kobayashi [**18**]).

Let $\gamma(t)$, $t \in [t_0, t_1]$ be a piece-wise smooth, closed curve on P. Recall, a piece-wise smooth curve $\tilde\gamma^a(t) \subset Y_a$ is a *horizontal lift* of γ if $\rho_a \circ \tilde\gamma^a(t) = \gamma(t)$ and $\theta^a(\frac{d}{dt}\tilde\gamma^a(t)) = 0$, whenever the velocity vector is defined. The *holonomy* $\mathrm{Hol}^a(\gamma)$ is an element $g \in S^1$, such that $g \cdot \tilde\gamma^a(t_0) = \tilde\gamma^a(t_1)$.

LEMMA 3.1. [**33**] *Let* $\gamma_s(t) = \Gamma(t,s)$ *be a variation of* $\gamma : [0,1] \to P$ *with fixed endpoints and let* $c : [0,1] \to P$ *be an arbitrary curve connecting* $x_0 = \gamma(0)$ *and* $x_1 = \gamma(1)$. *We have a family of closed orbits* $\bar\gamma_s = \gamma_s \cdot c^{-1}$. *The derivative of* $\mathrm{Hol}^a(\bar\gamma_s)$ *is given by:*

$$\frac{d\,\mathrm{Hol}^a(\bar\gamma_s)}{ds}\Big|_{s=0} = \int_0^1 \beta^a\big(\dot\gamma(t), \delta\gamma(t)\big) dt \cdot \frac{\partial}{\partial y^a}.$$

Consider the equation (2.3), where $\partial H/\partial t = 0$. Let M be a regular component of $H^{-1}(h)$ and $\Omega^h(x_0, x_1)$ be a space of regular paths $\gamma : [0,1] \to M$ that connect points x_1 and x_2.

For every $\gamma \in \Omega^h(x_0, x_1)$ define a picewise smooth, closed path $\bar{\gamma} = \gamma \cdot c^{-1} : [0,2] \to M$, where $c \in \Omega^h(x_0, x_1)$ is fixed. We call

$$A_{\mathbb{T}^n} : \Omega^h(x_0, x_1) \longrightarrow \mathbb{T}^n, \quad \gamma \longmapsto (\mathrm{Hol}^1(\bar{\gamma}), \dots, \mathrm{Hol}^n(\bar{\gamma}))$$

a *torus valued reduced action.* From Lemma 3.1 we have:

$$\lambda\Big(\frac{d}{ds} A_{\mathbb{T}^n}(\gamma_s)\big|_{s=0}\Big) = \sum_{a=1}^n \mu_a \int_0^1 \beta^a(\dot{\gamma}(t), \delta\gamma(t))\, dt = \int_0^1 \omega(\dot{\gamma}(t), \delta\gamma(t))\, dt.$$

whence, we obtain the following principle of stationary isoenergetic action on the nonexact symplectic manifolds.

Theorem 3.2. *A curve $\gamma \in \Omega^h(x_0, x_1)$ is an integral curve of the characteristic line bundle $\mathcal{L}_M$ if and only if*

$$\frac{d}{ds}\Big(\Lambda \circ A_{\mathbb{T}^n}(\gamma_s)\Big)\Big|_{s=0} = \lambda\Big(\frac{d}{ds} A_{\mathbb{T}^n}(\gamma_s)\big|_{s=0}\Big) = 0$$

for all variations $\gamma_s \in \Omega^h(x_0, x_1)$.

For the completeness of the exposition we include:

Proof of Lemma 3.1. In local trivializations

$$\rho^{-1}(U_i) \cong U_i \times S^1(x_i, y_i \bmod 2\pi),$$

we have local connection 1-forms α_i on U_i such that $\theta = \alpha_i + dy_i$ (the index a is omitted). The transition functions between fiber coordinates and connection 1-forms are given by

$$y_j = y_i + g_{ij}(x), \quad \alpha_i = \alpha_j + dg_{ij}, \quad g^a_{ij} : U_i \cap U_j \to S^1. \tag{3.4}$$

On the other hand, the curvature 2-form is invariant: $\beta = d\alpha_i = d\alpha_j$.

Suppose $\gamma_s([t_0, t_1]) \subset U_i$, $s \in [0, \epsilon]$. The local expression for $\tilde{\gamma}_s$ reads

$$\tilde{\gamma}_s(t) = (\gamma_s(t), y_i(t,s)), \quad \theta_i\Big(\frac{d}{dt}\tilde{\gamma}(t)\Big) = 0 \Longleftrightarrow \alpha_i(\dot{\gamma}(t)) + \dot{y}_i(t,s) = 0.$$

Therefore

$$y_i(t_1, s) = y_i(t_0, s) - \int_{t_0}^{t_1} \alpha_i(\dot{\gamma}_s(t))\, dt. \tag{3.5}$$

By taking the differential of (3.5) at $s = 0$ and applying (2.4) we get

$$\delta y_i(t_1) + \alpha_i(\delta\gamma(t_1)) = \delta y_i(t_0) + \alpha_i(\delta\gamma(t_0)) + \int_{t_0}^{t_1} \beta(\dot{\gamma}(t), \delta\gamma(t))\, dt, \tag{3.6}$$

where $\delta_i y(t) = \frac{d}{ds} y_i(t,s)|_{s=0}$, $\delta\gamma(t) = \frac{d}{ds}\gamma_s(t)|_{s=0}$.

Now, assume $t_0 < t_0' < t_1 < t_1'$, $\gamma_s([t_0, t_1]) \subset U_i$ and $\gamma_s([t_0', t_1']) \subset U_j$. The transformations (3.4) imply

$$\delta y_i(t) + \alpha_i(\delta\gamma(t)) = \delta y_j(t) + \alpha_j(\delta\gamma(t)), \qquad t \in [t_0', t_1]. \tag{3.7}$$

By combining (3.6) and (3.7), it follows

$$\delta y_j(t_1') + \alpha_j(\delta\gamma(t_1')) = \delta y_i(t_0) + \alpha_i(\delta\gamma(t_0)) + \int_{t_0}^{t_1'} \beta(\dot{\gamma}(t), \delta\gamma(t))\, dt. \tag{3.8}$$

Let $\bar{\gamma}_s = \gamma_s \cdot c^{-1} : [0,2] \to M$ and let $U_1, \dots, U_l$ be local charts, such that

$$\bar{\gamma}_s([t_{i-1}, t_i]) \subset U_i, \quad 0 = t_0 < t_1 < \cdots < t_k = 1 < t_{k+1} < \cdots < t_l = 2, \quad s \in [0, \epsilon].$$

From the relation (3.8) and $\delta\gamma(0) = \delta\gamma(1) = 0 = \delta\bar{\gamma}(t) = 0$, $t \in [1,2]$, we get

$$\delta\bar{y}_k(1) - \delta y_1(0) = \int_0^1 \beta(\dot{\gamma}(t), \delta\gamma(t))\, dt, \quad \delta\bar{y}_l(2) - \delta\bar{y}_k(1) = 0.$$

We can suppose that the horizontal lifts of all curves start from the same point in Y. Then $\delta y_1(0) = 0$. This proves the statement. □

3.5. Reduced action for magnetic flows. Let us return to the magnetic equations (3.2), where $H(q,p)$ is an arbitrary smooth function and σ is not exact. Let M be a regular component of $H(q,p)^{-1}(h)$ and let $\pi : T^*Q \to Q$ be the natural projection.

As in Theorem 2.2, we need not to fix endpoints in the fiber directions. Consider a class of regular curves γ lying on M and connecting the subspaces $T^*_{q_0}Q$ and $T^*_{q_1}Q$, such that the projection $\pi(\gamma)$ is homotopic to c:

$$\Omega^h_c(q_0, q_1) = \big\{\gamma : [0,1] \to M \mid \pi(\gamma(0)) = q_0,\, \pi(\gamma(1)) = q_1, \pi(\gamma) \sim c\big\},$$

and a class of all regular paths connecting $T^*_{q_0}Q$ and $T^*_{q_1}Q$ and lying in M:

$$\Omega^h(q_0, q_1) = \bigcup_c \Omega^h_c(q_0, q_1),$$

where we take representatives $c : [0,1] \to Q$ for all nonhomotopic paths connecting q_0 and q_1.

Theorem 3.3. *Assume $\sigma|_{\pi_2(Q)} = 0$. The phase trajectories of the magnetic equations* (3.2) *in the class of curves $\Omega^h_c(q_0, q_1)$ are extremals of the reduced action*

$$A : \Omega^h(q_0, q_1) \to \mathbb{R}, \quad A(\gamma)|_{\Omega^h_c(q_0,q_1)} = \int_\gamma p\, dq + \int_D f^*_\gamma \sigma,$$

where $f_\gamma : D \to Q$ is smooth for $|z| < 1$, continuous on D and

$$\pi(\gamma(t)) = f_\gamma\big(\exp\big(\sqrt{-1}\pi t\big)\big), \quad c(t) = f_\gamma\big(\exp\big(\sqrt{-1}\pi(2-t)\big)\big), \quad t \in [0,1].$$

If $\sigma|_{\pi_2(Q)} \neq 0$, we can use a combination of the usual reduced action and a torus valued action with respect to the form σ. Suppose $\sigma = \sum_{a=1}^n \mu_a \beta^a$, where β^a are 2-forms, representing integrals cohomology classes in Q. We take the decomposition with minimal n. As above, to σ we associate principal S^1-bundles L_a over Q having the connections θ^a with curvature forms β^a, $a = 1, \dots, n$.

Let us fix $c : [0,1] \to Q$, $c(0) = q_0$, $c(1) = q_1$. For every $\gamma \in \Omega^h(q_0, q_1)$, we associate a picewise smooth, closed path $\underline{\gamma} = \pi(\gamma) \cdot c^{-1} : [0,2] \to Q$. Define

$$B_{\mathbb{T}^n} : \Omega^h(q_0, q_1) \longrightarrow \mathbb{T}^n, \quad \gamma \longmapsto (\mathrm{Hol}^1(\underline{\gamma}), \dots, \mathrm{Hol}^n(\underline{\gamma})),$$

where Hol^a is the holonomy of the bundle $L_a \to Q$. Let $\upsilon = \sum_{a=1}^n \mu_a dy^a$ be a 1-form on $\mathbb{T}^n$, considered as a differential of a multi-valued function Υ: $d\Upsilon = \upsilon$.

THEOREM 3.4. *A curve $\gamma \in \Omega^h(q_0, q_1)$ is an integral curve of the characteristic line bundle $\mathcal{L}_M$ if and only if*

$$\frac{d}{ds}\left(\int_{\gamma_s} p\,dq - \Upsilon \circ B_{\mathbb{T}^n}(\gamma_s)\right)\Big|_{s=0} = 0$$

for all variations $\gamma_s \in \Omega^h(q_0, q_1)$.

REMARK 3.1. For various approaches to the existence problem of closed magnetic orbits, see [**9, 31**] and references therein. Integrable magnetic geodesic flows on homogeneous spaces can be found in [**6**].

4. Isoenergetic hypersurfaces of contact type

4.1. A *contact form* α on a $(2n+1)$-dimensional manifold M is a Pfaffian form satisfying $\alpha \wedge (d\alpha)^n \neq 0$. By a *contact manifold* $(M, \mathcal{H})$ we mean a connected $(2n+1)$-dimensional manifold M equipped with a nonintegrable *contact* (or *horizontal*) *distribution* $\mathcal{H}$, locally defined by a contact form: $\mathcal{H}|_U = \ker \alpha|_U$, U is an open set in M [**20**]. A contact manifold $(M, \mathcal{H})$ is *co-oriented (or strictly) contact* if $\mathcal{H}$ is defined by a global contact form α. For a given contact form α, the *Reeb vector field* Z is a vector field uniquely defined by $i_Z\alpha = 1$, $i_Z d\alpha = 0$.

4.2. In studying the existence problem of closed Hamiltonian trajectories on a fixed isoenergetic surface, Weinstein introduced the following concept [**35**]. An orientable hypersurface M of a symplectic manifold (P, ω) is of *contact type* if there exist a 1-form α on M satisfying $d\alpha = j^*\omega$, $\alpha(\xi) \neq 0$, $\xi \in \mathcal{L}_M$, $\xi \neq 0$, where $j : M \to P$ is the inclusion. If (M, α) is of contact type, since $\mathcal{L} = \ker \omega_M$, the kernel of α $\mathcal{H} = \{\xi \in T_xM \mid \alpha(\xi) = 0,\ x \in M\}$ is a $(2n-2)$-dimensional nonintegrable distribution on which $d\alpha = \omega$ is nondegenerate. Consequently, $\alpha \wedge d\alpha^{n-1}$ is a volume form on M and $(M, \mathcal{H})$ is a co-oriented contact manifold.

Now, let $(P, \omega = d\alpha)$ be an exact symplectic manifold. Consider a regular component M of an isoenergetic surface $H^{-1}(h)$ (H does not depend on time). If $\alpha(X_H)|_M \neq 0$ then M is of contact type. We say that M is of *contact type with respect to α.*

If M is of contact type with respect to α, then α has no zeros in some open neighborhood of M. Contrary, suppose that an 1-form α has no zeros in some open neighborhood of M. Then, from the nondegeneracy of ω, there exists a unique vector field E such that

$$i_E\omega = \alpha. \tag{4.1}$$

The vector field E has no zeros. From Cartan's formula, the condition $i_E\omega = \alpha$ is equivalent to $L_E\omega = \omega$, i.e., E is the *Liouville vector field* of ω. We have (e.g., see Libermann and Marle [**20**]):

LEMMA 4.1. *A regular connected component M of an isoenergetic surface $H^{-1}(h)$ is of contact type with respect to α if and only if the Liouville vector field defined by* (4.1) *is transverse to M.*

Proof. Since $i_E\omega^n = n\alpha\wedge d\alpha^{n-1}$, the kernel of $\alpha\wedge d\alpha^{n-1}$ is the vector bundle generated by E. Therefore $\alpha\wedge d\alpha^{n-1}|_M$ is a volume form on M at x if and only if $E(x)\notin T_xM$. $\square$

Let M be of contact type with respect to α and let Z be the corresponding Reeb vector field on M: $i_Z d\alpha|_M = 0$, $\alpha(Z) = 1$.

Since Z is a section of $\ker d\alpha|_M$, it is proportional to $X_H|_M$: $Z = \mathcal{N} X_H|_M$, $\mathcal{N}\neq 0$. Consequently, the flow of Z can be seen as a flow of $X_H|_M$ after a time reparametrization $dt = \mathcal{N} d\tau$:

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = X_H(x)\cdot\mathcal{N}(x) = Z(x), \qquad x\in M. \tag{4.2}$$

Alternatively, we can change the Hamiltonian H. Extend $\mathcal{N}$ to a neighborhood of M. Then

$$X_{\mathcal{N}(H-h)}(x) = \mathcal{N}(x)X_H(x), \qquad x\in M. \tag{4.3}$$

Based on observations (4.2), (4.3), we have the following statement.

Lemma 4.2. *The function $H_0 = \frac{H-h}{E(H)}$ has M as an invariant surface and the Hamiltonian vector field $X_{H_0}|_M$ is equal to the Reeb field Z. If ρ is any smooth function of a real variable, such that $\rho'(\lambda) = 1$, then $\rho(H_0+\lambda)$ has the same property. In particular, for $\rho(x) = -1/(4x)$, $\lambda = -1/2$, we get*

$$H_J = \frac{E(H)}{4h-4H+2E(H)}, \quad H_J|_M = \frac{1}{2}, \quad Z = X_{H_J}|_M. \tag{4.4}$$

Proof. According to (2.2), (4.1), we have

$$\alpha(X_F) = \omega(E, X_F) = dF(E) = E(F), \quad F\in C^\infty(P).$$

Thus, $Z = X_H/E(H)|_M$, i.e., $\mathcal{N} = 1/E(H)$. It is clear that $H_0|_M = 0$, while (4.3) implies $Z = X_{H_0}|_M$.

Let ρ is a smooth function, such that $\rho'(\lambda) = 1$. Then $\rho(H_0+\lambda)|_M = \rho(\lambda)$ and $E(\rho(H_0+\lambda))|_M = \rho'(\lambda)E(H_0) = 1$. $\square$

4.3. Exact magnetic flows. Consider a natural mechanical system given by Hamiltonian function (3.1). The canonical 1-form pdq is different from zero outside the zero section $\{p=0\}$, where we have the *standard Liouville vector field* $E = \sum_i p_i\partial/\partial p_i$ on T^*Q.

Since $E(H) = \langle p, p-\theta\rangle$, a regular hypersurface $M_h = H^{-1}(h)$ is of contact type with respect to pdq within a region

$$\begin{aligned} M_{0,h} &= \big\{\langle p-\theta, p-\theta\rangle + 2V(q) = 2h,\ \langle p, p-\theta\rangle \neq 0\big\} \\ &= \big\{\langle p-\theta, p-\theta\rangle + 2V(q) = 2h,\ \langle p, p\rangle \neq 2V + \langle\theta,\theta\rangle - 2h\big\} \subset T^*Q_h. \end{aligned}$$

Note that the equation $\langle p, p-\theta\rangle = 0|_q$, $\theta_q\neq 0$, defines an ellipsoid in T^*_qQ. Assume $h_* = \max_{q\in Q}\big(V(q)+\frac{1}{2}\langle\theta,\theta\rangle\big) < \infty$ (for example, h_* exists if Q is compact). Then regular hypersurfaces $M = H^{-1}(h)$, for $h > h_*$, are of contact type with respect to pdq. The function (4.4) has the form

$$H_J(q,p)=\frac{\langle p-\theta,p\rangle}{4(h-V(q))+2\langle\theta,p\rangle}.$$

In particular, if $\theta\equiv 0$, H_J is the Hamiltonian function of the geodesic flow of Jacobi's metric (2.7) and M_h is the corresponding co-sphere bundle over Q.

REMARK 4.1. The function $\mathcal{N}$ in the time reparametrization (4.2) equals $\mathcal{N}=1/E(H)$, $E(H)=\langle p,p-\theta\rangle=2(h-V(q))|_M$. That is, $dt=d\tau/2(h-V)$, which agrees with Corollary 2.2 (where the time parameter dt of the original system is denoted by $d\tau$, $d\tau=ds/\sqrt{2(h-V)}=ds_J/2(h-V)$, and ds_J is the natural parameter of Jacobi'metric).

5. Examples: contact flows and integrable systems

5.1. Harmonic oscillators. Consider the simplest integrable system - the system of n independent harmonic oscillators defined by the Hamiltonian function

$$H=\sum_i F_i,\quad F_i=\frac{1}{2}(a_iq_i^2+b_ip_i^2),\quad i=1,\dots,n,$$

in the standard symplectic linear space $\mathbb{R}^{2n}(q,p)$. Here we suppose that the products a_ib_i, $i=1,\dots,n$ are positive.

By the use of the first integrals $F_i=c_i$, a generic solution of the equations

$$\dot q_i=b_ip_i,\qquad \dot p_i=-a_iq_i,\qquad i=1,\dots,n \tag{5.1}$$

can be written in the form

$$q_i(t)=\sqrt{\frac{2c_i}{a_i}}\cos\left(\omega_it+\varphi_i^0\right),\quad p_i(t)=-\sqrt{\frac{2c_i}{b_i}}\sin\left(\omega_it+\varphi_i^0\right),\quad \omega_i=\sqrt{a_ib_i},$$

where $\varphi_i^0\in[0,2\pi)$ are determined from the initial conditions. Assume

$$\begin{aligned}&A_k=a_{r_1+\dots+r_{k-1}+1}=\dots=a_{r_1+\dots+r_k},\\&B_k=b_{r_1+\dots+r_{k-1}+1}=\dots=b_{r_1+\dots+r_k},\\&1\leqslant k\leqslant s,\quad r_1+\dots+r_s=n,\quad r_0=0\end{aligned}$$

and that the frequencies $\sqrt{A_1B_1},\sqrt{A_2B_2},\dots,\sqrt{A_sB_s}$ are independent over $\mathbb{Q}$.

Due to the $U(r_1)\times\dots\times U(r_s)$-symmetry, the system (5.1) has additional Noether integrals

$$\begin{aligned}&F_{ij}^k=A_kq_iq_j+B_kp_ip_j,\qquad G_{ij}^k=q_jp_i-p_jq_i,\\&r_1+\dots+r_{k-1}+1\leqslant i<j\leqslant r_1+\dots+r_k,\qquad k=1,\dots,s,\end{aligned}$$

implying the noncommutative integrability of the system [**25, 22**]. Generic trajectories fill up densely invariant s-dimensional invariant isotropic tori generated by the Hamiltonian vector fields of integrals

$$H_1=F_1+\dots+F_{r_1},\ \dots,\ H_s=F_{r_1+\dots+r_{s-1}+1}+\dots+F_{r_1+\dots+r_s}.$$

The quadric $M_h=H^{-1}(h)$, $h\neq 0$ is of contact type with respect to the canonical 1-form $p\,dq$ outside $p=0$, where we have a well defined Jacobi's metric.

However, if instead of $p\,dq$, we take

$$\alpha = \sum_{i=1}^{n} p_i\,dq_i - \frac{1}{2}d\Big(\sum_{i=1}^{n} p_i q_i\Big) = \frac{1}{2}\sum_i p_i\,dq_i - q_i\,dp_i, \tag{5.2}$$

then $d\alpha = d(p\,dq) = dp \wedge dq$ and the only zero of α is at the origin 0. The corresponding Liouville vector field is

$$E = \frac{1}{2}\sum_i q_i\frac{\partial}{\partial q_i} + p_i\frac{\partial}{\partial p_i}.$$

Since $E(H) = h|_{M_h}$, the quadric M_h is of contact type with respect to α and the Reeb flow on M_h is $Z = h^{-1}X_H|_{M_h}$.

The above construction provides natural examples of contact structures on quadrics within $\mathbb{R}^{2n}$ having the integrable Reeb flows with s-dimensional invariant tori, for any $s = 1,\dots,n$. The case $s = n$ corresponds to contact commutative integrability introduced by Banyaga and Molino [**3**] (see also [**16, 7**]), while for $s < n$ we have contact noncommutative integrability recently proposed in [**14**].

By taking all parameters to be positive ($a_i, b_i > 0$, $i = 1,\dots,n$), after rescaling of M_h to a sphere S^{2n-1}, we get K-contact structures on a sphere S^{2n-1} given by Yamazaki (see Example 2.3 in [**37**]). In particular, for $a_1 = a_2 = \dots = a_n = b_n = 1$ we have the *standard contact structure* on a sphere $S^{n-1} = H^{-1}(1/2)$ with the Reeb flow which defines the Hopf fibration (e.g., see [**20**]).

Remark 5.1. A modification of the canonical form $p\,dq$ given by (5.2) can be applied for starshaped hypersurfaces in $\mathbb{R}^{2n}$. More generally, consider a regular isoenergetic hypersurface $M_h = H^{-1}$ in $(T^*Q(q,p), dp\wedge dq)$. It is of contact type if there exist a closed 1-form φ on M_h such that $p\,dq(X_H|_{M_h}) + \varphi(X_H|_{M_h}) \neq 0$. If M_h is compact, then the required 1-form φ exists if and only if $\int_{M_h} p\,dq(X_H)d\mu \neq 0$ for every invariant probability measure μ with zero homology (see Appendix B in [**9**]). In particular, for a compact regular energy surface $M_h = H^{-1}(h)$ in the standard symplectic linear space $(R^{2n}(q,p), dq\wedge dq)$ we have the following sufficient conditions. Suppose:

(i) $p\,dq(X_H) > 0$, for $p \neq 0$, $(q,p) \in M$;

(ii) if $M \cap \{p = 0\} \neq \emptyset$, then $\frac{\partial}{\partial q}H(q,0) \neq 0$ at the points $(q,0) \in M$.

Then M_h is of contact type with respect to

$$\alpha = \sum_{i=1}^{n} p_i dq_i - \epsilon d\Big(\sum_{i=1}^{n} p_i \frac{\partial}{\partial q_i} H(q,0)\Big),$$

for a certain parameter ϵ (see [**13**]).

5.2. The regularization of Kepler's problem. The motion of a particle in the central potential filed is described by the Hamiltonian function

$$H : \mathbb{R}^{2n}_* = \mathbb{R}^{2n} \setminus \{q = 0\} \to \mathbb{R}, \qquad H(q,p) = \frac{|p|^2}{2} - \frac{\gamma}{|q|},$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product in $\mathbb{R}^n$. Moser's regularization of Kepler's problem (see [**23**]) can be interpreted in contact terms as follows.

Let $M_h = \{H = h\} \subset \mathbb{R}^{2n}_*$ be an isoenergetic hypersurface. Let us interchange the roll of q and p and consider the form $\alpha = -\sum_{i=1}^n q_i dp_i$ and the associated Liouville vector field $E = \sum_{i=1}^n q_i \frac{\partial}{\partial q_i}$.

Since $E(H) = \gamma/|q|$, M_h is of contact type with respect to α. According to Lemma 4.2, the Reeb flow on M_h can be seen as a Hamiltonian flow of

$$H_0 = (|p|^2 - 2h)|q|/2\gamma - 1.$$

In order to get a smooth Hamiltonian we can take $F = (H_0 + 1)^2/2$ (Lemma 4.2):

$$F(q,p) = \frac{(|p|^2 - 2h)^2}{8\gamma^2}|q|^2.$$

Then $F|_{M_h} = \frac{1}{2}$, $Z = X_F|_{M_h}$ and, moreover, X_F is defined on the whole $\mathbb{R}^{2n}$.

Assume $h < 0$. The Hamiltonian $F(q,p)$ can be interpreted as a geodesic flow of the metric proportional to

$$ds_h^2 = \frac{dp_1^2 + \cdots + dp_n^2}{(2h - |p|^2)^2}.$$

It represents the round sphere metric obtained by a stereographic projection (see Moser [**23**]). Thus, for $h < 0$, there exists a compact contact manifold $\bar{M}_h = M_h \cup S^n$ (a co-sphere bundle over S^n) with a Reeb vector field $\bar{Z}$, which is a smooth extension of Z. In particular, for $n = 2$, $\bar{M}_h \cong \mathbb{RP}^3$. On $\mathbb{RP}^3$ we have a *standard contact structure*, obtained from the standard contact structure on S^3 via antipodal mapping.

Note that for $h > 0$, the metric ds_h^2 is defined within the ball of radius $\sqrt{2h}$ and represents Poincaré's model of the Lobachevsky space.

The contact regularization of the restricted 3-body problem is given in [**1**].

5.3. The Maupertuis principle and geodesic flows on a sphere. It is well known that the standard metric on a rotational surface and on an ellipsoid have the geodesic flows integrable by means of an integral polynomial in momenta of the first (Clairaut) and the second degree (Jacobi) [**2**]. A natural question is the existence of a metric on a sphere S^2 with polynomial integral which can not be reduced to linear or quadratic one. The first examples are given in [**5**]. Namely, the motion of a rigid body about a fixed point in the presence of the gravitation field admits SO(2)-reduction (rotations about the direction of gravitational field) to a natural mechanical system on S^2. Starting from the integrable Kovalevskaya and Goryachev–Chaplygin cases and taking the corresponding Jacobi's metrics, we get the metrics with additional integrals of 4-th and 3-th degrees, respectively.

We proceed with a celebrated Neumann system. The Neumann system describes the motion of a particle on a sphere $\langle q, q\rangle = 1$ with respect to the quadratic potential $V(q) = \frac{1}{2}\langle Aq, q\rangle$, $A = \operatorname{diag}(a_1, \dots, a_n)$ (we assume that A is positive definite). The Hamiltonian of the system is:

$$H_N(q,p) = \tfrac{1}{2}\langle p, p\rangle + \tfrac{1}{2}\langle Ax, x\rangle\,. \tag{5.3}$$

Here, the cotangent bundle of a sphere T^*S^{n-1} is realized as a submanifold P of $\mathbb{R}^{2n}$ given by the constraints

$$F_1 \equiv \langle q, q\rangle = 1, \quad F_2 \equiv \langle q, p\rangle = 0. \tag{5.4}$$

The canonical symplectic form on $P \cong T^*S^{n-1}$ is a restriction of the standard symplectic form $dp \wedge dq$ to P. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$. The Hamiltonian vector field $X_H|_P$ reads

$$X_H(q,p)|_P = X_H(q,p) - \lambda_1 X_{F_1}(q,p) - \lambda_2 X_{F_2}(q,p), \quad (q,p) \in P,$$

where the Lagrange multipliers are determined from the condition that $X_H|_P$ is tangent to P (e.g., see [**24**]).

There is a well known Knörrer's correspondence between the trajectories $q(t)$ of the Neumann system (5.3) restricted to the zero level set of the integral

$$H(q,p) = \tfrac{1}{2}\left(\langle A^{-1}q, q\rangle\langle A^{-1}p, p\rangle - \langle A^{-1}q, p\rangle^2 - \langle A^{-1}q, q\rangle\right). \tag{5.5}$$

and the geodesic lines on an ellipsoid $E_1^{n-1} = \{x \in \mathbb{R}^n \mid \langle x, Ax\rangle = 1\}$ by the use of a time reparametrization and the Gauss mapping $q = Ax/|Ax|$ [**17**].

Recently, by using optimal control techniques, Jurdjevic obtain a similar statement for the flow of the system defined by the Hamiltonian (5.5) [**15**].

We give the interpretation of Jurdjevic's time change by the use of Maupertuis principle. Since the potential $V(q) = -\frac{1}{2}\langle A^{-1}q, q\rangle$ is negative, the isoenergetic surface

$$M_0 = \{H|_P = 0\} \subset P \cong T^*S^{n-1} \tag{5.6}$$

is of contact type with respect to $p\,dq|_P$. The Reeb vector field Z equals to the Hamilonian vector field of

$$H_J = \frac{1}{4\langle A^{-1}q, q\rangle}\left(\langle A^{-1}q, q\rangle\langle A^{-1}p, p\rangle - \langle A^{-1}q, p\rangle^2\right)\big|_P \tag{5.7}$$

(the Hamiltonian of the corresponding Jacobi's metric).

The Legendre transformation of a function of the form (5.7) in the presence of constraints (5.4) is given in [**11**] (see Theorem 2 [**11**] and interchange the role of the tangent and cotangent bundles of a sphere). As a result, we obtain the Lagrangian function $L(q,\dot q) = \frac{1}{2}\langle A\dot q, \dot q\rangle|_{S^{n-1}}$.

Remarkably, after the linear coordinate transformation $x = \sqrt{A}q$, $L(q,\dot q)$ becomes the Lagrangian $L(x,\dot x) = \frac{1}{2}\langle \dot x, \dot x\rangle$ of the standard metric on the ellipsoid

$$E_2^{n-1} = \{x \in \mathbb{R}^n \mid \langle A^{-1}x, x\rangle = 1\}.$$

Recall that the Reeb flow on M_0 can be seen as a time reparametrization of the original Hamiltonian flow (see Remark 4.1). We can summarize the consideration above in the following statement.

PROPOSITION 5.1. [**15**] *Under the time substitution $dt = d\tau/2\langle A^{-1}q, q\rangle$ and the linear transformation $x = \sqrt{A}q$, the q-components of the trajectories of the system defined by the Hamiltonian function* (5.5) *that lie on the zero energy level* (5.6)*, become geodesic lines of the standard metric on the ellipsoid E_2^{n-1}.*

Further interesting examples of transformations related to the Maupertuis Principle, which map a given integrable system into another one are given in [**32**].

Acknowledgments. I am grateful to Professor Jurdjevic for providing a preprint of the paper [**15**] and to the referee for valuable remarks. This research was supported by the Serbian Ministry of Science Project 174020, Geometry and Topology of Manifolds, Classical Mechanics and Integrable Dynamical Systems.

References

[1] P. Albers, U. Frauenfelder, O. van Koert, G. P. Paternain, *The contact geometry of the restricted 3-body problem*, Comm. Pure Appl. Math. **65**:2 (2012), 229–263; arXiv:1010.2140v1 [math.SG]

[2] В. И. Арнольд, *Математические методы классической механики*, Наука, Москва, 1974; English translation: V. I. Arnol'd, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, 1978.

[3] A. Banyaga, P. Molino, *Géométrie des formes de contact complétement intégrables de type torique*, Séminare Gaston Darboux, Montpellier (1991–92), 1–25. English translation: A. Banyaga, P. Molino, *Complete Integrability in Contact Geometry*, Penn State preprint PM 197, 1996.

[4] А. Билимовић, *Рационална механика. Том 2 (Механика система)*, Научна књига, Београд, 1951.

[5] А. В. Болсинов, В. В. Козлов, А. Т. Фоменко, *Принцип Мопертюи и геодезические потоки на сфери, возникающие из интегрируемых случаев динамики тверого тела*, Успехи Мат. Наук **50**:3 (1995), 3–32; English translation: A. V. Bolsinov, V. V. Kozlov, A. T. Fomenko, *The Maupertuis principle and geodesic flow on the sphere arising from integrable cases in the dynamic of a rigid body*, Russian Math. Surv. **50** (1995) 473–501.

[6] A. V. Bolsinov, B. Jovanović, *Magnetic Flows on Homogeneous Spaces*, Com. Mat. Helv. **83**:3 (2008), 679-Ű700; arXiv: math-ph/0609005.

[7] C. P. Boyer, *Completely integrable contact Hamiltonian systems and toric contact structures on $S^2 \times S^3$*, SIGMA **7** (2011), 058, 22 pages; arXiv: 1101.5587 [math.SG]

[8] H. Cendra, J. E. Marsden, S. Pekarsky, T. S. Ratiu, *Variational principles for Lie-Poisson and Hamilton–Poincaré equations*, Moscow Math. J. **3**:3 (2003), 833–867.

[9] G. Contreras, L. Macarini, G. P. Paternain, *Periodic orbits for exact magnetic flows on surfaces*, Int. Math. Res. Not. **8** (2004), 361-Ű387.

[10] В. Драговић, Д. Милинковић, *Анализа на многострукостима*, Математички факултет, Београд.

[11] Y. Fedorov, B. Jovanović, *Hamiltonization of the Generalized Veselova LR System*, Reg. Chaot. Dyn. **14**:4–5 (2009), 495–505.

[12] P. A. Griffits, *Exterior Differential Systems and the Calculus of Variations*, Progress in Math. 25, Birkhäuser, Boston, Mass., 1983.

[13] H. Hofer, E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser, 1994.

[14] B. Jovanović, *Noncommutative integrability and action angle variables in contact geometry*, to appear in J. Sympl. Geometry; arXiv:1103.3611 [math.SG]

[15] V. Jurdjevic, *Optimal control on Lie groups and integrable Hamiltonian systems*, Regul. Chaotic Dyn. **16**:5 (2011), 514-Ű535.

[16] B. Khesin, S. Tabachnikov, *Contact complete integrability*, Regul. Chaot. Dyn., Special Issue: Valery Vasilievich Kozlov 60 (2010); arXiv:0910.0375 [math.SG].

[17] H. Knörrer, *Geodesics on quadrics and a mechanical problem of C. Neumann*, J. Reine Angew. Math. **334** (1982), 69–78.

[18] S. Kobayashi, *Principal fibre bundles with the* 1*-dimensional toroidal group* Tohoku Math. J. (2)**8** (1956), 29-Ű45.

[19] В. В. Козлов, *Вариационное исчисление в целом и классическая механика*, Успехи Мат. Наук **40**:2(242) (1982), 33Ű-60.
[20] P. Libermann, C. Marle, *Symplectic Geometry, Analytical Mechanics*, Riedel, Dordrecht, 1987.
[21] J. E. Marsden, T. S. Ratiu, *Introduction to Mechanics and Symmetry*, 2nd edition, Springer, 1999.
[22] А. С. Мищенко, А. Т. Фоменко, *Обобщенный метод Лиувилля интегрирования гамилтоновых систем* , Функц. анал. прилож. **12**:2 (1978), 46–56; English translation: A. S. Mishchenko, A. T. Fomenko, *Generalized Liouville method of integration of Hamiltonian systems.* Funct. Anal. Appl. **12** (1978), 113–121.
[23] J. Moser, *Regularization of Kepler's problem and the averaging method on a manifold*, Comm. Pure Appl. Math. **23** (1970) 606–636.
[24] J. Moser, *Geometry of quadric and spectral theory*, In: Chern Symposium 1979, Berlin–Heidelberg–New York, 147–188, 1980.
[25] Н. Н. Нехорошев, *Переменные действие–угол и их обобщения* , Тр. Моск. Мат. О.-ва. **26** (1972), 181–198; English translation: N. N. Nehoroshev, *Action-angle variables and their generalization*, Trans. Mosc. Math. Soc. **26** (1972), 180–198.
[26] С. П. Новиков, *Гамильтонов формализм и многозначный аналог теории Морса* , Успехи Мат. Наук **37**:5(227) (1982), 3Ű-49.
[27] H. Poencaré, *Les méthodes nouvelles de la méchanique céleste III. Invariant intégraux. Solutions périodiques du deuxieme genre. Solutions doublement asymptotiques*, Gauthier-Villars, Paris, 1899.
[28] Л. С. Полак (ед.), *Вариационные принципы механики (Сборник статей классиков науки)* , Физматгиз, Москва, 1959.
[29] P. H. Rabinowitz, *Periodic solutions of Hamiltonian systems*, Comm. Pure. Appl. Math. **31** (1978), 157Ű-184.
[30] G. Romano, R. Barretta, A. Barretta, *On Maupertuis principle in dynamics*, Rep. Math. Phys. **63**:3 (2009), 331–346.
[31] A. I. Taimanov, *Periodic magnetic geodesics on almost every energy level via variational methods*, Regular and Chaotic Dynamics **15** (2010), 598-605, arXiv:1001.2677
[32] A. V. Tsiganov, *The Maupertuis principle and canonical transformations of the extended phase space*, J. Nonlinear Math. Phys. 8:1 (2001), 157-Ű182.
[33] G. M. Tuynman, *Un principle variationnel pour les variétés symplectiques*, C. R. Acad. Sci. Paris Sér. I Math. **326**:3 (1998), 339–342.
[34] A. Weinstein, *Bifurcations and Hamiltonian's principle*, Math. Z. **159** (1978), 235–248.
[35] A. Weinstein, *On the hypotheses of Rabinowitz' periodic orbit theorems*, J. Differential Equations **33**:3 (1979), 353Ű-358.
[36] E. T. Whittaker, *A Treatise on the Analytic Dynamics of Particles and Rigid Bodies*, Cambridge, 1904.
[37] T. Yamazaki, *A construction of K-contact manifolds by a fiber join*, Tohoku Math. J. **51** (1999) 433–446.

Mathematical Institute SANU
Serbian Academy of Sciences and Arts
Kneza Mihaila 36, 11000 Belgrade
Serbia
bozaj@mi.sanu.ac.rs

(Received 04 05 2011)
(Revised 19 03 2012)