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GENERALIZED LAGRANGE–D'ALEMBERT PRINCIPLE

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ABSTRACT. The major issues in the analysis of the motion of a constrained dynamic system are to determine this motion and calculate constraint forces. In the analytical mechanics, only the first of the two problems is analysed. Here, the problem is solved simultaneously using: 1) Principle of liberation of constraints; 2) Principle of generalized virtual displacement; 3) Idea of ideal constraints; 4) Concept of generalized and "supplementary" generalized coordinates. The Lagrange–D'Alembert principle of virtual work is generalized introducing virtual displacement as vectorial sum of the classical virtual displacement and virtual displacement in the "supplementary" directions. From such principle of virtual work we derived Lagrange equations of the second kind and equations of dynamical equilibrium in the "supplementary" directions. Constrained forces are calculated from the equations of dynamic equilibrium. At the same time, this principle can be used for consideration of equilibrium of system of material particles. This principle simultaneously gives the connection between applied forces at equilibrium state and the constrained forces. Finally, the principle is applied to a few particular problems.

1. Introduction

There are two problems in analyses of motion of the dynamical systems under action of the constraints. The first problem is to determine the motion of the system, and the second to calculate the reaction forces of the constraints during the motion.

In the analytical mechanics, see for example [**3**], [**1**], [**6**] and [**2**], these two tasks are separated by the notions of ideal constraints and Lagrange–D'Alembert principle of virtual work (In 1743, Jean Le Rond d'Alembert published his important *Traité de dynamique*, a fundamental treatise on dynamics containing the famous "d'Alembert's principle"). According to the notion of ideal constraints, the virtual work of the reactions forces is equal to zero, and the work vanishes in the Lagrange–D'Alembert principle of virtual work. Therefore, the reaction forces are eliminated from further analyses of motion, which is based on the Lagrange–D'Alembert principle. If there are some reaction forces whose virtual work is not equal to zero, then

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those forces are added to the given set of forces and the corresponding constraints are called non-ideal. Reaction forces of ideal constraints are absent in the Lagrange equations of the second kind. In fact, analytical mechanics considers only the motion of a system, while for finding reaction forces the equations of Newtonian mechanics or Lagrange equations of the first kind must be used [**6**], [**5**] and [**4**].

Nevertheless, the Lagrange–D'Alembert principle is frequently used for finding reaction forces of ideal constraints. According to that procedure, the mechanical system frees itself of a particular constraint and performs a virtual displacement in direction consistent with the removed constraint. Calculation of virtual work on that displacement yields the corresponding reaction force.

It is this author's opinion that any principle of dynamics must simultaneously give answer to both mentioned questions of mechanical motion under constraints. In that sense, the Lagrange–D'Alembert principle requires some generalization.

In this paper, the basic problem for dynamical systems is analysed using the following ideas:

(1) Virtual displacements and supplementary virtual displacements,
(2) Principle of liberation of constraints,
(3) Ideal constraints,
(4) Generalized coordinates and supplementary "generalized" coordinates.

The supplementary virtual displacements are introduced in directions which are consistent with the removed constraints. Because the directions are in normal directions to the constraints, the virtual displacements are called normal virtual displacements. Also, supplementary generalized coordinates are introduced, which are measured along this normal virtual displacements, and whose number is equal to the number of constraints. In the Lagrange–D'Alembert principle of virtual work, the classical virtual displacements are replaced by vectorial sum of the classical and normal virtual displacements. From such generalized Lagrange–D'Alembert principle of virtual work, the Lagrange equations of the second kind and the equations of dynamical equilibrium in the normal directions are derived. From the equations of dynamical equilibrium the constraints forces are calculated. Hence, using this principle, the equations of Newtonian mechanics are not necessary for finding the reaction forces. This principle can be used for consideration of equilibrium of material particles. In that case, this principle simultaneously gives the connection between the applied forces at equilibrium state and the constraint forces. Finally, the principle is applied to a few particular problems.

Now, the Lagrange–D'Alembert principle is used [**7**]–[**30**] in analysis of different problems in conservative, nonconservative, holonomic and nonholonomic mechanics and different areas of physics. But, in this papers there is no generalization of the principle in the sense of the present paper.

Here, a small Greek index takes values from 1 to n, the capital Greek index from 1 to s, small italic index from 1 to N and capital italic index from 1 to $n+s$. Every repeated capital italic index, small Greek index or capital Greek index means sum with respect to that index. Bold face letters are vectors.

2. Generalized Virtual Displacement

Let us consider motion of a dynamical system with N material particles M_i, whose masses are m_i, and which is under the action of a given set of forces $\mathbf{F}_i$, and subject to s holonomic constraints

$$f_\Delta(t,\, x_i,\, y_i,\, z_i) = 0, \quad \Delta = 1, \dots, s. \tag{2.1}$$

Here t is time and x_i, y_i and z_i are the Cartesian coordinates of a particle M_i. If particles are liberated from the constraints, then their actions on particles M_i are replaced by the reaction forces $\mathbf{R}_i$. The system has $n = 3N - s$ degrees of freedom and q_α $(\alpha = 1, \dots, n)$ are the corresponding generalized coordinates.

Let the position of any particle M_i, of the system, which during the motion is defined by its position vector $\mathbf{r}_i$ in the three dimensional space be $\mathbf{r}_i = x_i\mathbf{i}+y_i\mathbf{j}+z_i\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors of Cartesian coordinate system x, y, and z.

Virtual changes of the constraint equations (2.1) are

$$\delta f_\Delta = \sum_i \mathbf{l}_{i\Delta} \cdot \delta\mathbf{r}_i = 0, \tag{2.2}$$

where

$$\mathbf{l}_{i\Delta} = \mathrm{grad}_i\, f_\Delta = \frac{\partial f_\Delta}{\partial x_i}\mathbf{i} + \frac{\partial f_\Delta}{\partial y_i}\mathbf{j} + \frac{\partial f_\Delta}{\partial z_i}\mathbf{k}, \tag{2.3}$$

and $\delta\mathbf{r}_i = \delta x_i\mathbf{i} + \delta y_i\mathbf{j} + \delta z_i\mathbf{k}$ is the virtual change of the position vector. It is a well known fact that vectors (2.3) are perpendicular to the constraints.

Let us liberate every particle M_i in the system of the constraints and permit to every particle displacements $\mathbf{l}_{i\Delta}q_{n+\Delta}$ in the perpendicular directions $\mathbf{l}_{i\Delta}$, where $q_{n+\Delta}$ are supplementary generalized coordinates. In our analysis the independent parameters $q_{n+\Delta}$ are considered as functions of time. For motion along constraints (2.1), the parameters $q_{n+\Delta}$ and their derivatives $\dot{q}_{n+\Delta}$ and $\ddot{q}_{n+\Delta}$ are equal to zero, i.e., $q_{n+\Delta} = 0$, $\dot{q}_{n+\Delta} = 0$, $\ddot{q}_{n+\Delta} = 0$. Any quantity, which is calculated for the motion along the constraints, is denoted by lower index 0.

The position vector of the so liberated particle M_i of the constraints is

$$\boldsymbol{\rho}_i(t, q_U) = \mathbf{r}_i(t, q_\alpha) + \mathbf{l}_{i\Delta}(t, q_\alpha)q_{n+\Delta}, \tag{2.4}$$

whose virtual change is $\delta\boldsymbol{\rho}_i = \delta\mathbf{r}_i + \delta\mathbf{l}_{i\Delta}q_{n+\Delta} + \mathbf{l}_{i\Delta}\delta q_{n+\Delta}$. For the motion of the particle along the constraints, we have

$$(\delta\boldsymbol{\rho}_i)_0 = \delta\mathbf{r}_i + \mathbf{l}_{i\Delta}\delta q_{n+\Delta}. \tag{2.5}$$

Relations (2.5) define generalized virtual displacement of any particle M_i. Let us decompose the generalized virtual displacement into the following two parts

$$(\delta\boldsymbol{\rho}_i)_0 = (\delta\boldsymbol{\rho}_i)_{0T} + (\delta\boldsymbol{\rho}_i)_{0N}, \tag{2.6}$$

where

$$(\delta\boldsymbol{\rho}_i)_{0T} = \delta\mathbf{r}_i \tag{2.7}$$

is the classical virtual displacement in the tangential directions to the constraints, while

$$(\delta \boldsymbol{\rho}_i)_{0N} = \mathbf{l}_{i\Delta}\delta q_{n+\Delta}, \tag{2.8}$$

is virtual displacement in perpendiculars to the constraints.

Calculating sum $\sum_i (\delta \boldsymbol{\rho}_i)_{0T} \cdot (\delta \boldsymbol{\rho}_i)_{0N}$, and using (2.2) we have

$$\sum_i (\delta \boldsymbol{\rho}_i)_{0T} \cdot (\delta \boldsymbol{\rho}_i)_{0N} = \delta q_{n+\Delta} \sum_i \mathbf{l}_{i\Delta} \cdot \delta \mathbf{r}_i = 0.$$

In the sense to this relation, two parts $\delta \mathbf{r}_i$ and $\mathbf{l}_{i\Delta}\delta q_{n+\Delta}$ of the generalized virtual displacement $(\delta \boldsymbol{\rho}_i)_0$ are mutually orthogonal, and decomposition (2.6) of the generalized virtual displacement is justifiable.

3. Constraints forces

Here, the classical definition of the ideal constraints (see for example [**3**, p. 218]) is accepted. This means that the virtual work of all reaction forces of the constraints on the tangential virtual displacements must be zero, i.e.,

$$\sum_i (\mathbf{R}_i)_0 \cdot (\delta \boldsymbol{\rho}_i)_{0T} = 0. \tag{3.1}$$

The position vectors of the material particles M_i are functions of time and the generalized coordinates $\mathbf{r}_i = \mathbf{r}_i(t, q_\alpha)$, and virtual displacements (2.7) in tangential planes to the constraints are

$$(\delta \boldsymbol{\rho}_i)_{0T} = \delta \mathbf{r}_i = \frac{\partial \mathbf{r}_i}{\partial q_\alpha}\delta q_\alpha. \tag{3.2}$$

Substituting the virtual displacements into orthogonality condition (2.2), and because the virtual changes δq_α are mutually independent and different from zero, the following condition is valid

$$\sum_i \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \cdot \mathbf{l}_{i\Delta} = 0. \tag{3.3}$$

Combining (3.1) and (3.2), and using the fact that virtual changes δq_α are mutually independent and different from zero, we obtain the following conditions

$$\sum_i (\mathbf{R}_i)_0 \frac{\partial \mathbf{r}_i}{\partial q_\alpha} = 0, \tag{3.4}$$

which must be satisfied by the reaction forces of the ideal constraints. Comparison of (3.3) and (3.4) suggests the following form for the reaction forces

$$(\mathbf{R}_i)_0 = (R^{-1})_\Delta \mathbf{l}_{i\Delta}, \tag{3.5}$$

where $(R^{-1})_\Delta$ are mutually independent quantities. The condition (3.4) is identically satisfied for arbitrary values of the quantities $(R^{-1})_\Delta$.

Let us introduce a second-order system $b_{\Delta\Gamma}$ by relations

$$\sum_i \mathbf{l}_{i\Delta} \cdot \mathbf{l}_{i\Gamma} = b_{\Delta\Gamma}, \tag{3.6}$$

where $b_{\Delta\Gamma} = b_{\Gamma\Delta}$, which means that the system is symmetric. The corresponding inverse system $(b^{-1})_{\Delta\Omega}$ is defined by

$$(b^{-1})_{\Delta\Omega} b_{\Delta\Gamma} = \delta_{\Omega\Gamma} \tag{3.7}$$

where $\delta_{\Omega\Gamma}$ are the Kronecker symbols, that means $\delta_{\Omega\Gamma} = 0$ for $\Omega \neq \Gamma$ and $\delta_{\Omega\Gamma} = 1$ for $\Omega = \Gamma$. Using the second-order system $b_{\Delta\Gamma}$ the inverse vectors R_Δ and $(R^{-1})_\Gamma$ are connected with the relation

$$R_\Delta = (R^{-1})_\Gamma b_{\Gamma\Delta}. \tag{3.8}$$

4. Generalized Forces

The virtual work of all applied forces $\mathbf{F}_i$ on the generalized virtual displacements (2.6) is

$$\delta A^{\mathbf{F}} = \sum_i (\mathbf{F}_i)_0 \cdot (\delta \boldsymbol{\rho}_i)_0. \tag{4.1}$$

Using (2.6), (2.8) and (3.2), the virtual work becomes

$$\delta A^{\mathbf{F}} = Q_\alpha \delta q_\alpha + N_\Delta \delta q_{n+\Delta}, \tag{4.2}$$

where $Q_\alpha = \sum_i \left(\mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_\alpha}\right)_0$, are the classical generalized forces, and

$$N_\Delta = \sum_i (\mathbf{F}_i)_0 \cdot \mathbf{l}_{i\Delta} \tag{4.3}$$

are generalized forces in the perpendicular directions. If all applied forces $\mathbf{F}_i$ have the potential energy $\Pi(t, q_U)$, then the generalized forces are

$$Q_\alpha = -\frac{\partial \Pi}{\partial q_\alpha}, \quad N_\Delta = -\frac{\partial \Pi}{\partial q_\Delta}. \tag{4.4}$$

5. Kinetic Energy of the System

Position in the space of every particle M_i of the system, which is liberated from the corresponding constraints is defined by the vectors $\boldsymbol{\rho}_i$ (2.4). The kinetic energy of the system free of constraints is

$$E_k = \frac{1}{2} \sum_i m_i \dot{\boldsymbol{\rho}}_i \cdot \dot{\boldsymbol{\rho}}_i. \tag{5.1}$$

The velocity of the particle free from constraints (2.1) is obtained as the first derivative of the vector $\boldsymbol{\rho}_i$ in to time

$$\dot{\boldsymbol{\rho}}_i = \dot{\mathbf{r}}_i + \dot{\mathbf{l}}_{i\Delta} q_{n+\Delta} + \mathbf{l}_{i\Delta} \dot{q}_{n+\Delta}. \tag{5.2}$$

Here, $\dot{\mathbf{r}}_i$ is the velocity of the particle which is not free from constraints. Using (5.2), kinetic energy (5.1) becomes

$$E_k = (E_k)_0 + \tilde{E}_k + NLT, \tag{5.3}$$

where $(E_k)_0 = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i$ is kinetic energy of the system which is not free from constraints, and $\tilde{E}_k = \sum_i m_i \dot{\mathbf{r}}_i \cdot (\dot{\mathbf{l}}_{i\Delta} q_{n+\Delta} + \mathbf{l}_{i\Delta} \dot{q}_{n+\Delta})$ is the part of the kinetic energy linear to $q_{n+\Delta}$ and $\dot{q}_{n+\Delta}$. The term NLT is nonlinear in $q_{n+\Delta}$ and $\dot{q}_{n+\Delta}$.

The term NLT is not important for analysis and therefore it is omitted in the further considerations.

6. Generalized D'Alembert Principle

For the motion of a particle M_i of the system under action of a given force $\mathbf{F}_i$ and a reaction force $\mathbf{R}_i$, the second Newton law of motion under action of the constraints means that $(\mathbf{F}_i+\mathbf{R}_i-m_i\ddot{\boldsymbol{\rho}}_i)_0 = 0$. Multiplying the previous equations by virtual displacements $(\delta\boldsymbol{\rho}_i)_0$ (2.6) and summing up such equations for all particles, i.e., over the index i, we have $\sum_i(\mathbf{F}_i + \mathbf{R}_i - m_i\ddot{\boldsymbol{\rho}}_i)_0 \cdot (\delta\boldsymbol{\rho}_i)_0 = 0$, or

$$\delta A^{\mathbf{I}} + \delta A^{\mathbf{F}} + \delta A^{\mathbf{R}} = 0, \tag{6.1}$$

where

$$\delta A^{\mathbf{I}} = -\sum_i (m_i\ddot{\boldsymbol{\rho}}_i \cdot \delta\boldsymbol{\rho}_i)_0 \tag{6.2}$$

is the virtual work of the inertial forces, $\delta A^{\mathbf{R}} = \sum_i(\mathbf{R}_i \cdot \delta\boldsymbol{\rho}_i)_0$ is the virtual work of reaction forces and $\delta A^{\mathbf{F}}$ is virtual work of active forces given by (4.1). Equation (6.1) is a generalized form of the Lagrange–D'Alembert principle of virtual work. The meaning of the generalized Lagrange–D'Alembert principle of virtual work (6.1) is that during the motion of a mechanical system the sum of the virtual works of inertial, active and reaction forces is equal to zero.

Using (2.6) and (2.7) the virtual work of reaction forces is

$$\delta A^{\mathbf{R}} = \sum_i (\mathbf{R}_i)_0 \delta\mathbf{r}_i + \delta q_{n+\Delta} \sum_i (\mathbf{R}_i)_0 \mathbf{I}_{i\Delta}. \tag{6.3}$$

The first term in (6.3) is equal to zero according to (3.1) and the previous relation becomes

$$\delta A^{\mathbf{R}} = R_\Delta \delta q_{n+\Delta}, \tag{6.4}$$

where the generalized reaction forces are $R_\Delta = \sum_i(\mathbf{R}_i)_0\mathbf{I}_{i\Delta}$.

7. Lagrange equations

According to (2.4) $\boldsymbol{\rho}_i$ is a function of time t and of all generalized coordinates q_U, $U = 1, \dots, n+s$. Using the well known relations for such functions

$$\frac{\partial\dot{\boldsymbol{\rho}}_i}{\partial\dot{q}_U} = \frac{\partial\boldsymbol{\rho}_i}{\partial q_U}, \quad \frac{d}{dt}\frac{\partial\boldsymbol{\rho}_i}{\partial q_U} = \frac{\partial\dot{\boldsymbol{\rho}}_i}{\partial q_U},$$

virtual work of inertial forces (6.2) becomes

$$\delta A^{\mathbf{I}} = Z_U \delta q_U, \quad \text{where} \quad Z_U = \left(\frac{\partial E_k}{\partial q_U} - \frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}_U}\right). \tag{7.1}$$

Now, using (4.2), (6.4), and (7.1), the generalized form of the Lagrange–D'Alembert principle of virtual work in generalized coordinates (6.1) becomes

$$(Z_\alpha + Q_\alpha)\delta q_\alpha + (Z_{n+\Delta} + N_\Delta + R_\Delta)\delta q_{n+\Delta} = 0. \tag{7.2}$$

Because the generalized coordinates and the supplementary generalized coordinates are mutually independent, and their virtual changes are also independent and different from zero, i.e., $\delta q_\alpha \neq 0$ and $\delta q_{n+\Delta} \neq 0$, virtual work δA (7.2) is equal to zero for arbitrary virtual changes δq_α and $\delta q_{n+\Delta}$ only if

$$\left(\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}_\alpha} - \frac{\partial E_k}{\partial q_\alpha}\right)_0 = Q_\alpha, \tag{7.3}$$

and

$$\left(\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}_{n+\Delta}} - \frac{\partial E_k}{\partial q_{n+\Delta}}\right)_0 = N_\Delta + R_\Delta. \tag{7.4}$$

System of equations (7.3) are well known Lagrange equations of motion for the dynamical system, which are called the Lagrange equations of the second kind. Equations (7.4) form another system of equations from which, using (3.5), (3.8), we calculate the reaction forces on every particle of a system as

$$(\mathbf{R}_i)_0 = \left[\left(\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}_{n+\Delta}} - \frac{\partial E_k}{\partial q_{n+\Delta}}\right)_0 - N_\Delta\right](b^{-1})_{\Delta\Omega}\mathbf{l}_{i\Omega}. \tag{7.5}$$

For this reason the equations (7.5) can be called equations of dynamic equilibrium in normal directions.

8. Examples

Example 1. Let us consider the motion of a material point with mass m on a smooth surface whose equation is $F_1 = F(x, y, z)$, where x, y and z are the Cartesian coordinates of the point and the axis z is in up direction. This motion has two degrees of freedom. The normal vector (2.3) is

$$\mathbf{l}_{11} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}, \tag{8.1}$$

where the low index means partial derivative with respect to that coordinate. Using (3.6) we have

$$b_{11} = F_x^2 + F_y^2 + F_z^2, \quad (b^{-1})_{11} = \frac{1}{F_x^2 + F_y^2 + F_z^2}. \tag{8.2}$$

The kinetic energy (5.3) of the particle is

$$E_k = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + (\dot{x}\dot{F}_x + \dot{y}\dot{F}_y + \dot{z}\dot{F}_z)mq_3, \tag{8.3}$$

where q_3 is the displacement in normal direction (8.1). There is only one active force $\mathbf{F} = -mg\mathbf{k}$, where g is gravity acceleration. Hence, the generalized force in normal direction (4.3) is $N_3 = -mgF_z$. Now, applying (7.4) to (8.3) for the coordinate q_3 we have $R_3 = mgF_z - m(\dot{x}\dot{F}_x + \dot{y}\dot{F}_y + \dot{z}\dot{F}_z)$, and using (7.5) and (8.2) we have the reaction force

$$\mathbf{R} = \frac{R_3}{F_x^2 + F_y^2 + F_z^2}(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}).$$

If the surface is a sphere with radius r, then $F = x^2+y^2+z^2-r^2$, and the reaction force is

$$\mathbf{R} = \frac{m[gz - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)]}{r^2}\mathbf{r},$$

where $\mathbf{r}$ is the position vector of the point.

EXAMPLE 2. Let us consider a mathematical pendulum of the length a and mass m. The corresponding constraint is $f_1 = x^2+y^2-a^2 = 0$, where x is horizontal axis and y vertical oriented down. From this relation we have the gradient vector (2.3)

$$\mathbf{I}_{11} = 2x\mathbf{i}+2y\mathbf{j}.$$

The problem has one degree of freedom, the angle α between the pendulum and the vertical direction. In that case

$$\mathbf{I}_{11} = 2a\sin\alpha\mathbf{i} + 2a\cos\alpha\mathbf{j}, \tag{8.4}$$

and velocity of the particle $\dot{\mathbf{r}} = (a\cos\alpha\mathbf{i} - a\sin\alpha\mathbf{j})\dot{\alpha}$. Here, there is only gravity applied force $\mathbf{F} = mg\mathbf{j}$, where g is the gravity acceleration. The corresponding potential energy is $\Pi = mga(1-\cos\alpha)$. Therefore, the generalized forces (4.4) and (4.3) are $Q_\alpha = -mga\sin\alpha$, $N_\alpha = 2mga\cos\alpha$. The kinetic energy (5.1) has the form $E_k = \frac{m}{2}(a\dot{\alpha})^2 + 2m(a\dot{\alpha})^2 q_2$, where q_2 is the additional generalized coordinate. Second-order symmetric system (3.6), using (8.4) becomes and its inverse (3.7) $(b^{-1})_{11} = 1/4a^2$. Finally, from (7.3) and (7.5) we have the equation of motion $\ddot{\alpha} + \frac{g}{a}\sin\alpha = 0$, and the reaction force to the particle

$$\mathbf{R} = -m(a\dot{\alpha}^2 + g\cos\alpha)(\sin\alpha\mathbf{i} + \cos\alpha\mathbf{j}).$$

EXAMPLE 3. Let us consider the motion in a vertical plane of two equal rods OA and AB equal masses m and lengths l connected by a cylindrical joint at the point A, while the rod OA is connected to unmovable cylindrical joint at the end of rod at the point O. This system has two degrees of freedom and the angles φ and ψ between the rod and vertical direction are the generalized coordinates. Let x be the horizontal and y the vertical axes. The centers of gravity for the rods are the points C_1 and C. The gravity forces are

$$\mathbf{F}_{c1} = mg\mathbf{j}, \quad F_c = mg\mathbf{j}, \tag{8.5}$$

where g is the gravity acceleration. If we want to determine forces at cylindrical joints then we must remove them. Let x_1 and y_1 be the motion of the rods and with respect to joint O, and x_2 and y_2 motion of A of the rod AB with respect to the joint point of the rod OA. The corresponding equations of constraints are

$$\begin{aligned} F_{1x_1} &= x_1 = 0, \quad F_{1y_1} = y_1 = 0, \\ F_{2x_2} &= x_2 = 0, \quad F_{2y_2} = y_2 = 0. \end{aligned} \tag{8.6}$$

The kinetic energy of the system is

$$E_k = \frac{1}{2}J_{c1}\dot{\varphi}^2 + \frac{m}{2}v_{c1}^2 + \frac{1}{2}J_c\dot{\psi}^2 + \frac{m}{2}v_c^2,$$

where J_{c1} and J_c are the corresponding moments of inertia and v_{c1} and v_c are velocities of the points C_1 and C. The part of the kinetic energy linear with respect x_1, y_1 x_2, y_2, $\dot{x}_1$, $\dot{y}_1$ $\dot{x}_2$, $\dot{y}_2$ is

$$\tilde{E}_k = \frac{m}{2} l(\dot{x}_1\dot{\varphi}\cos\varphi - \dot{y}_1\dot{\varphi}\sin\varphi) + ml\Big[(\dot{x}_1+\dot{x}_2)\Big(\dot{\varphi}\cos\varphi + \frac{\dot{\psi}}{2}\cos\psi\Big) - (\dot{y}_1+\dot{y}_2)\Big(\dot{\varphi}\sin\varphi + \frac{\dot{\psi}}{2}\sin\psi\Big)\Big]. \tag{8.7}$$

The virtual work of active forces (8.5) is

$$\delta A^{\mathbf{F}} = 2mg\delta y_1 + mg\delta y_2 - mg\frac{3}{2}l\sin\varphi\delta\varphi - mgl\sin\psi\delta\psi,$$

and the generalized forces in perpendicular directions are

$$N_{x_1} = 0, \quad N_{y_1} = 2mg, \quad N_{x_2} = 0, \quad N_{y_2} = mg. \tag{8.8}$$

Applying (7.4) to (8.7) and using (8.8) we have

$$\begin{aligned}
R_{x_1} &= \frac{d}{dt}\Big[\frac{ml}{2}\dot{\varphi}\cos\varphi + ml\Big(\dot{\varphi}\cos\varphi + \frac{\dot{\psi}}{2}\cos\psi\Big)\Big],\\
R_{y_1} &= -2mg - \frac{d}{dt}\Big[\frac{ml}{2}\dot{\varphi}\sin\varphi + ml\Big(\dot{\varphi}\sin\varphi + \frac{\dot{\psi}}{2}\sin\psi\Big)\Big],\\
R_{x_2} &= \frac{d}{dt}\Big[ml\Big(\dot{\varphi}\cos\varphi + \frac{\dot{\psi}}{2}\cos\psi\Big)\Big],\\
R_{y_2} &= -mg - \frac{d}{dt}\Big[ml\Big(\dot{\varphi}\sin\varphi + \frac{\dot{\psi}}{2}\sin\psi\Big)\Big].
\end{aligned} \tag{8.9}$$

The vectors in perpendicular directions to the constraints (8.6) are $\mathbf{l}_{1x_1} = \mathbf{i}$, $\mathbf{l}_{1y_1} = \mathbf{j}$, $\mathbf{l}_{2x_2} = \mathbf{i}$, $\mathbf{l}_{2y_2} = \mathbf{j}$. Hence the system $b_{\Delta\Gamma}$ and its inverse system $(b^{-1})_{\Delta\Gamma}$ are 4×4 unit matrixes, and using (3.8) we have $(R^{-1})_{x_1} = R_{x_1}$, $(R^{-1})_{y_1} = R_{y_1}$, $(R^{-1})_{x_2} = R_{x_2}$, $(R^{-1})_{y_2} = R_{y_2}$. Therefore, the vectors of reaction forces at joints at the points O and A are $\mathbf{R}_O = R_{x_1}\mathbf{i} + R_{y_1}\mathbf{j}$, $\mathbf{R}_A = R_{x_2}\mathbf{i} + R_{y_2}\mathbf{j}$, where R_{x_1}, R_{y_1}, R_{x_2} and R_{y_2} are given by (8.9).

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