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ALGEBRO-GEOMETRIC APPROACH TO THE YANG–BAXTER EQUATION AND RELATED TOPICS

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Abstract. We review the results of algebro-geometric approach to 4×4 solutions of the Yang–Baxter equation. We emphasis some further geometric properties, connected with the double-reflection theorem, the Poncelet porism and the Euler–Chasles correspondence. We present a list of classifications in Mathematical Physics with a similar geometric background, related to pencils of conics. In the conclusion, we introduce a notion of discriminantly factorizable polynomials as a result of a computational experiment with elementary n-valued groups.

1. Introduction

The aim of this paper is to present a circle of questions which arise in contemporary Mathematical Physics and Mechanics, in quite different contexts, but, as we observe, they all share the same geometric background. The central subject of our presentation is the Yang–Baxter equation, one of the main objects in Quantum inverse scattering method (see [**33**]) and in exactly solvable models in Lattice Statistical Mechanics (see [**4**]). The approach we follow here is an algebro-geometric one, and it goes back 30 years in the past, to a paper of Krichever [**29**]. The classification of so-called rank one solutions in the first nontrivial case of 4×4 matrices has been performed there, for the case of general position. Later, it has been completed in works of the author, [**13**], [**15**].

Here, we want to emphasis geometric ideas, which lie behind that classification, and which lead us to the Euler–Chasles correspondence and to the study of pencils of conics.

We notice a similar situation in two more recent subjects: in the classification of integrable quad-graphs by Adler, Bobenko and Suris, see [**2**], and a new classification of discriminantly separable polynomials of degree two in each of the three variables from [**22**].

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Dedicated to the memory of Academician Anton Bilimovich (1879–1970).

Yet another, fourth case, goes more in the past, to 1954: this is the well-known Petrov classification (see [**31**], [**32**]) in the study of exact solutions of Einstein's field equations. Mathematically, it describes the possible algebraic symmetries of the Weyl tensor at a point in a Lorentzian manifold. A relationship between the Petrov classification and the pencils of conics has been elaborated in [**9**]. The Penrose diagram (see [**30**], and the diagrams (8.1) and (8.3) below) is a simple but effective illustration of the Petrov classification.

The Penrose type scheme (see (8.1) and (8.3)) may serve as a perfect light motif for all four stories of modern Mathematical Physics, we are going to talk about.

We conclude the paper with a piece of Experimental Mathematics. A general case of discriminantly separable polynomials has one more interpretation in terms of 2-valued groups associated with elliptic curves (see [**19**], [**22**]). In [**19**] it has been shown that an elementary 2-valued group p_2 from [**6**] also has discriminant separability property. In the last section we examine, as a simple computational experiment, the discriminants of higher elementary n-valued groups. It appears that the discriminants *are not separable any more, but they are still factorizable.* We hope to reach a better understanding of this interesting phenomena, and possibly find some dynamical applications.

2. The Euler–Chasles correspondence. Baxter's *R*-matrix

Symmetric $(2-2)$-correspondences, of the form

$$E: \quad ax^2y^2 + b(x^2y + xy^2) + c(x^2 + y^2) + 2dxy + e(x + y) + f = 0 \tag{2.1}$$

play an important role in geometry, especially in connection with classical theorems of Poncelet and Darboux, as well as in the theory of ordinary differential equations, and addition theorems. The first crucial step in the understanding of the last relation in later contexts goes back to Euler, while the geometric part is associated to Chasles and Darboux.

In 1766 Euler proved the following theorem, as the starting point of the addition theory for elliptic functions.

Theorem 2.1 (Euler 1766). *For the general symmetric $(2-2)$-correspondence (2.1) there exists an even elliptic function ϕ of the second degree and a constant shift c such that $u = \phi(z)$, $v = \phi(z \pm c)$.*

Thus, the usual name for relation (2.1) is the *Euler–Chasles correspondence.*

In the modern science, the Euler–Chasles correspondence occupies an important place in a yet another context related to statistical and quantum mechanics. It has been a cornerstone in a remarkable book of Baxter (see [**4**, p. 471]).

The aim of this paper is to elaborate connections between these various ideas coming from geometry, classical mechanics and statistical and quantum mechanics. It parallels and develops the ideas we presented in our book [**23**].

Baxter derived an elliptic parametrization of a symmetric biquadratic in his book, by reproving the Euler theorem. By using a projective Möbius transformation both on x and y, the given biquadratic reduces to the canonical form

$$x^2y^2 + c_1(x^2 + y^2) + 2d_1xy + 1 = 0.$$

The last quadratic equation in x can be solved:

$$y = -\frac{dx \pm \sqrt{-c_1 + (d_1^2 - 1 - c_1^2)x^2 - c_1 x^4}}{c_1 + x^2}.$$

The square root is a polynomial of fourth degree in x, which can be transformed to a perfect square by use of Jacobian elliptic functions. In the change of variables $x = k^{1/2}\,\mathrm{sn}\,u$, the modulus k satisfies $k + k^{-1} = (d_1^2 - c_1^2 - 1)/c_1$. Well-known identities of the Jacobian functions and the addition formula for $\mathrm{sn}(u-v)$ are used to transform the argument of the discriminant

$$-c_1(1 - (k + k^{-1})x^2 + x^4) = -c_1(1 - \mathrm{sn}^2 u)(1 - k^2\,\mathrm{sn}^2 u) = -c\,\mathrm{cn}^2 u \cdot \mathrm{dn}\,u.$$

Taking a parameter η such that $c_1 = -1/k\,\mathrm{sn}^2\eta$, one gets $d_1 = \mathrm{cn}\,\eta\,\mathrm{dn}\,\eta/(k\,\mathrm{sn}^2\eta)$ and finally, the formula for y:

$$y = k^{1/2}\frac{\mathrm{sn}\,u \cdot \mathrm{cn}\,\eta \cdot \mathrm{dn}\,\eta \pm \mathrm{sn}\,\eta \cdot \mathrm{cn}\,u \cdot \mathrm{dn}\,u}{1 - k^2\,\mathrm{sn}^2 u \cdot \mathrm{sn}^2\eta} = k^{1/2}\,\mathrm{sn}(u \pm \eta).$$

Baxter comes to a parametrization of the canonical form of a biquadratic:

$$x = k^{1/2}\,\mathrm{sn}\,u, \quad y = k^{1/2}\,\mathrm{sn}\,v,$$

where $v = u - \eta$ or $v = u + \eta$. The final form of parametrization:

$$x = \phi(u), \quad y = \phi(u \pm \eta)$$

with an elliptic function ϕ, is obtained from the Jacobian sn function by use of the projective transformations.

Baxter came along this way to his celebrated R-matrix, which played a crucial role in his solutions of the XYZ and the Eight Vertex Model. The Baxter R-matrix is a 4×4 matrix $R_b(t,h)$ of the form

$$R_b(t,h) = \begin{pmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{pmatrix}, \tag{2.2}$$

where $a = \mathrm{sn}(t + 2h)$, $b = \mathrm{sn}\,t$, $c = \mathrm{sn}\,2h$, $d = k\,\mathrm{sn}\,2h \cdot \mathrm{sn}\,t \cdot \mathrm{sn}(t + 2h)$. It is a solution of *the Yang–Baxter equation*

$$R^{12}(t_1 - t_2, h)R^{13}(t_1, h)R^{23}(t_2, h) = R^{23}(t_2, h)(R^{13}(t_1, h)R^{12}(t_1 - t_2, h).$$

Here t is the so called *spectral parameter* and h is the *Planck constant.* Here we assume that $R(t,h)$ is a linear operator from $V \otimes V$ to $V \otimes V$ and

$$R^{ij}(t,h) : V \otimes V \otimes V \to V \otimes V \otimes V$$

is an operator acting on the i-th and j-th components as $R(t,h)$ and as identity on the third component. For example $R^{12}(t,h) = R \otimes Id$. In the first nontrivial case, the matrix $R(t,h)$ is 4×4 and the space V is two-dimensional. Even in this case, the quantum Yang–Baxter equation is highly nontrivial. It represents strongly over-determined system of 64 third degree equations on 16 unknown functions. It is not obvious at all why the solutions exist.

The Yang–Baxter equation is a paradigm of a modern addition formula, and it is a landmark of Mathematical Physics in the last 25 years.

If the h dependence satisfies the quasi-classical property $R = I + hr + O(h^2)$, then the classical r-matrix $r = r(t)$ satisfies the co-called *classical Yang–Baxter equation*. Classification of the solutions of the classical Yang–Baxter equation was done by Belavin and Drinfeld in 1982. The problem of classification of the quantum R matrices is still unsolved. However, some classification results have been obtained in the basic 4×4 case by Krichever (see [**29**]) and following his ideas in [**13, 14, 15**]. Before we pass to the exposition of Krichever's ideas, let us briefly recall the basic definitions of the Heisenberg XYZ model, and the double reflection theorem.

3. Heisenberg quantum feromagnetic model

The Heisenberg ferromagnetic model [**27**] is defined by its Hamiltonian

$$H = -\sum_{i=1}^{N} \left(X\sigma_{i+1}^x \sigma_i^x + Y\sigma_{i+1}^y \sigma_i^y + Z\sigma_{i+1}^z \sigma_i^z \right),$$

where the operator H maps $V^{\otimes N}$ to $V^{\otimes N}$, $V = \mathbb{C}^2$ and σ_i^x denotes the operator which acts as the Pauli matrix on i-th V and as the identity on the other components. Following Heisenberg's definition of the model in 1928, Bethe solved the simplest case $X = Y = Z$ in 1931. The next step was done by Yang in 1967 by solving the XXZ case, obtained by putting $X = Y$. The final step was done by Baxter, who solved the general XYZ problem in 1971 (see [**4**]).

Both, Yang and Baxter exploited the connection with statistical mechanics on the plane lattice. The first one used the six-vertex model and the second one the eight-vertex model. Denote by L and L' local transition matrices

$$L, L' : W \otimes V \to W \otimes V,$$

and by R a solution of the Yang–Baxter equation acting on $V \otimes V$. The key role is played by the matrix R. In Yang case, that was R_y, the XXZ-R matrix of the form

$$R_y(t,h) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix},$$

where a, b, c are trigonometrical functions obtained when k tends to 0 from corresponding functions in the Baxter matrix (see equation (2.2)).

The fundamental point, in both Yang's and Baxter's approaches, was the relation between two tensors Λ_1 and Λ_2 in $W \otimes V \otimes V$, which we are going to call *the Yang equation*:

$$\Lambda_1 = \Lambda_2, \tag{3.1}$$

where

$$\begin{aligned} \Lambda_1 &= \Lambda_{1pq\beta}^{ij\alpha} = L'^{k\gamma}_{p\beta} L^{l\alpha}_{q\gamma} R^{ij}_{kl} \\ \Lambda_2 &= \Lambda_{2pq\beta}^{ij\alpha} = R^{kl}_{pq} L^{i\gamma}_{k\beta} L'^{j\alpha}_{l\gamma}. \end{aligned} \tag{3.2}$$

We assume summation over repeated indices and we use the convention that Latin indices indicate the space V, while the Greek ones are reserved for W. In the simplest case, when W and V are two-dimensional, the Yang equation is again overdetermined system of 64 equations of degree three in 48 unknowns.

Denote by $T = \prod_{n=1}^{N} L_n$, where $L_n : W^{\otimes N} \otimes V \to W^{\otimes N} \otimes V$, acts as L on the n-th W and V and as identity otherwise. From the Yang equation (3.1, 3.2) it follows $R(T \otimes T') = (T' \otimes T)R$. Therefore, all the operators $\mathrm{tr}_V T$ commute. The connection between the Heisenberg model and the vertex models mentioned above lies in the fact that the Hamiltonian operator H commutes with all the operators $\mathrm{tr}_V T$. Thus, they have the same eigenvectors.

The Algebraic Bethe Ansatz (ABA) is a method of a formal construction of those eigenvectors. There is a very nice presentation of the Algebraic Bethe Ansatz in the work of Takhtadzhyan and Faddeev (see [**33**]). Starting with matrices R_y and R_b they found vectors X, Y, U, V which satisfy the relation

$$RX \otimes U = Y \otimes V. \tag{3.3}$$

They calculated the vectors X, Y, U, V in terms of theta-functions, and using computational machinery of theta-functions, they produced the eigenvectors of the Heisenberg model.

We are going to present here a sort of converse approach to the Algebraic Bethe Ansatz, see [**16**] and references therein. Our presentation of the Algebraic Bethe Ansatz applies uniformly to all 4×4 solutions of rank 1 of the Yang equation. It does not involve computations with theta functions, but uses geometry of the Euler–Chasles correspondence. It can be interpreted, as we are going to show in the next section, in terms of the billiard dynamics within ellipses and of the Poncelet geometry, see also [**23**].

4. Double reflection configuration

In this section, following [**12**], see also [**23**], we introduce a fundamental projective geometry configuration of double reflection in d-dimensional projective space over an arbitrary field of characteristic not equal to 2.

We assume a pencil of quadrics to be given.

First, we define a notion of reflection projectively, without metrics.

Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two quadrics, from the given pencil, or in other words, defining a pencil. Denote by u the tangent plane to $\mathcal{Q}_1$ at point x and by z the pole of u with respect to $\mathcal{Q}_2$. Suppose lines l_1 and l_2 intersect at x, and the plane containing these two lines meet u along l.

Definition 4.1. If lines l_1, l_2, xz, l are coplanar and harmonically conjugated, we say that rays l_1 and l_2 *obey the reflection law* at the point x of the quadric $\mathcal{Q}_1$ with respect to the confocal system which contains $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

If we introduce a coordinate system in which quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are confocal in the usual sense, reflection defined in this way is the same as the standard, metric one.

Theorem 4.1 (One Reflection Theorem). *Suppose rays l_1 and l_2 obey the reflection law at x of $\mathcal{Q}_1$ with respect to the confocal system determined by quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Let l_1 intersects $\mathcal{Q}_2$ at y_1' and y_1, u is a tangent plane to $\mathcal{Q}_1$ at x, and z its pole with respect to $\mathcal{Q}_2$. Then lines $y_1'z$ and y_1z respectively contain intersecting points y_2' and y_2 of ray l_2 with $\mathcal{Q}_2$. The converse is also true.*

Corollary 4.1. *Let rays l_1 and l_2 obey the reflection law of $\mathcal{Q}_1$ with respect to the confocal system determined by quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Then l_1 is tangent to $\mathcal{Q}_2$ if and only if is tangent l_2 to $\mathcal{Q}_2$; l_1 intersects $\mathcal{Q}_2$ at two points if and only if l_2 intersects $\mathcal{Q}_2$ at two points.*

The next assertion is crucial for our further considerations.

Theorem 4.2 (Double Reflection Theorem). *Suppose that $\mathcal{Q}_1$, $\mathcal{Q}_2$ are given quadrics and l_1 line intersecting $\mathcal{Q}_1$ at the point x_1 and $\mathcal{Q}_2$ at y_1. Let u_1, v_1 be tangent planes to $\mathcal{Q}_1$, $\mathcal{Q}_2$ at points x_1, y_1 respectively, and z_1, w_1 their with respect to $\mathcal{Q}_2$ and $\mathcal{Q}_1$. Denote by x_2 second intersecting point of the line w_1x_1 with $\mathcal{Q}_1$, by y_2 intersection of y_1z_1 with Q_2 and by l_2, l_1', l_2' lines x_1y_2, y_1x_2, x_2y_2. Then pairs l_1, l_2; l_1, l_1'; l_2, l_2'; l_1', l_2' obey the reflection law at points x_1 (of $\mathcal{Q}_1$), y_1 (of $\mathcal{Q}_2$), y_2 (of $\mathcal{Q}_2$), x_2 (of $\mathcal{Q}_1$) respectively.*

Corollary 4.2. *If the line l_1 is tangent to quadric $\mathcal{Q}'$ confocal with $\mathcal{Q}_1$ and $\mathcal{Q}_2$, then rays l_2, l_1', l_2' also touch $\mathcal{Q}'$.*

Now, we give a definition of a certain configuration which is going to play a central role in the paper. It is connected with so-called real and virtual reflections, but its properties remain in the projective case, too.

Let points X_1, X_2; Y_1, Y_2 belong to quadrics $\mathcal{Q}_1$, $\mathcal{Q}_2$ in $\mathbf{P}^d$.

Definition 4.2. We will say that the quadruple of points X_1, X_2, Y_1, Y_2 constitutes a *virtual reflection configuration* if pairs of lines X_1Y_1, X_1Y_2; X_2Y_1, X_2Y_2; X_1Y_1, X_2Y_1; X_1Y_2, X_2Y_2 satisfy the reflection law at points X_1, X_2 of $\mathcal{Q}_1$ and Y_1, Y_2 of $\mathcal{Q}_2$ respectively, with respect to the confocal system determined by $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

If, additionally, the tangent planes to $\mathcal{Q}_1, \mathcal{Q}_2$ at X_1, X_2; Y_1, Y_2 belong to a pencil, we say that these points constitute a *double reflection configuration.*

Let us also recall the classical notion of the Darboux coordinates in a projective plane.

Definition 4.3. Given a conic K in the plane, with a fixed rational parametrization. For a given point P in the plane, there are two tangents from P to the conic K. Denote the values of the rational parameter of the two points of tangency of the tangent lines with the conic K by (x_1, x_2). Then, we call the pair (x_1, x_2) *the Darboux coordinates* of the point P associated with the parametrized conic K.

Using the Darboux coordinates, we can visualize the Euler–Chasles correspondence (2.1) by the Figure 1.

This connects the Euler–Chasles correspondence with a billiard dynamic within a conic. In such a dynamics, for a given billiard trajectory, there is always a caustic,

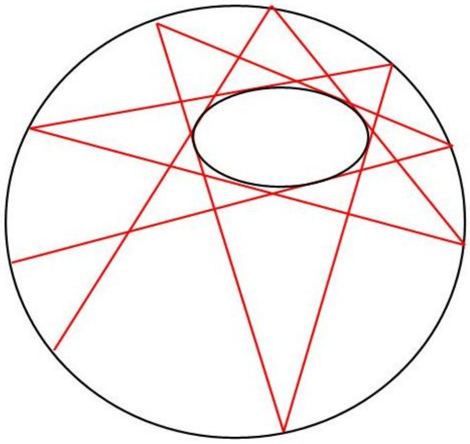

FIGURE 1. The Euler–Chasles correspondence

an additional "internal" conic, such that each segment of the trajectory, or a line supporting it, is tangent to the caustic. This caustic plays a role of the conic K in the definition of the Darboux coordinates.

In a case where caustic and the boundary are two confocal conics in an Euclidean plane, then the associated billiard dynamics is the classical one, where the angles of impact and reflection are equal.

For a relationship between the Euler–Chasles correspondence and 2-valued group action, see [**8**] and [**19**].

Given a pencil of conics in a projective plane, and four conics from the pencil K, C, C_1, C_2. If there is a triangle with sides tangent to K with vertices on C, C_1 and C_2, then, according to a general Poncelet theorem (see [**5**], [**23**]), there are infinitely many such triangles. We will call such triangles *Poncelet triangles.* We recall that by applying the double reflection theorem, one passes from one Poncelet triangle, to another one.

5. Krichever's algebro-geometric approach

Our, converse approach to the Algebraic Bethe Ansatz is based on ideas of Krichever (see [**29**]), on his classification of rank 1 4×4 solutions of the Yang equation in general situation and on classification of remaining cases, developed in [**13, 14, 15**]. Then, we connect these ideas with projective-geometry constructions from the previous Section.

5.1. Vacuum vectors and vacuum curves. Having in mind Baxter's considerations leading to the discovery of the Baxter R matrix and Faddeev–Takhtadzhyan study of the vectors of the form given by the equation (3.3), Krichever in

[**29**] suggested a sort of inverse approach, following the best traditions of the theory of the "finte-gap" integration (see [**25**]).

Krichever's method is based on the *vacuum vector representation* of an arbitrary $2n \times 2n$ matrix L. Such a matrix is understood as a 2×2 matrix with blocks of $n \times n$ matrices. In other words, $L = L^{i\alpha}_{j\beta}$ is a linear operator in the tensor product $C^n \otimes C^2$. *The vacuum vectors* X, Y, U, V satisfy, by definition, the relation

$$LX \otimes U = hY \otimes V,$$

or in coordinates

$$L^{i\alpha}_{j\beta} X_i U_\alpha = hY_j V_\beta,$$

where we assume now that the Latin indices run from 1 to n while the Greek ones from 1 to 2. We assume additionally the following convention for affine notation

$$X_n = Y_n = U_2 = V_2 = 1, \quad U_1 = u, \quad V_1 = v, \text{ and } \tilde{V} = (1 \ - v).$$

The vacuum vectors are parametrized by *the vacuum curve* Γ_L, which is defined by the affine equation

$$\Gamma_L : P_L(u, v) = \det(L^i_j) = \det(V^\beta L^{i\alpha}_{j\beta} U_\alpha) = 0.$$

The polynomial $P_L(u, v)$, called *the spectral polynomial* of the matrix L is of degree n in each variable. In general position, the genus of the curve Γ_L is equal $g = g(\Gamma_L) = (n-1)^2$ and Krichever proved that X understood as a meromorphic function on Γ_L is of degree $N = g + n - 1$. And, following ideology of "finite-gap" integration, Krichever proved the converse statement.

Theorem 5.1 (Krichever, [**29**]). *In the general position, the operator L is determined uniquely up to a constant factor by its spectral polynomial and by the meromorphic vector-functions X and Y on the vacuum curve with pole divisors D_X and D_Y of degree $n(n-1)$ which satisfy $D_X + D_U \sim D_Y + D_V$.*

5.2. General rank 1 solutions in (4×4) **case.** Now we specialize to the basic 4×4 case. The question is to describe analytical conditions, vacuum curves and vacuum vectors of three 4×4 matrices R, L and L' in order to satisfy the Yang equation, see equations (3.1, 3.2).

Denote by $P = P(u, v)$ and $P_1 = P_1(u, v)$ the spectral polynomial of given 4×4 matrices L and L'. They are polynomials of degree 2 in each variable and they define the vacuum curves $\Gamma = \Gamma_L : P(u, v) = 0$ and $\Gamma_1 = \Gamma_{L'} : P_1(u, v) = 0$. The vacuum curves are of genus not greater than 1.

Let us make one general observation concerning vacuum curves in 4×4 case. As one can easily see, each of them can naturally be understood as the intersection of two quadrics in $\mathbb{P}^3$. The first one is the Segre quadric, seen as embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ represented by $Y \otimes V$. The second one is the image of the Segre quadric represented by $X \otimes U$ by a linear map, induced by the linear operator under consideration.

In this subsection, following Krichever, we are going to consider only the general case, when those curves are elliptic ones. Thus, we have

$$\begin{aligned} L(X(u, v) \otimes U) &= h(u, v)(Y(u, v) \otimes V), \\ L'(X(u^1, v^1) \otimes U^1) &= h_1(u^1, v^1)(Y^1(u, v) \otimes V^1). \end{aligned} \tag{5.1}$$

Each of the tensors Λ_i which represents one of the sides of the Yang equation (see equations (3.1,3.2), is by itself a 2×2 matrix of blocks of 4×4 matrices. Denote corresponding spectral polynomials of degree four in each variable as Q_1 and Q_2 and the vacuum curves as $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$. Krichever's crucial observation is the following

THEOREM 5.2 (Krichever). *If 4×4 matrices L, L', R satisfy the Yang equation, then $(2-2)$ relations defined by the polynomials P and P_1 commute.*

From our experience with Euler–Chasles correspondences, we know that composition of two commuting ones is reducible. Although, we haven't considered symmetry of spectral polynomials yet, the same can be proven for their composition.

LEMMA 5.1. [**29**] *The polynomial $Q(u,w) = Q_1(u,w) = Q_2(u,w)$ is reducible.*

If the spectral polynomial Q is a perfect square, then the triplet (R, L, L') of solutions of the Yang equation is of *rank two*. Otherwise, it is a *solution of rank 1.*

In this subsection we proceed to consider rank 1 solutions. Let us consider the elliptic component $\hat{\Gamma}'$ of the vacuum curve $\hat{\Gamma}$ of the matrix Λ which contains pairs (u, w_1) and (u, w_4), using terminology of the proof of the last lemma. The curve $\hat{\Gamma}'$ is isomorphic to the vacuum curves Γ and Γ_1 and using uniformizing parameter x of the elliptic curve.

Denote by $(u(z), w(z))$, $(u(z), v(z))$ and $(u(z), \hat{v}(z))$ parametrizations of the curves $\hat{\Gamma}'$, Γ and Γ_1 respectively. Taking into account that $(v(z), w(z))$ also parametrizes Γ_1, we get

PROPOSITION 5.1. *There exist shifts η and η_1 on the elliptic curve, such that*

$$v(z) = u(z - \eta), \quad \hat{v}(z) = u(z - \eta_1).$$

From the equations (5.1) now we have

$$\begin{aligned} R(X(z-\eta_1) \otimes X^1(z)) &= g(z)(X^1(z-\eta) \otimes X(z)), \\ Y(z) &= X(z - \eta_2), \\ Y^1(z) &= X^1(z - \eta_2), \\ L(X(z) \otimes U(z)) &= h(z)(X(z+\eta_2) \otimes U(z-\eta)), \\ L^1(X^1(z) \otimes U(z)) &= h^1(z)(X^1(z+\eta_2) \otimes U(z-\eta_1)), \end{aligned} \tag{5.2}$$

where η_2 is a shift, which, as well as the shift η_1, differs from η by a half-period of the elliptic curve.

If we denote by G_X, G_{X^1}, G_U arbitrary invertible (2×2) matrices, then they define a *weak gauge transformations* which transform a triplet (R, L, L_1) solution of the Yang equation to a triplet $(\tilde{R}, \tilde{L}, \tilde{L}_1)$ by the formulae

$$\begin{aligned} \tilde{L} &= (G_X \otimes G_U) L (G_X^{-1} \otimes G_U^{-1}), \\ \tilde{L}_1 &= (G_{X^1} \otimes G_U) L_1 (G_{X^1}^{-1} \otimes G_U^{-1}), \\ \tilde{R} &= (G_{X^1} \otimes G_X) R (G_X^{-1} \otimes G_{X^1}^{-1}), \end{aligned}$$

which are again solutions of the Yang equation.

Denote by G_i the 2×2 matrices which correspond to shifts for half-periods: $U(z+1/2)=G_1U(z)$ and $U(z+1/2\tau)=fG_2U(z)$, where

$$G_1=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\quad G_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Then the shift of η_1 for half-periods transforms solutions according to the formula

$$T_i\colon (L,L_1R)\mapsto (L,(I\otimes G_i)L_1,R(G_i\otimes I)). \tag{5.3}$$

In the same way, the shift of η_2 for half-periods transforms solutions according to the formula

$$\hat{T}_i\colon (L,L_1R)\mapsto (L(G_i\otimes I),L_1(G_i\otimes I)). \tag{5.4}$$

Summarizing, we get

THEOREM 5.3 (Krichever). *Given an arbitrary elliptic curve Γ with three points η,η_1,η_2 which differ up to a half period and three meromorphic functions of degree two x,x^1,u. Then formulae* (5.2) *define solutions of the Yang equation. All 4×4 rank one general solutions of the Yang equation are of that form. Moreover, up to weak gauge transformations and transformations* (5.3) *and* (5.4) *all such solutions are equivalent to Baxter's matrix R_b.*

For the last part of the theorem exact formulae for vacuum vectors for the Baxter matrix from [**33**] have been used. Krichever also formulated a following

PROPOSITION 5.2. *In the case $\eta=\eta_1=\eta_2$ all 4×4 rank one general solutions of the Yang equation are weak-gauge equivalent to the Baxter solution R_b.*

5.3. Rank 1 solutions in nongeneral (4×4) **cases.** But, as it was observed in [**13**], [**15**], there exist 4×4 rank one solutions of the Yang–Baxter equation which are not equivalent to the Baxter solution. The modal example is the so-called Cherednik matrix R_{ch} (see [**11**]) with the formula

$$R_{ch}(t,h)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & b & c & 0\\ 0 & c & b & 0\\ d & 0 & 0 & 1\end{pmatrix},$$

where

$$b=\frac{\sinh t}{\sinh(t+h)},\quad c=\frac{\sinh t}{\sinh(t+h)},\quad d=-4\sinh t\cdot\sinh h.$$

It was calculated in [**13**] that the spectral polynomial is of the form

$$P_{ch}(u,v)=Au^2v^2+B(u^2+v^2)+Cuv \tag{5.5}$$

and analytic properties have been studied. It was proven that vacuum curve in this case is rational with ordinary double point. For such sort of solutions, a description as in Theorem 5.3 has been established in [**13**] where it was summarized with

THEOREM 5.4. *All* (4×4) *rank one solutions of the Yang equation with rational vacuum curve with ordinary double point are gauge equivalent to the Cherednik solution.*

The Cherednik and the Baxter solutions of the Yang–Baxter equation are essentially different and are not gauge equivalent. Moreover, the second one is $\mathbb{Z}_2$ symmetric, while the first one is not. It is interesting to note that, nevertheless, the first one can be obtained from the last one in a nontrivial gauge limit.

LEMMA 5.2. [**13**] *Denote by* $T(k)$ *the following family of matrices depending on the modulus* k *of the elliptic curve:*

$$T(k) = \begin{pmatrix} (-4/k)^{1/4} & 0 \\ 0 & (-k/4)^{1/4} \end{pmatrix}$$

and denote by $R_b(t, h; k)$ *the Baxter matrix. Then*

$$\lim_{k\to 0}(T(k)\otimes T(k))R_b(t, h; k)(T(k)^{-1}\otimes T(k)^{-1}) = R_{ch}(it, h).$$

A similar analysis of vacuum data as for the Cherednik solution, was done for the Yang solution R_y. In [**15**] it was shown that the vacuum curve consists of two rational components. For such kind of solutions the similar statement was proven there.

THEOREM 5.5. *All* (4×4) *rank one solutions of the Yang equation with vacuum curve reducible on two rational components are gauge equivalent to the Yang solution.*

It was observed in [**13**] that such an approach does not lead to a solution of the Yang–Baxter equation with rational vacuum curve with cusp singularity.

5.4. Relationship with Poncelet–Darboux theorems. Let us go back to considerations from [**29**], around Lemma 5.1 and Theorem 5.2. We are going to underline a couple of observations which have not been stressed there, see [**23**].

Denote by $P_2(u, w) = 0$ the relation which corresponds to $\hat{\Gamma}'$.

PROPOSITION 5.3. *If* $P(u, v) = 0$*,* $P_1(u, v) = 0$ *and* $P_2(u, v) = 0$ *are* $(2-2)$ *relations corresponding to rank one solutions of the Yang equation, then these correspondences are symmetric.*

Thus, these correspondences are of the Euler–Chasles type. According to geometric interpretations of such correspondences, we may associate to P a pair of conics (K, C), to P_1 a pair (K, C_1) and finally a pair (K, C_2) to P_2. Since correspondences commute, they form a pencil χ of conics. Since we assume in this subsection that underlying curve is elliptic, the pencil χ is of general type, having four distinct points in the critical divisor. In other words, the conics K, C, C_1 and C_2 intersect in four distinct points.

Having in mind the Darboux coordinates and related geometric interpretation, we come to the following

Theorem 5.6. *Suppose a general* 4×4 *rank one solution of the Yang equation is given. Then there exist a pencil of conics* χ *and four conics* $K, C, C_1, C_2 \in \chi$ *intersecting in four distinct points such that triples* (u, v, w) *and* $(u, \hat{v}, w)$ *satisfying* $P(u, v) = 0$*,* $P_1(v, w) = 0$ *and* $P_1(u, \hat{v}) = 0$*,* $P(\hat{v}, w) = 0$ *form sides of two Poncelet triangles circumscribed about* K *with vertices on* C*,* C_1 *and* C_2*. Moreover, the quadruple of lines* $(u, v, \hat{v}, w)$ *forms a double reflection configuration at* C *and* C_1*.*

Such type of pencils is sometimes denoted also as $(1, 1, 1, 1)$. In cases of 4×4 rank one solutions with rational vacuum curves, the situation is practically the same. One can easily calculate directly that spectral polynomials for the Cherednik R matrix (see equation (5.5)) and for the Yang R matrix are symmetric.

Proposition 5.4. (a) *In the case of Cherednik type solutions, corresponding pencil of conics has one double critical point and two ordinary ones–type* $(2, 1, 1)$*. Conics are tangent in one point and intersect in two other distinct points.*

(b) *In the case of the Yang type solutions, the corresponding pencil of conics is bitangential: it has two double critical point. Conics are tangent in two points-type* $(2, 2)$*.*

(c) *In both rational cases, the statement about two Poncelet triangles and double reflection configuration holds as in Theorem* (5.6)*.*

It would be interesting to check if geometrically possible degeneration of a pencil to a superosculating one, the one of type (4), can give some nontrivial contribution to solutions of the Yang–Baxter equation.

In all cases, elliptic or rational, the vacuum vector representation of rank one 4×4 solutions of the Yang–Baxter equation has the following form

$$LX_l \otimes U_l = hX_{l+1} \otimes U_{l-1},$$

where all the functions are meromorphic on a vacuum curve Γ of degree two and we use notation $X_{l+n} = X_l \circ \Psi^n$. Here Ψ is an automorphism of the vacuum curve Γ and geometric interpretation of the last formula is the following: *the tangent to conic* K *with Darboux coordinate* x_l *reflects* n *times of the conic* C_1 *and gives a new tangent to* K *with Darboux coordinate* x_{l+n}*.*

The last formula with the last interpretation plays a crucial role in our presentation of the Algebraic Bethe Ansatz.

6. Algebraic Bethe Ansatz and Vacuum Vectors

Our starting point, following [**16**], is the last formula giving general vacuum vector representation of all rank one solutions of the Yang–Baxter equation in 4×4 case.

As in [**16**], together with vacuum vector representation, we consider *the vacuum covector representation*:

$$A^j B^\beta L^{i\alpha}_{j\beta} = lC^i D^\alpha.$$

We will use more general notion of spectral polynomial. Let P^{ij}_L denotes polynomial in two variables obtained as determinant of the matrix generated by L contracting the i-th bottom and j-th top index. The spectral polynomial we used up to now, in this notation becomes P_L^{22}.

It can easily be shown that vacuum covectors, under the same analytic conditions as vacuum vectors, uniquely define matrix L. Thus, there is a relation between vacuum vectors and vacuum covectors.

LEMMA 6.1. [**16**] *If a* 4×4 *matrix* L *satisfies the condition*

$$L(X_l \otimes U_l) = h(X_{l+1} \otimes U_{l-1}),$$

then

$$(\tilde{X}_{l+1} \otimes \tilde{U}_{l+1})L = g(\tilde{X}_{l+2} \otimes \tilde{U}_l).$$

We use notation $X = [x\ 1]^t$ *and* $\tilde{X} = [1\ \ -x]$.

Let us just mention that the proof of the last lemma extends automatically to arbitrary 4×4 matrices.

The matrices L, solutions of the Yang equation, are local transition matrices. As it was suggested in [**33**] we change them according to the formula

$$L_n^l(\lambda) = M_{n+l}^{-1}(\lambda)L_n(\lambda)M_{n+l-1} = \begin{pmatrix} \alpha_n^l(\lambda) & \beta_n^l(\lambda) \\ \gamma_n^l(\lambda) & \delta_n^l(\lambda) \end{pmatrix},$$

where the matrix M_l has vacuum vectors as columns:

$$M_l = \begin{pmatrix} x_l & x_{l+1} \\ 1 & 1 \end{pmatrix}.$$

By this transformation, the monodromy matrix $T(\lambda) = \prod_{n=1}^N L_n(\lambda)$ transforms into $T_N^l(\lambda) = M_{N+l}^{-1}T(\lambda)M_l$. Denote the elements of the last matrix by $A_N^l(\lambda)$, $B_N^l(\lambda)$, $C_N^l(\lambda)$, $D_N^l(\lambda)$. The aim of the ABA method is to find local vacuum vectors ω^l independent of λ such that

$$\gamma_n^l(\lambda)\omega_n^l = 0, \quad \alpha_n^l(\lambda)\omega_n^l = g(\lambda)\omega_n^{l-1}, \quad \delta_n^l(\lambda)\omega_n^l = g'(\lambda)\omega_n^{l+1}.$$

Having the last relations satisfied, vector $\Omega_N^l = \omega_1^l \otimes \cdots \otimes \omega_N^l$ would satisfy

$$A_N^l(\lambda)\Omega_N^l = g^N(\lambda)\Omega_N^{l-1}, \quad D_N^l(\lambda)\Omega_N^l = g'^N(\lambda)\Omega_N^{l+1}, \quad C_N^l(\lambda)\Omega_N^l = 0$$

and provide a family of generating vectors.

THEOREM 6.1. [**16**] *The following relations are valid*

$$\gamma_n^l(\lambda)U_l = 0, \quad \alpha_n^l(\lambda)U_l = g(\lambda)U_{l-1}, \quad \delta_n^l(\lambda)\omega_n^l = g'(\lambda)U_{l+1}.$$

It is also necessary to calculate images of shifted vacuum vectors. The formulae are given in the following

LEMMA 6.2. [**16**] *The image of the shifted vacuum vector is a combination of two shifted vacuum vectors:*

$$LX_{l+1} \otimes U_l = hX_{l+1} \otimes U_l + h'X_l \otimes U_{l+1}.$$

The proof follows from Lemma (6.1) using the same arguments as at the end of the proof of the last Theorem.

From the last two Lemmas one can easily derive commuting relations between matrix elements of the transfer matrix. We need them for application of the ABA method and we state them in the following

PROPOSITION 6.1. *The matrix elements of the transfer matrix commute according to the formulae*

$$\begin{aligned} B^k_{l+1}(\lambda)B^{k+1}_l(\mu) &= B^k_{l+1}(\mu)B^{k+1}_l(\lambda), \\ B^k_{l-2}(\lambda)A^{k+1}_{l-1}(\mu) &= h'A^k_l(\mu)B^{k+1}_{l-1}(\lambda) + h''B^k_{l-2}(\mu)A^{k+1}_{l-1}(\lambda), \\ B^{k+2}_l(\lambda)D^{k+1}_{l-1}(\mu) &= k'D^k_l(\mu)B^{k+1}_{l-1}(\lambda) + k''B^{k+2}_l(\mu)D^{k+1}_{l-1}(\lambda). \end{aligned}$$

Using the last statements, one can derive relations among λ_i in order that the sum of vectors

$$\Psi_l(\lambda_1, \ldots, \lambda_n) = B^{l-1}_{l+1}(\lambda_1) \cdots B^{l-n}_{l+n}(\lambda_n)\Omega^{l-n}_N$$

be an eigen-vector of the operator $\operatorname{tr} T(\lambda) = A^l_l(\lambda) + D^l_l(\lambda)$. For more details see [**33**].

7. Rank 2 solutions in (4×4) case

The main example of solutions of rank two is the Felderhof R matrix, see [**26, 29**]. We will use the following its parametrization, (see [**3**]):

$$F_f(\phi|q, p|k) = \begin{pmatrix} b_1 & 0 & 0 & d \\ 0 & b_2 & c & 0 \\ 0 & c & b_3 & 0 \\ d & 0 & 0 & b_0 \end{pmatrix},$$

where

$$\begin{gathered} b_0 = \rho(1 - pqe(\phi)), \quad b_1 = \rho(e(\phi) - pq), \\ b_2 = \rho(q - pe(\phi)), \quad b_3 = \rho(p - qe(\phi)), \\ c = \frac{i\rho}{2 \cdot \operatorname{sn}(\phi/2)} \sqrt{(1-p^2)(1-q^2)}\big(1 - e(\phi)\big), \\ d = \frac{k\rho}{2} \sqrt{(1-p^2)(1-q^2)}\big(1 - e(\phi)\big) \cdot \operatorname{sn}(\phi/2), \\ e(\phi) = \operatorname{cn}\phi + i \cdot \operatorname{sn}\phi, \end{gathered}$$

where ρ is a trivial common constant, p, q are arbitrary constants and cn and sn are Jacobian elliptic functions of modulus k.

The key property of the Felderhof R-matrix is *the free-fermion condition*

$$b_0 b_1 + b_2 b_3 = c^2 + d^2.$$

The free-fermion six-vertex R-matrix F_{XXZ} is given by the limit:

$$F_{XXZ}(\phi|p, q) = \lim_{k \to 0} F_f(\phi|p, q|k).$$

To get a free-fermion analogous of the Cherednik R matrix, one applies the trick of Lemma 5.2. More detailed, denoting by $T(k)$ the family of matrices

$$T(k) = \begin{pmatrix} (-1/k)^{1/4} & 0 \\ 0 & (-k)^{1/4} \end{pmatrix},$$

we have

THEOREM 7.1. [**17**] *A rank two solution of the Yang–Baxter equation* $F_1(\phi|p,q)$ *is obtained in a limit*

$$F_1(\phi|p,q) = \lim_{k\to 0}(T(k)^{-1}\otimes T(k)^{-1})F_f(\phi|p,q|k)(T(k)\otimes T(k)),$$

with explicit formulae

$$F_1(\phi|p,q) = \begin{pmatrix} \hat{b}_1 & 0 & 0 & 0 \\ 0 & \hat{b}_2 & \hat{c} & 0 \\ 0 & \hat{c} & \hat{b}_3 & 0 \\ \hat{d} & 0 & 0 & \hat{b}_0 \end{pmatrix}$$

where

$$\begin{aligned} \hat{b}_0 &= \rho(1-pq\hat{e}(\phi)), \quad \hat{b}_1 = \rho(\hat{e}(\phi)-pq), \\ \hat{b}_2 &= \rho(q-p\hat{e}(\phi)), \quad \hat{b}_3 = \rho(p-q\hat{e}(\phi)), \\ c &= \frac{i\rho}{2\cdot\sin(\phi/2)}\sqrt{(1-p^2)(1-q^2)}\big(1-\hat{e}(\phi)\big), \\ d &= \frac{k\rho}{2}\sqrt{(1-p^2)(1-q^2)}\big(1-e(\phi)\big)\cdot\sin(\phi/2), \\ e(\phi) &= \cos\phi + i\cdot\sin\phi. \end{aligned}$$

PROPOSITION 7.1. *The vacuum curves of the rank two R-matrices are of the following types:*

(a) [**29**] *The vacuum curve of the Felderhof R matrix is elliptical.*
(b) [**17**] *The vacuum curve of the six-vertex free-fermion R matrix consists of two rational components.*
(c) [**17**] *The vacuum curve of the free-fermion R matrix* F_1 *is rational with ordinary double point.*

Consider two polynomials of type P_a, the vacuum polynomials in the general rank two case:

$$P_{a1} = u^2v^2 + A_1u^2 + B_1v^2 + 1, \quad P_{a2} = u^2v^2 + A_2u^2 + B_2v^2 + 1.$$

Krichever proved that they induce $(2-2)$ correspondences which commute if and only if $A_1 + B_1 = A_2 + B_2$. The same statement, with associated polynomials is true in rational cases, (see [**17**]).

The previous considerations were devoted to the basic, most simple 4×4 case of solutions of the Yang–Baxter equation. The real challenge is to develop something similar for higher dimensions. Some attempts were done, for example see [**18**]. But there a very specific situation was considered connected with the so called Potts model.

In general, essential problem is that Krichever's vacuum vector approach works only for even-dimensional matrices.

8. Pencils of conics and the Penrose diagram

8.1. Pencils of conics. We will denote pencils of conics of general $(1, 1, 1, 1)$ type as [A], with four simple common points of intersection. The case with two simple points of intersection and one double with a common tangent at that point is denoted $(1, 1, 2)$ as [B]. The next case with two double points of intersection and with a common tangent in each of them is $(2, 2)$, denoted as [C]. The case $(1, 3)$, denoted as [D] is defined by one simple and one triple point of intersection. Finally (4), the case of one quadriple point, is denoted as [E]. Fig. 2 (see [**5**] and [**22**]) illustrates possible configurations of pencils of conics.

Following [**20**], [**21**], we will code the process of transition from a more general pencil to a more special one by a Penrose-type diagram:

$$\begin{tikzcd} A \arrow[d] \arrow[dr] & & \\ B \arrow[r] \arrow[d] \arrow[dr] & C \arrow[d] \arrow[dr] & \\ D \arrow[r] & E \arrow[r] & O \end{tikzcd} \tag{8.1}$$

8.2. Integrable quad graphs. Now, we will start with basic ideas of the theory of integrable systems on quad graphs from works of Adler, Bobenko, Suris [**1**], [**2**]. We will use the notation $\mathcal{P}_d^n(K)$ for the set of polynomials in d variables of degree at most n in each of variables, over the field K.

Recall that the basic building blocks of systems on quad-graphs are the equations on quadrilaterals of the form

$$Q(x_1, x_2, x_3, x_4) = 0 \tag{8.2}$$

where $Q \in \mathcal{P}_4^1$ is a multiaffine polynomial.

Equations of type (8.2) are called *quad-equations*. The field variables x_i are assigned to four vertices of a quadrilateral as in Fig. 3. Equation (8.2) can be solved for each variable, and the solution is a rational function of the other three variables. A solution (x_1, x_2, x_3, x_4) of equation (8.2) is *singular* with respect to x_i if it also satisfy the equation $Q_{x_i}(x_1, x_2, x_3, x_4) = 0$.

Following [**2**] one considers the idea of integrability as consistency, see Fig. 4. One assigns six quad-equations to the faces of coordinate cube. The system is said to be *3D-consistent* if three values for x_{123} obtained from equations on right, back and top faces coincide for arbitrary initial data x, x_1, x_2, x_3.

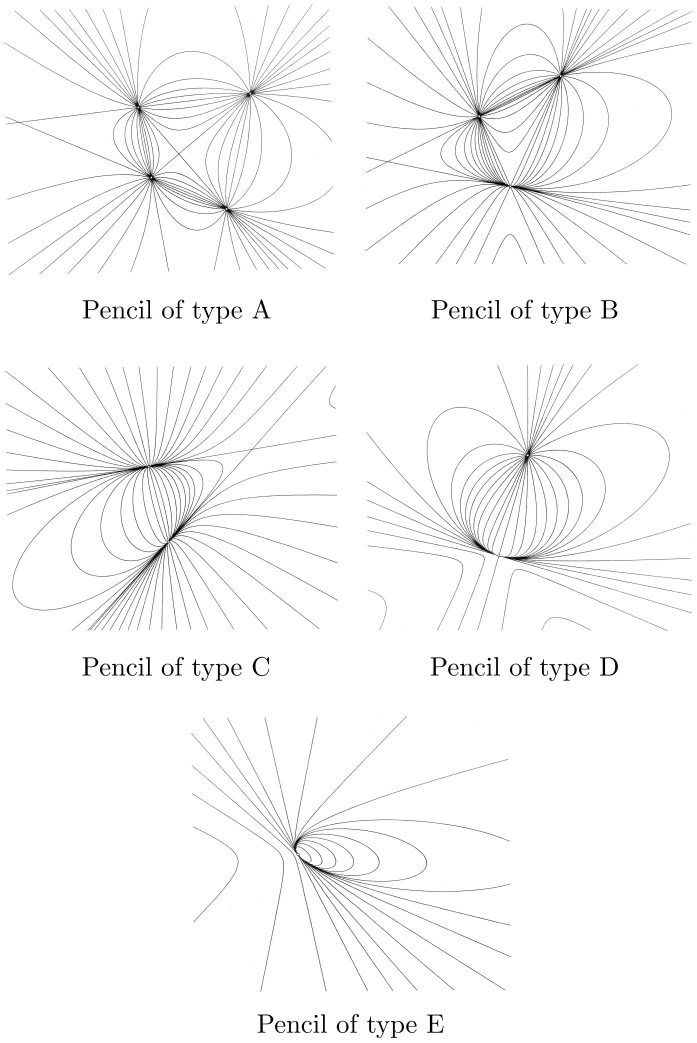

FIGURE 2

Then, applying discriminant-like operators introduced in [**2**]

$$\delta_{x,y} : \mathcal{P}_4^1 \to \mathcal{P}_2^2, \quad \delta_x : \mathcal{P}_2^2 \to \mathcal{P}_1^4$$

by formulae

$$\delta_{x,y}(Q) = Q_x Q_y - Q Q_{xy}, \quad \delta_x(h) = h_x^2 - 2hh_{xx},$$

there is a descent from the faces to the edges and then to the vertices of the cube: from a multiaffine polynomial $Q(x_1, x_2, x_3, x_4)$ to a biquadratic polynomial $h(x_i, x_j) := \delta_{x_k,x_l}(Q(x_i, x_j, x_k, x_l))$ and further to a polynomial $P(x_i) = \delta_{x_j}(h(x_i, x_j))$ of one variable of degree up to four.

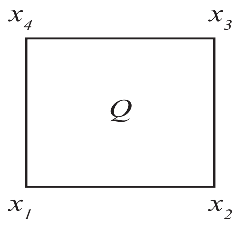

FIGURE 3. An elementary quadrilateral. A quad-equation $Q(x_1, x_2, x_3, x_4) = 0$.

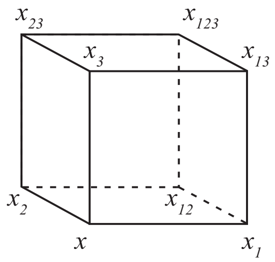

FIGURE 4. A 3D consistency.

A biquadratic polynomial $h(x, y) \in \mathcal{P}_2^2$ is said to be *nondegenerate* if no polynomial in its equivalence class with respect to fractional linear transformations is divisible by a factor of the form $x - c$ or $y - c$, with $c = \text{const}$. A multiaffine function $Q(x_1, x_2, x_3, x_4) \in \mathcal{P}_4^1$ is said to be of *type* Q if all four of its accompanying biquadratic polynomials h^{jk} are nondegenerate. Otherwise, it is of *type* H. Previous notions were introduced in [**2**], where the following classification list of multiaffine polynomials of type Q has been obtained:

$$\begin{aligned}
Q_A &= \operatorname{sn}(\alpha)\operatorname{sn}(\beta)\operatorname{sn}(\alpha+\beta)(k^2 x_1x_2x_3x_4 + 1) - \operatorname{sn}(\alpha)(x_1x_2 + x_3x_4)\\
&\quad - \operatorname{sn}(\beta)(x_1x_4 + x_2x_3) + \operatorname{sn}(\alpha+\beta)(x_1x_3 + x_2x_4),\\
Q_B &= (\alpha - \alpha^{-1})(x_1x_2 + x_3x_4) + (\beta - \beta^{-1})(x_1x_4 + x_2x_3)\\
&\quad - (\alpha\beta - \alpha^{-1}\beta^{-1})(x_1x_3 + x_2x_4) + \tfrac{\delta}{4}(\alpha - \alpha^{-1})(\beta - \beta^{-1})(\alpha\beta - \alpha^{-1}\beta^{-1}),
\end{aligned}$$

with $\delta \neq 0$, and from the last relation, for $\delta = 0$ one gets

$$\begin{aligned}
Q_C &= (\alpha - \alpha^{-1})(x_1x_2 + x_3x_4) + (\beta - \beta^{-1})(x_1x_4 + x_2x_3)\\
&\quad - (\alpha\beta - \alpha^{-1}\beta^{-1})(x_1x_3 + x_2x_4),\\
Q_D &= \alpha(x_1 - x_4)(x_2 - x_3) + \beta(x_1 - x_2)(x_4 - x_3)\\
&\quad - \alpha\beta(\alpha+\beta)(x_1 + x_2 + x_3 + x_4) + \alpha\beta(\alpha+\beta)(\alpha^2 + \alpha\beta + \beta^2),\\
Q_E &= \alpha(x_1 - x_4)(x_2 - x_3) + \beta(x_1 - x_2)(x_4 - x_3) - \delta\alpha\beta(\alpha+\beta).
\end{aligned}$$

8.3. Discriminantly separable polynomials. The notion of discriminantly separable polynomials has been introduced in [**19**]. A family of such polynomials has been constructed there as pencil equations from the theory of conics

$$\mathcal{F}(w, x_1, x_2) = 0,$$

where w, x_1, x_2 are the pencil parameter and the Darboux coordinates respectively. The key algebraic property of the pencil equation, as quadratic equation in each of three variables w, x_1, x_2 is: *all three of its discriminants are expressed as products of two polynomials in one variable each*:

$$\begin{aligned}\mathcal{D}_w(\mathcal{F})(x_1, x_2) &= P(x_1)P(x_2),\\ \mathcal{D}_{x_1}(\mathcal{F})(w, x_2) &= J(w)P(x_2),\\ \mathcal{D}_{x_2}(\mathcal{F})(w, x_1) &= P(x_1)J(w),\end{aligned}$$

where J, P are polynomials of degree up to 4, and the elliptic curves

$$\Gamma_1 : y^2 = P(x), \quad \Gamma_2 : y^2 = J(s)$$

are isomorphic (see Proposition 1 of [**19**]) .

In this section we will present briefly a classification of strongly discriminantly separable polynomials $\mathcal{F}(x_1, x_2, x_3) \in \mathcal{P}_3^2$, which are those with $J = P$, modulo a gauge group of the following fractional-linear transformations

$$x_i \mapsto \frac{ax_i + b}{cx_i + d}, \quad i = 1, 2, 3,$$

from [**22**], where more details can be found.

Let $\mathcal{F}(x_1, x_2, x_3) = \sum_{i,j,k=0}^{2} a_{ijk}x_1^i x_2^j x_3^k$ be a strongly discriminantly separable polynomial with $\mathcal{D}_{x_i}\mathcal{F}(x_j, x_k) = P(x_j)P(x_k)$, $(i, j, k) = c.p.(1, 2, 3)$.

Then, the classification of such polynomials, following [**22**], goes along the study of structure of zeros of a nonzero polynomial $P \in \mathcal{P}_1^4$. There are five cases: [A] with four simple zeros; [B] with a double zero and two simple zeros; [C] corresponds to polynomials with two double zeros; [D] is the case of one triple and one simple zero; finally, [E] is the case of one zero of degree four. The corresponding families of polynomials $\mathcal{F}_A$, $\mathcal{F}_B$, $\mathcal{F}_{C1}$, $\mathcal{F}_{C2}$, $\mathcal{F}_D$, $\mathcal{F}_{E1}$, $\mathcal{F}_{E2}$, $\mathcal{F}_{E3}$, $\mathcal{F}_{E4}$ are listed in Theorem 4 of [**22**]. The case [A], of general position, as it has been proven in [**22**] corresponds to a 2 valued group (from [**7**]) associated with an elliptic curve $y^2 = P(x)$.

Here, we are giving another example.

Example 8.1 (B). (1,1,2): two simple zeros and one double zero, for canonical form $P(x) = x^2 - \epsilon^2$, with $\epsilon \neq 0$, the corresponding discriminantly separable polynomial is $\mathcal{F}_B = x_1x_2x_3 + \frac{\epsilon}{2}(x_1^2 + x_2^2 + x_3^2 - \epsilon^2)$.

8.4. From discriminant separability to quad graph integrability. The relationship between the discriminantly separable polynomials of degree two in each of three variables, and integrable quad-graphs of Adler, Bobenko and Suris has been established in [**22**]. The key point is the following formula, which defines an h, a biquadratic ingredient of quad-graph integrability, starting form a discriminantly separable polynomial $\mathcal{F}$:

$$\hat{h}(x_1, x_2, \alpha) = \frac{\mathcal{F}(x_1, x_2, \alpha)}{\sqrt{P(\alpha)}}.$$

As an example, we are going to present here only the case [B].

EXAMPLE 8.2. The system for h_B leads to

$$h_{22} = 0, \quad h_{21} = h_{12} = 0, \quad h_{01} = h_{10} = 0,$$
$$h_{02} = h_{20}, \quad h_{11} = \pm\sqrt{1 + 4h_{20}^2}, \quad h_{00} = e^2/4h_{20}.$$

Thus h_{20} is a free parameter of the system, an arbitrary function of α. In [**2**], it appears that they used the following choice for h_{20}:

$$h_{20} = \frac{\alpha}{1 - \alpha^2}.$$

If one wants to get h_B

$$\hat{h}_B(x_1, x_2, \alpha) = \frac{\mathcal{F}_B(x_1, x_2, \alpha)}{\sqrt{P_B(\alpha)}},$$

then the appropriate expression for the free parameter is

$$\hat{h}_{20} = \frac{e}{2\sqrt{\alpha^2 - e^2}}$$

and we get

$$\begin{aligned}\hat{h}_B(x_1, x_2, \alpha) &= \Big(\frac{e}{2}(x_1^2 + x_2^2 + \alpha^2) + \alpha x_1 x_2 - \frac{e^3}{2}\Big)\Big/\sqrt{\alpha^2 - e^2} \\ &= F_B(x_1, x_2, \alpha)/\sqrt{\alpha^2 - e^2}.\end{aligned}$$

One can easily calculate the corresponding multiaffine polynomial $Q_B \in \mathcal{P}_4^1$:

$$\begin{aligned}\hat{Q}_B = {} & \sqrt{\beta_1^2 - e^2}(x_1x_4 + x_2x_3) + \sqrt{\alpha_1^2 - e^2}(x_1x_2 + x_3x_4) \\ & + \frac{\alpha_1\sqrt{\beta_1^2 - e^2} + \beta_1\sqrt{\alpha_1^2 - e^2}}{e}(x_1x_3 + x_2x_4) \\ & - \frac{\sqrt{\beta_1^2 - e^2}\sqrt{\alpha_1^2 - e^2}(\alpha_1\sqrt{\beta_1^2 - e^2} + \beta_1\sqrt{\alpha_1^2 - e^2})}{e}.\end{aligned}$$

Let us note that integrable quad graphs related to pencils of higher dimensional quadrics have been studied recently in [**24**]. There, a role of multiaffine quad equations is played by the double reflection configurations.

8.5. Petrov classification. As we mentioned in the Introduction, the Petrov 1954 classification gives a description of possible algebraic symmetries of the Weyl tensor at a point in a Lorentzian manifold (see [**31**],[**32**]). Its popularity is connected with applications to the theory of relativity, in the study of the exact solutions of the Einstein field equations.

The Weyl tensor, as a fourth rank $(2, 2)$-tensor, evaluated at some point, acts on the space of bivectors at that point as a linear operator:

$$W : Y^{\alpha\beta} \mapsto \tfrac{1}{2} W^{\alpha\beta}_{pq} Y^{pq}.$$

The associated eigenvalues and eigenbivectors are defined by the equation

$$\tfrac{1}{2} W^{\alpha\beta}_{pq} Y^{pq} = \lambda Y^{\alpha\beta}.$$

In the case of basic four dimensional spacetimes, the space of antisymmetric bivectors at each point is six-dimensional, but, moreover, due to the symmetries of the Weyl tensor, eigenbivectors lie in a four dimensional subset. Thus, the Weyl tensor at a point has at most four linearly independent eigenbivectors. The eigenbivectors of the Weyl tensor can occur with multiplicities, which indicate a kind of algebraic symmetry of the tensor at the point. The multiplicities reflect the structure of zeros of a certain polynomial of degree four. The eigenbivectors are associated with null vectors in the original spacetime, the principal null directions at point. According to the Petrov classification theorem, there are six possible types of algebraic symmetry, the six Petrov types:

[I] –four simple principal null directions;
[II] –two simple principal null directions and one double;
[D] –two double principal null directions;
[III] –one simple and one triple principal null direction;
[N] –one quadruple principal null direction;
[O] –corresponds to the case where the Weyl tensor vanishes.

Of course, in different points the same tensor can have different Petrov types.

A Weyl tensor of type I at some point is also called *algebraically general*; otherwise, it is called *algebraically special*. The classification of the Petrov types, has been schematically presented by original Penrose diagram (see [**30**]):

(8.3)
$$\begin{tikzcd} I \arrow[d] \arrow[dr] & & \\ II \arrow[r] \arrow[d] \arrow[dr] & D \arrow[d] \arrow[dr] & \\ III \arrow[r] & N \arrow[r] & O \end{tikzcd}$$

which served as a motivation for our diagram (8.1), see also [**20**], [**21**].

9. Experimental Mathematics: From elementary n-valued groups to discriminant factoriziblity

The list of elementary n-valued groups has been done in [**6**]. For a fixed n, corresponding n-valued group is defined by a symmetric polynomial $p_n \in \mathcal{P}_3^n$.

The elementary symmetric functions of three variables are denoted as s_1, s_2, s_3:

$$s_1 = x + y + z, \quad s_2 = xy + xz + yz, \quad s_3 = xyz.$$

The first example is 2-valued group, defined by $p_2(x, y, z) = 0$, where

$$p_2(z, x, y) = (x + y + z)^2 - 4(xy + yz + zx).$$

As it has been observed in [**19**], the polynomial $p_2(z, x, y)$ is discriminantly separable. The discriminants satisfy relations

$$\mathcal{D}_z(p_2)(x, y) = P(x)P(y), \quad \mathcal{D}_x(p_2)(y, z) = P(y)P(z), \quad D_y(p_2)(x, z) = P(x)P(z),$$

where $P(x) = 2x$. Moreover, it has been shown in [**19**] that the polynomial p_2 as discriminantly separable, generates a case of generalized Kowalevski system of differential equations, with $K = 0$.

Now, we can produce a small mathematical experiment with the next cases of elementary n valued groups, with small n.

Example 9.1 ($n = 3$).

$$\begin{aligned} p_3 &= s_1^3 - 3^3 s_3 \\ Dp_3 &= y^2 x^2 (x - y)^2. \end{aligned}$$

Example 9.2 ($n = 4$).

$$\begin{aligned} p_4 &= s_1^4 - 2^3 s_1^2 s_2 + 2^4 s_2^2 - 2^7 s_1 s_3 \\ Dp_4 &= y^3 x^3 (x - y)^2 (y + 4x)^2 (4y + x)^2. \end{aligned}$$

Example 9.3 ($n = 5$).

$$\begin{aligned} p_5 &= s_1^5 - 5^4 s_1^2 s_3 + 5^5 s_2 s_3 \\ Dp_5 &= y^4 x^4 (x - y)^4 (-y^2 - 11xy + x^2)^2 (-y^2 + 11xy + x^2)^2. \end{aligned}$$

We see that the polynomials p_3, p_4, p_5 are not discriminantly separable any more. But, we observe an amazing factoriziblity property of their discriminants. It looks as a challenging and interesting problem to turn this simple mathematical experiment into a mathematical theory, to formulate and prove an exact statement about the observed phenomena, and to possibly relate it to some kind of integrability.

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