PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 90(104) (2011), 85–98 DOI: 10.2298/PIM1104085L

CHAOS EXPANSION METHODS FOR STOCHASTIC DIFFERENTIAL EQUATIONS INVOLVING THE MALLIAVIN DERIVATIVE–PART II

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Communicated by Stevan Pilipović

Abstract. We solve stochastic differential equations involving the Malliavin derivative and the fractional Malliavin derivative by means of a chaos expansion on a general white noise space (Gaussian, Poissonian, fractional Gaussian and fractional Poissonian white noise space). There exist unitary mappings between the Gaussian and Poissonian white noise spaces, which can be applied in solving SDEs.

1. Introduction

This paper represents the conclusion of the results stated in Part I [**11**], where we introduced four types of white noise spaces: Gaussian, Poissonian, fractional Gaussian and fractional Poissonian depending on a Hurst parameter $H \in (0,1)$. Generalized stochastic processes, such as the Brownian motion, white noise or Poissonian noise, are given in the form of their chaos expansion in terms of the Fourier–Hermite and Fourier–Charlier polynomials. In [**11**] we showed that there exist unitary mappings between the Gaussian and Poissonian spaces, as well as between the regular spaces ($H = \frac{1}{2}$) and their fractional counterparts ($H \in (0,1)$).

In this paper we focus on some examples of stochastic differential equations involving the Malliavin derivative, the Ornstein–Uhlenbeck operator and their fractional versions. All equations we solve can be interpreted on all four types of white noise spaces. We provide a general method of solving, using the Wiener–Itô chaos decomposition form, also known as the propagator method (see [**12, 13, 14, 21**]). It is used to set all coefficients in the chaos expansion on the left-hand side of

2010 *Mathematics Subject Classification:* 60H40, 60H10, 60H07, 60G20.

Key words and phrases: Gaussian white noise space, Poissonian white noise space, fractional Gaussian white noise space, fractional Poissonian white noise space, series expansion, Malliavin derivative, Skorokhod integral, Ornstein-Uhlenbeck operator, stochastic differential equation.

This paper was supported by the project *Functional analysis, ODEs and PDEs with singularities*, No. 17024, financed by the Ministry of Science, Republic of Serbia.

the equation equal to the corresponding coefficients on the right-hand side of the equation, resulting in an infinite system of deterministic equations. Solving these equations and summing up the solutions into a series expansion, one obtains the solution of the initial equation, provided the series converges in some q-weighted space.

The paper is organized in the following manner: In Section 2 we review the notation and some results on chaos expansions and Malliavin derivatives obtained in Part I of the paper. Section 3 is devoted to solving some classes of stochastic differential equations which are driven by the Malliavin derivative and the Ornstein–Uhlenbeck operator.

2. Overview

In this section we provide the most important notation needed for further understanding. For details we refer to [**11**] and the references cited therein.

Let $h_n, \xi_{n+1}, n \in \mathbb{N}_0$ denote the family of Hermite polynomials and Hermite functions respectively. Consider the Schwartz spaces of tempered distributions $S'(\mathbb{R}) = \bigcup_{l\in\mathbb{N}_0} S_{-l}(\mathbb{R})$, where

$$S_{-l}(\mathbb{R}) = \left\{ f = \sum_{k=1}^{\infty} b_k\, \xi_k :\ \|f\|^2_{-l} = \sum_{k=1}^{\infty} b_k^2 (2k)^{-l} < \infty \right\},\ l \in \mathbb{N}_0,$$

and the spaces of distributions with exponential growth introduced in [**20**] $\exp S'(\mathbb{R}) = \bigcup_{l\in\mathbb{N}_0} \exp S_{-l}(\mathbb{R})$, where

$$\exp S_{-l}(\mathbb{R}) = \left\{ f = \sum_{k=1}^{\infty} b_k\, \xi_k :\ \|f\|^2_{\exp,-l} = \sum_{k=1}^{\infty} b_k^2 e^{-2kl} < \infty \right\},\ l \in \mathbb{N}_0.$$

Let $\mathcal{I} = (\mathbb{N}_0^{\mathbb{N}})_c$ denote the set of sequences of non-negative integers which have finitely many nonzero components $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m, 0, 0, \dots)$, $\alpha_i \in \mathbb{N}_0$, $i = 1, 2, \dots, m$, $m \in \mathbb{N}$. The k-th unit vector $\varepsilon^{(k)} = (0, \cdots, 0, 1, 0, \cdots)$, $k \in \mathbb{N}$ is the sequence of zeros with the number 1 as the k-th component. The length of a multi-index $\alpha \in \mathcal{I}$ is defined as $|\alpha| = \sum_{k=1}^{\infty} \alpha_k$ and $\alpha! = \prod_{k=1}^{\infty} \alpha_k!$. Let $(2\mathbb{N})^{\alpha} = \prod_{k=1}^{\infty} (2k)^{\alpha_k}$.

The basic probability space is $(\Omega, \mathcal{F}, P)$ $(S'(\mathbb{R}), \mathcal{B}, P)$, where $S'(\mathbb{R})$ denotes the space of tempered distributions, $\mathcal{B}$ the Borel sigma-algebra generated by the weak topology on $S'(\mathbb{R})$ and P denotes the unique probability measure on $(S'(\mathbb{R}), \mathcal{B})$ corresponding to a given characteristic function.

2.1. Chaos expansions on white noise spaces. Suppose that $L^2(P) = L^2(S'(\mathbb{R}), \mathcal{B}, P)$ is the Hilbert space of square integrable functions on $S'(\mathbb{R})$ with respect to the measure P and let K_α, $\alpha \in \mathcal{I}$, denote the orthogonal basis of $L^2(P)$.

THEOREM 2.1 (Wiener–Itô chaos expansion). *Every element $F \in L^2(P)$ has a unique representation of the form $F(\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha K_\alpha(\omega)$, $c_\alpha \in \mathbb{R}$, such that*

$$\|F\|^2_{L^2(P)} = \sum_{\alpha\in\mathcal{I}} c_\alpha^2 \|K_\alpha\|^2_{L^2(P)} < \infty.$$

In [**11**] we considered two important cases of the measure P: the Gaussian measure μ and the Poissonian measure ν. Their fractional versions, for a fixed Hurst parameter $H \in (0,1)$, are obtained using the mapping $M = M^{(H)} : \mathcal{S}(\mathbb{R}) \to L^2(\mathbb{R}) \cap C^\infty(\mathbb{R})$ defined by $\widehat{Mf}(y) = |y|^{\frac{1}{2}-H}\,\widehat{f}(y)$, $y \in \mathbb{R}$, $f \in S(\mathbb{R})$, which has the property $M^{-1} = M^{(1-H)}$. This map extends by linearity and continuity to $\mathcal{S}'(\mathbb{R})$, and thus by setting $L^2(P_H) = L^2(P \circ M^{-1}) = \{G : \Omega \to \mathbb{R};\ G \circ M \in L^2(P)\}$ one obtains the fractional version of the square integrable random variables on the white noise space $L^2(P)$, i.e., $L^2(\mu_H)$ in the fractional Gaussian case and $L^2(\nu_H)$ in the fractional Poissonian case.

The orthogonal basis of the four white noise spaces $L^2(P)$ is thus obtained in the following manner:

Table 1.

white noise space	classical		fractional	
	Gaussian	Poissonian	Gaussian	Poissonian
measure P	μ	ν	μ_H	ν_H
basis K_α	H_α	C_α	$\widetilde{\mathcal{H}}_\alpha$	$\widetilde{\mathcal{C}}_\alpha$
basis $\mathfrak{e}_k$	ξ_k	ξ_k	e_k	e_k

where the family of Fourier–Hermite polynomials is defined by

$$H_\alpha(\omega) = \prod_{k=1}^{\infty} h_{\alpha_k}(\langle \omega, \xi_k \rangle), \quad \alpha \in \mathcal{I},$$

and the family of Fourier–Charlier polynomials is defined by

$$C_\alpha(\omega) = C_{|\alpha|}(\omega; \underbrace{\xi_1, \dots, \xi_1}_{\alpha_1}, \dots, \underbrace{\xi_m, \dots, \xi_m}_{\alpha_m}), \quad \alpha = (\alpha_1, \dots, \alpha_m, 0, 0, \dots) \in \mathcal{I},$$

where

$$C_k(\omega; \varphi_1, \dots, \varphi_k) = \frac{\partial^k}{\partial u_1 \dots \partial u_k} \exp\left[\left\langle \omega, \log\left(1 + \sum_{j=1}^{k} u_j \varphi_j\right) - \sum_{j=1}^{k} u_j \int_{\mathbb{R}} \varphi_j(y)\,dy \right\rangle\right]\Bigg|_{u_1 = \dots = u_k = 0},$$

for $k \in \mathbb{N}$ and $\varphi_j \in S(\mathbb{R})$.

In the fractional spaces the bases are given by $\widetilde{\mathcal{H}}_\alpha(\omega) = \prod_{k=1}^{\infty} h_{\alpha_k}(\langle \omega, e_k \rangle)$, $\alpha \in \mathcal{I}$ and

$$\widetilde{\mathcal{C}}_\alpha(\omega) = C_{|\alpha|}(\omega; \underbrace{e_1, \dots, e_1}_{\alpha_1}, \dots, \underbrace{e_m, \dots, e_m}_{\alpha_m}), \quad \alpha = (\alpha_1, \dots, \alpha_m, 0, 0, \dots) \in \mathcal{I},$$

where $e_i = M^{-1}\xi_i$, $i \in \mathbb{N}$.

The spaces of generalized random variables are introduced by imposing weights on the convergence condition in the Wiener–Itô chaos expansion and thus weakening the $L^2(P)$-norm. For a weight sequence $q_\alpha > 1$, $\alpha \in \mathcal{I}$, let $(Q)_{-1}^P = \bigcup_{p \in \mathbb{N}_0} (Q)_{-1,-p}^P$ be the inductive limit of the spaces

$$(Q)^P_{-1,-p} = \Big\{ F = \sum_{\alpha\in\mathcal{I}} b_\alpha K_\alpha :\ \|F\|^2_{(Q)^P_{-1,-p}} = \sum_{\alpha\in\mathcal{I}} b_\alpha^2\, q_\alpha^{-p} < \infty \Big\},\ \ p \in \mathbb{N}_0.$$

Two important special cases are given by weights of the form: $q_\alpha = (2\mathbb{N})^\alpha$ and $q_\alpha = e^{(2\mathbb{N})^\alpha}$. In these cases we denote the q-weighted spaces by $(S)^P_{-1}$ and $\exp(S)^P_{-1}$ respectively.

We consider generalized stochastic processes of type (I) as linear and continuous mappings from a topological vector space X into the space of q-weighted generalized functions $(Q)^P_{-1}$ i.e., elements of $\mathcal{L}(X,(Q)^P_{-1})$. If at least one of the spaces X or $(Q)^P_{-1}$ is nuclear, then $\mathcal{L}(X,(Q)^P_{-1}) \cong X' \otimes (Q)^P_{-1}$.

THEOREM 2.2. *Let $X = \bigcap_{k=0}^\infty X_k$ be a nuclear space endowed with a family of seminorms $\{\|\cdot\|_k;\ k \in \mathbb{N}_0\}$ and let $X' = \bigcup_{k=0}^\infty X_{-k}$ be its topological dual. Generalized stochastic processes as elements of $X'\otimes(Q)^P_{-1}$ have a chaos expansion of the form*

$$u = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes K_\alpha, \quad f_\alpha \in X_{-k},\ \alpha\in\mathcal{I}, \tag{2.1}$$

where $k\in\mathbb{N}_0$ does not depend on $\alpha\in\mathcal{I}$, and there exists $p\in\mathbb{N}_0$ such that

$$\|u\|^2_{X'\otimes(Q)^P_{-1,-p}} = \sum_{\alpha\in\mathcal{I}} \|f_\alpha\|^2_{-k} q_\alpha^{-p} < \infty. \tag{2.2}$$

The expectation of the process u is given by the zeroth order chaos expansion coefficient $E(u) = f_{(0,0,0,\dots)}$.

In [**11**] we have constructed two unitary mappings: the mapping $\mathcal{U}$ acting between the Gaussian and Poissonian spaces, and the mapping $\mathcal{M}$ acting between a regular space and its fractional version. The essence of both operators is to establish a mapping between the orthogonal bases of the corresponding spaces, leaving the coefficients of the chaos expansion unaffected. Let $\mathcal{U}:\ X\otimes(Q)^\mu_{-1} \to X\otimes(Q)^\nu_{-1}$ be defined by

$$\mathcal{U}\Big[\sum_{\alpha\in\mathcal{I}} u_\alpha\otimes H_\alpha\Big] = \sum_{\alpha\in\mathcal{I}} u_\alpha\otimes C_\alpha, \quad u_\alpha\in X,\ \alpha\in\mathcal{I}, \tag{2.3}$$

and let $\mathcal{M}:\ X\otimes(Q)^{\mu_H}_{-1} \to X\otimes(Q)^\mu_{-1}$ be defined by

$$\mathcal{M}\Big[\sum_{\alpha\in\mathcal{I}} v_\alpha\otimes \widetilde{\mathcal{H}}_\alpha\Big] = \sum_{\alpha\in\mathcal{I}} v_\alpha\otimes H_\alpha, \quad v_\alpha\in X,\ \alpha\in\mathcal{I}. \tag{2.4}$$

This resulted in the following commutative diagram:

$$\begin{tikzcd}
L^2(\mu) \arrow[r, "\mathcal{M}^{-1}"] \arrow[d, "\mathcal{U}"'] \arrow[dr, "\mathcal{U}\circ\mathcal{M}^{-1}", "\mathcal{M}^{-1}\circ\mathcal{U}"'] & L^2(\mu_H) \arrow[d, "\mathcal{U}"] \\
L^2(\nu) \arrow[r, "\mathcal{M}^{-1}"'] & L^2(\nu_H)
\end{tikzcd}$$

Diagram 1

Another important operator is obtained by extending $M : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ into $\mathbf{M} = M \otimes Id : \mathcal{S}'(\mathbb{R}) \otimes (Q)_{-1}^{P} \to \mathcal{S}'(\mathbb{R}) \otimes (Q)_{-1}^{P}$ given by

$$\mathbf{M}\bigg(\sum_{\alpha\in\mathcal{I}} a_\alpha(t) \otimes K_\alpha(\omega)\bigg) = \sum_{\alpha\in\mathcal{I}} Ma_\alpha(t) \otimes K_\alpha(\omega).$$

2.2. The Malliavin derivative and the Ornstein–Uhlenbeck operator. In this subsection P will denote either the Gaussian measure μ or the Poissonian measure ν, and P_H will denote its corresponding fractional measure μ_H or ν_H. The notation $(Q)_{-1}^{P}$ will refer to either $(S)_{-1}^{P}$ or $\exp(S)_{-1}^{P}$, and $(Q)_{-1}^{P_H}$ to either $(S)_{-1}^{P_H}$ or $\exp(S)_{-1}^{P_H}$.

Definition 2.1. The Malliavin derivative of a process $u \in \mathrm{Dom}(\mathbb{D})$ of the form $u = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes K_\alpha$, $f_\alpha \in X$, $\alpha \in \mathcal{I}$ is defined by

$$\mathbb{D}u = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha_k\, f_\alpha \otimes \mathfrak{e}_k \otimes K_{\alpha-\varepsilon^{(k)}},$$

where $\mathrm{Dom}(\mathbb{D})$ is defined as an appropriate subset of $X \otimes (Q)_{-1}^{P}$ and called the domain of the Malliavin derivative.

Note that $\mathbb{D}u$ is a generalized stochastic process with values in a distribution space i.e., $\mathbb{D}u \in X \otimes \mathcal{S}'(\mathbb{R}) \otimes (S)_{-1}^{P}$ or $\mathbb{D}u \in X \otimes \exp S'(\mathbb{R}) \otimes \exp(S)_{-1}^{P}$. In [**11**] we gave a characterization of the domain and the codomain in both cases.

Definition 2.2. The Skorokhod integral of a process $F \in \mathrm{Dom}(\delta)$ of the form $F = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes v_\alpha \otimes K_\alpha$, $f_\alpha \in X$, $\alpha \in \mathcal{I}$ and $v_\alpha \in S'(\mathbb{R})$ or $v_\alpha \in \exp S'(\mathbb{R})$ respectively, is defined by $\delta(F) = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} v_{\alpha,k}\, f_\alpha \otimes K_{\alpha+\varepsilon^{(k)}}$, where $v_\alpha = \sum_{k\in\mathbb{N}} v_{\alpha,k}\, \mathfrak{e}_k$, $v_{\alpha,k} \in \mathbb{R}$ is the expansion of v_α in $S'(\mathbb{R})$ or $\exp S'(\mathbb{R})$ respectively.

Note that $\mathrm{Dom}(\delta) \subset X \otimes \mathcal{S}'(\mathbb{R}) \otimes (S)_{-1}^{P}$, resp. $\mathrm{Dom}(\delta) \subset X \otimes \exp S'(\mathbb{R}) \otimes \exp(S)_{-1}^{P}$ and that $\delta(F) \in X \otimes (S)_{-1}^{P}$, resp. $\delta(F) \in X \otimes \exp(S)_{-1}^{P}$. For a detailed characterization of the domain we refer to [**11**].

Definition 2.3. The composition $\mathcal{R} = \delta \circ \mathbb{D}$ is called the Ornstein–Uhlenbeck operator. For $u \in \mathrm{Dom}(\mathbb{D})$ its action is given by $\mathcal{R}u = \sum_{\alpha\in\mathcal{I}} |\alpha| u_\alpha \otimes K_\alpha$, but its action can be extended to a larger set, i.e., $\mathrm{Dom}(\mathcal{R}) \supset \mathrm{Dom}(\mathbb{D})$ in a general case.

The Hermite i.e., the Charlier polynomials are eigenfunctions of $\mathcal{R}$ and the corresponding eigenvalues are $|\alpha|$, $\alpha \in \mathcal{I}$, i.e., $\mathcal{R}(K_\alpha) = |\alpha| K_\alpha$.

Definition 2.4. The fractional Malliavin derivative of $F = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes K_\alpha \in X \otimes (Q)_{-1}^{P}$ is defined by $\mathbb{D}^{(H)}F = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha_k\, f_\alpha \otimes M^{-1}\mathfrak{e}_k \otimes K_{\alpha-\varepsilon^{(k)}}$.

Theorem 2.3. *Let $\mathbb{D}$ and $\mathbb{D}^{(H)}$ denote the Malliavin derivative, respectively the fractional Malliavin derivative on $X \otimes (Q)_{-1}^{P}$. Let $\widetilde{\mathbb{D}}$ denote the Malliavin derivative on $X \otimes (Q)_{-1}^{P_H}$. Then,*

$$\mathbb{D}^{(H)}F = \mathbf{M}^{-1} \circ \mathbb{D}F = \mathcal{M} \circ \widetilde{\mathbb{D}} \circ \mathcal{M}^{-1}F, \tag{2.5}$$

for all $F \in \mathrm{Dom}(\mathbb{D})$.

DEFINITION 2.5. Let $\delta : X\otimes\mathcal{S}'(\mathbb{R})\otimes(Q)_{-1}^{P} \to X\otimes(Q)_{-1}^{P}$ denote the Skorokhod integral. The fractional Skorokhod integral $\delta^{(H)} : X\otimes\mathcal{S}'(\mathbb{R})\otimes(Q)_{-1}^{P} \to X\otimes(Q)_{-1}^{P}$ is defined for every $F \in \mathrm{Dom}(\delta)$ by $\delta^{(H)}F = \delta\circ\mathbf{M}F$.

For the Ornstein–Uhlenbeck operator we note that its fractional version coincides with the regular one, i.e., $\mathcal{R}^{(H)} = \delta^{(H)}\circ\mathbb{D}^{(H)} = \delta\circ\mathbf{M}\circ\mathbf{M}^{-1}\circ\mathbb{D} = \delta\circ\mathbb{D} = \mathcal{R}$.

For further information on fractional white noise spaces and Malliavin derivatives we refer to the basic literature [**1**]–[**9**], [**16**]–[**19**], [**22**], [**23**].

3. Stochastic differential equations

In this section we investigate the existence of solutions for stochastic differential equations involving the Malliavin derivative and the Ornstein–Uhlenbeck operator. The method of Wiener–Itô chaos expansions, used to set all coefficients in the chaos expansion on the left-hand side of the equation equal to the corresponding coefficients on the right-hand side of the equation, is a general and useful tool, also known as the propagator method. With this method we reduce a problem to an infinite system of deterministic equations. Summing up all coefficients of the expansion and proving convergence in an appropriate weight space, one obtains the solution of the initial equation. Other types of equations investigated by the same method can be found in several papers: [**10**], [**12**]–[**15**], [**21**], [**24**]–[**26**].

All stochastic equations solved in this section can be interpreted, by the use of the isometric transformations $\mathcal{U}$ and $\mathcal{M}$ defined in (2.3) and (2.4), in all four white noise spaces we have considered so far. Also, due to Theorem 2.3 the Malliavin derivative and the Skorokhod integral can be interpreted as their fractional counterparts in the corresponding fractional white noise space. With this argumentation we state the equations and solve them in a white noise space of general type.

3.1. Equations with the Malliavin derivative. Denote by $r = r(\alpha) = \min\{k\in\mathbb{N} : \alpha_k \neq 0\}$, for nonzero $\alpha\in\mathcal{I}$. Then the first nonzero component of α is the rth component α_r, i.e., $\alpha = (0,0,\dots,0,\alpha_r,\dots,\alpha_m,0,0,\dots)$. Denote by $\alpha_{\varepsilon^{(r)}}$ the multi-index with all components equal to the corresponding components of α, except the r-th, which is $\alpha_r - 1$. We call $\alpha_{\varepsilon^{(r)}}$ the *representative* of α and write

$$\alpha = \alpha_{\varepsilon^{(r)}} + \varepsilon^{(r)}, \quad \alpha\in\mathcal{I},\ |\alpha|>0. \tag{3.1}$$

For example, the first nonzero component of $\alpha = (0,0,2,1,0,5,0,0,\dots)$ is its third component. It follows that $r = 3$, $\alpha_r = 2$ and the representative of α is $\alpha_{\varepsilon^{(r)}} = \alpha - \varepsilon^{(3)} = (0,0,1,1,0,5,0,0,\dots)$.

The set $\mathcal{K}_\alpha = \{\beta\in\mathcal{I} : \alpha = \beta+\varepsilon^{(j)},\ \text{for some } j\in\mathbb{N}\}$, $\alpha\in\mathcal{I}$, $|\alpha|>0$ is nonempty, because $\alpha_{\varepsilon^{(r)}}\in\mathcal{K}_\alpha$. Moreover, if $\alpha = n\varepsilon^{(r)}$, $n\in\mathbb{N}$, then $\mathrm{Card}(\mathcal{K}_\alpha) = 1$ and in all other cases $\mathrm{Card}(\mathcal{K}_\alpha) > 1$. For example if $\alpha = (0,1,3,0,0,5,0,\dots)$, then the set $\mathcal{K}_\alpha$ has three elements

$$\mathcal{K}_\alpha = \{\alpha_{\varepsilon^{(2)}} = (0,0,3,0,0,5,0,\dots), (0,1,2,0,0,5,0,\dots), (0,1,3,0,0,4,0,\dots)\}.$$

3.2. A first order equation. Let us consider an equation of the form

$$\left\{\begin{array}{ll} \mathbb{D}u = h, & h \in X \otimes \mathcal{S}'(\mathbb{R}) \otimes (S)_{-1} \\ Eu = \widetilde{u}_0, & \widetilde{u}_0 \in X \end{array}\right. . \tag{3.2}$$

THEOREM 3.1. *Let* $h = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} h_{\alpha,k} \otimes \mathfrak{e}_k \otimes K_\alpha \in X \otimes \mathcal{S}'(\mathbb{R}) \otimes (S)_{-1}$, *with coefficients* $h_{\alpha,k} \in X$ *such that*

$$\frac{1}{\alpha_r}\, h_{\alpha_{\varepsilon^{(r)}},r} = \frac{1}{\alpha_j}\, h_{\beta,\,j}\,, \tag{3.3}$$

for the representative $\alpha_{\varepsilon^{(r)}}$ *of* $\alpha \in \mathcal{I}$, $|\alpha| > 0$ *and all* $\beta \in \mathcal{K}_\alpha$, *such that* $\alpha = \beta + \varepsilon^{(j)}$, *for* $j \geqslant r$, $r \in \mathbb{N}$. *Then, equation* (3.2) *has a unique solution in* $X \otimes (S)_{-1}$. *The chaos expansion of the generalized stochastic process, which represents the unique solution of equation* (3.2) *is given by*

$$u = \widetilde{u}_0 + \sum_{\alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}\in\mathcal{I}} \frac{1}{\alpha_r}\, h_{\alpha_{\varepsilon^{(r)}},r} \otimes K_\alpha. \tag{3.4}$$

PROOF. We seek the solution in the form $u = \widetilde{u}_0 + \sum_{\substack{\alpha\in\mathcal{I}\\|\alpha|>0}} u_\alpha \otimes K_\alpha$. Thus,

$$\mathbb{D}\Bigg(\widetilde{u}_0 + \sum_{\substack{\alpha\in\mathcal{I}\\|\alpha|>0}} u_\alpha \otimes K_\alpha\Bigg) = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} h_{\alpha,k} \otimes \mathfrak{e}_k \otimes K_\alpha$$

$$\sum_{\substack{\alpha\in\mathcal{I}\\|\alpha|>0}} \Bigg(\sum_{k\in\mathbb{N}} \alpha_k u_\alpha \otimes \mathfrak{e}_k\Bigg) \otimes K_{\alpha-\varepsilon^{(k)}} = \sum_{\alpha\in\mathcal{I}} \Bigg(\sum_{k\in\mathbb{N}} h_{\alpha,k} \otimes \mathfrak{e}_k\Bigg) \otimes K_\alpha$$

$$\sum_{\alpha\in\mathcal{I}} \Bigg(\sum_{k\in\mathbb{N}} (\alpha_k+1) u_{\alpha+\varepsilon^{(k)}} \otimes \mathfrak{e}_k\Bigg) \otimes K_\alpha = \sum_{\alpha\in\mathcal{I}} \Bigg(\sum_{k\in\mathbb{N}} h_{\alpha,k} \otimes \mathfrak{e}_k\Bigg) \otimes K_\alpha$$

Due to uniqueness of the Wiener-Itô chaos expansion it follows that, for all $\alpha \in \mathcal{I}$

$$\sum_{k\in\mathbb{N}} (\alpha_k+1) u_{\alpha+\varepsilon^{(k)}} \otimes \mathfrak{e}_k = \sum_{k\in\mathbb{N}} h_{\alpha,k} \otimes \mathfrak{e}_k.$$

Due to the uniqueness of the series expansion in $\mathcal{S}'(\mathbb{R})$ we obtain a family of deterministic equations

$$u_{\alpha+\varepsilon^{(k)}} = \frac{1}{\alpha_k+1}\, h_{\alpha,k}, \quad \text{for all} \quad \alpha \in \mathcal{I},\ k \in \mathbb{N}, \tag{3.5}$$

from which we can calculate u_α, by induction on the length of α. For $\alpha = (0,0,0,\dots)$, the equations in (3.5) reduce to $u_{\varepsilon^{(k)}} = h_{\alpha,k}$, $\alpha \in \mathcal{I}$, $k \in \mathbb{N}$, i.e.,

$$\begin{aligned} u_{(1,0,0,\dots)} &= h_{(0,0,0,\dots),1} \\ u_{(0,1,0,\dots)} &= h_{(0,0,0,\dots),2} \\ u_{(0,0,1,0\dots)} &= h_{(0,0,0,\dots),3} \\ &\ \vdots \end{aligned}$$

and we obtain the coefficients u_α for α of length one. Note, u_α are obtained in the terms of $h_{\alpha_{\varepsilon^{(r)}},r} = h_{(0,0,0,\dots),r}$, $r \in \mathbb{N}$.

For $|\alpha| = 1$ the multi-indices are of the form $\alpha = \varepsilon^{(j)}$, $j \in \mathbb{N}$, so several cases occur. For $j = 1$, $\alpha = \varepsilon^{(1)} = (1, 0, 0, \dots)$, we have

$$\begin{aligned} u_{(2,0,0,\dots)} &= \tfrac{1}{2} h_{(1,0,0,\dots),1} \\ u_{(1,1,0,\dots)} &= h_{(1,0,0,\dots),2} \\ u_{(1,0,1,0\dots)} &= h_{(1,0,0,\dots),3} \\ u_{(1,0,0,1,0\dots)} &= h_{(1,0,0,\dots),4} \\ &\vdots \end{aligned} \tag{3.6}$$

Continuing, for $j = 2$, $\alpha = \varepsilon^{(2)} = (0, 1, 0, \dots)$ the equations in (3.5) reduce to

$$\begin{aligned} u_{(1,1,0,0,\dots)} &= h_{(0,1,0,0,\dots),1} \\ u_{(0,2,0,\dots)} &= \tfrac{1}{2} h_{(0,1,0,0,\dots),2} \\ u_{(0,1,1,0\dots)} &= h_{(0,1,0,0,\dots),3} \\ u_{(0,1,0,1,0\dots)} &= h_{(0,1,0,0,\dots),4} \\ &\vdots \end{aligned} \tag{3.7}$$

and then, for $\alpha = \varepsilon^{(3)} = (0, 0, 1, 0, \dots)$ we obtain

$$\begin{aligned} u_{(1,0,1,0,\dots)} &= h_{(0,0,1,0,\dots),1} \\ u_{(0,1,1,0,\dots)} &= h_{(0,0,1,0,\dots),2} \\ u_{(0,0,2,0\dots)} &= \tfrac{1}{2} h_{(0,0,1,0,\dots),3} \\ u_{(0,0,1,1,0\dots)} &= h_{(0,0,1,0,\dots),4} \\ &\vdots \end{aligned} \tag{3.8}$$

The coefficient $u_{(1,1,0,0,\dots)}$ appears in systems (3.6) and (3.7) and thus the additional condition $h_{(1,0,0,\dots),2} = h_{(0,1,0,0,\dots),1}$ has to hold in order to have a solvable system. Also, from expressions for $u_{(0,1,1,0,\dots)}$ and $u_{(0,1,0,1,\dots)}$ in (3.7) and (3.8) we obtain conditions $h_{(0,1,0,\dots),3} = h_{(0,0,1,0,\dots),2}$ and $h_{(0,0,0,1,0,\dots),2} = h_{(0,1,0,0,\dots),4}$ respectively, which need to be satisfied, in order to have a unique u_α. In the same manner we obtain all coefficients u_α, for α of the length two, expressed as a function of $h_{\alpha_{\varepsilon^{(r)}},r}$.

Let now $|\alpha| = 2$. Then different combinations for the multi-indices occur: if we choose $\alpha = (1, 1, 0, 0, \dots)$ then (3.5) transforms into the system

$$\begin{aligned} u_{(2,1,0,0,\dots)} &= \tfrac{1}{2} h_{(1,1,0,0,\dots),1} \\ u_{(1,2,0,\dots)} &= \tfrac{1}{2} h_{(1,1,0,0,\dots),2} \\ u_{(1,1,1,0\dots)} &= h_{(1,1,0,0,\dots),3} \\ u_{(1,1,0,1,0\dots)} &= h_{(1,1,0,0,\dots),4} \\ &\vdots \end{aligned} \tag{3.9}$$

and if we choose $\alpha = (1, 0, 1, 0, 0, \dots)$, then the equations in (3.5) transform into

$$\begin{aligned} u_{(2,0,1,0,\dots)} &= \tfrac{1}{2} h_{(1,0,1,0,0,\dots),1} \\ u_{(1,1,1,0,\dots)} &= h_{(1,0,1,0,0,\dots),2} \\ u_{(1,0,2,0\dots)} &= \tfrac{1}{2} h_{(1,0,1,0,0,\dots),3} \\ u_{(1,0,1,1,0\dots)} &= h_{(1,0,1,0,0,\dots),4} \\ &\vdots \end{aligned} \tag{3.10}$$

We continue with $\alpha = (0,1,1,0,0,\dots)$ and $\alpha = (2,0,0,\dots)$ and obtain the systems

$$\begin{aligned} u_{(1,1,1,0,\dots)} &= h_{(0,1,1,0,0,\dots),1} \\ u_{(0,2,1,0,\dots)} &= \tfrac{1}{2} h_{(0,1,1,0,0,\dots),2} \\ u_{(0,1,2,0\dots)} &= \tfrac{1}{2} h_{(0,1,1,0,0,\dots),3} \\ u_{(0,1,1,1,0\dots)} &= h_{(0,1,1,0,0,\dots),4} \\ &\vdots \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} u_{(3,0,0,\dots)} &= \tfrac{1}{3} h_{(2,0,0,\dots),1} \\ u_{(2,1,0,\dots)} &= h_{(2,0,0,\dots),2} \\ u_{(2,0,1,0\dots)} &= h_{(2,0,0,\dots),3} \\ u_{(2,0,0,1,0\dots)} &= h_{(2,0,0,\dots),4} \\ &\vdots \end{aligned} \tag{3.12}$$

respectively. For $\alpha = (0,2,0,0,\dots)$ the system (3.5) transforms into

$$\begin{aligned} u_{(1,2,0,0,\dots)} &= h_{(0,2,0,0,\dots),1} \\ u_{(0,3,0,\dots)} &= \tfrac{1}{3} h_{(0,2,0,0,\dots),2} \\ u_{(0,2,1,0\dots)} &= h_{(0,2,0,0,\dots),3} \\ u_{(0,2,0,1,0\dots)} &= h_{(0,2,0,0,\dots),4} \\ &\vdots \end{aligned} \tag{3.13}$$

Combining with the previous results, we obtain u_α for $|\alpha| = 3$. Two different representations of $u_{(2,1,0,0,\dots)}$ are given in systems (3.9) and (3.12), so an additional condition $\frac{1}{2} h_{(1,1,0,0,\dots),1} = h_{(2,0,0,0,\dots),2}$ follows. We express $u_{(2,1,0,0,\dots)} = \frac{1}{2} h_{(1,1,0,0,\dots),1}$ in form of the representative of the multi-index $\alpha = (2,1,0,0,\dots)$. Since the coefficient $u_{(1,2,0,\dots)}$ appears both in (3.9) and (3.13), we receive another condition $\frac{1}{2} h_{(1,1,0,0,\dots),2} = h_{(0,2,0,0,\dots),1}$, and express $u_{(1,2,0,\dots)} = h_{(0,2,0,0,\dots),1}$ by its representative. From (3.9), (3.10) and (3.11) we obtain $u_{(1,1,1,0,\dots)} = h_{(0,1,1,0,0,\dots),1}$ and the condition $h_{(1,1,0,0,\dots),3} = h_{(1,0,1,0,\dots),2} = h_{(0,1,1,0,0,\dots),1}$. Then $\frac{1}{2} h_{(0,1,1,0,\dots),2} = h_{(0,2,0,\dots),3}$ follows from (3.11) and (3.13), and $u_{(0,2,1,0,\dots)} = \frac{1}{2} h_{(0,1,1,0,\dots),2}$.

We proceed by the same procedure for all multi-index lengths to obtain u_α.

If the set $\mathcal{K}_\alpha$, $\alpha \in \mathcal{I}$, has at least one more element besides the representative $\alpha_{\varepsilon^{(r)}}$ of α, then the condition for the process h is given in the form (3.3). We obtain the coefficients u_α of the solution as functions of the representative $\alpha_{\varepsilon^{(r)}}$

$$u_\alpha = \frac{1}{\alpha_r} h_{\alpha_{\varepsilon^{(r)}}, r}, \quad \text{for } |\alpha| \neq 0, \ \alpha = \alpha_{\varepsilon^{(r)}} + \varepsilon^{(r)},$$

and the form of the solution (3.4).

It remains to prove convergence of the solution (3.4) in $X \otimes (S)_{-1}$. Let $h \in X \otimes S_{-p}(\mathbb{R}) \otimes (S)_{-1,-p}$. Then, there exists $p > 0$ such that

$$\|h\|^2_{X \otimes S_{-p}(\mathbb{R}) \otimes (S)_{-1,-p}} = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \|h_{\alpha,k}\|^2_X \,(2k)^{-p}\,(2\mathbb{N})^{-p\alpha} < \infty.$$

Note that for $\widetilde{u}_0 \in X$ we have $\|\widetilde{u}_0\|_X = \|\widetilde{u}_0\|_{X\otimes(S)_{-1,-q}}$ for all $q > 0$. Then, the convergence follows from

$$\begin{aligned}\|u\|^2_{X\otimes(S)_{-1,-2p}} &= \|\widetilde{u}_0\|^2_{X\otimes(S)_{-1,-2p}} + \sum_{\substack{\alpha\in\mathcal{I},|\alpha|>0,\\ \alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}}} \frac{1}{\alpha_r^2}\, \|h_{\alpha_{\varepsilon^{(r)}},r}\|^2_X (2\mathbb{N})^{-2p(\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)})}\\ &\leqslant \|\widetilde{u}_0\|^2_{X\otimes(S)_{-1,-2p}} + \sum_{\alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}} \|h_{\alpha_{\varepsilon^{(r)}},r}\|^2_X\, (2r)^{-p}\, (2\mathbb{N})^{-p\alpha}\\ &\leqslant \|\widetilde{u}_0\|^2_{X\otimes(S)_{-1,-2p}} + \sum_{\alpha\in\mathcal{I}}\sum_{r\in\mathbb{N}} \|h_{\alpha,r}\|^2_X\, (2r)^{-p}(2\mathbb{N})^{-p\alpha} < \infty,\end{aligned}$$

where we used the fact that $(2\mathbb{N})^{p\varepsilon^{(r)}}(2\mathbb{N})^{-p\alpha} \leqslant 1$ for all $\alpha \in \mathcal{I}$, $r \in \mathbb{N}$. $\square$

3.2.1. *Special cases.* Assume that the process h is expressed as a product $h = c \otimes g$, $c \in \mathcal{S}'(\mathbb{R})$ and $g \in X \otimes (S)_{-1}$.

THEOREM 3.2. *Let $c = \sum_{k\in\mathbb{N}} c_k\, \mathfrak{e}_k \in \mathcal{S}'(\mathbb{R})$ and $g = \sum_{\alpha\in\mathcal{I}} g_\alpha \otimes K_\alpha \in X \otimes (S)_{-1}$ with coefficients $g_\alpha \in X$ such that*

$$\frac{1}{\alpha_r}\, g_{\alpha_{\varepsilon^{(r)}}}\, c_r = \frac{1}{\alpha_j}\, g_\beta c_j, \tag{3.14}$$

holds for all $\beta \in \mathcal{K}_\alpha$, $j \geqslant r$, $r \in \mathbb{N}$, and their representative $\alpha_{\varepsilon^{(r)}}$. Then

$$\mathbb{D}u = c \otimes g, \quad Eu = \widetilde{u}_0, \quad \widetilde{u}_0 \in X, \tag{3.15}$$

has a unique solution in $X \otimes (S)_{-1}$ given by

$$u = \widetilde{u}_0 + \sum_{\alpha=\alpha_{\varepsilon^{(r)}}+\varepsilon^{(r)}\in\mathcal{I}} \frac{1}{\alpha_r}\, g_{\alpha_{\varepsilon^{(r)}}}\, c_r \otimes K_\alpha. \tag{3.16}$$

PROOF. Providing an analogue procedure as in the previous theorem, we reduce equation (3.15) to a family of deterministic equations

$$u_{\alpha+\varepsilon^{(k)}} = \frac{1}{\alpha_k+1}\, g_\alpha\, c_k, \quad \text{for all } \alpha \in \mathcal{I},\ k \in \mathbb{N}, \tag{3.17}$$

from which, by induction on $|\alpha|$, we obtain the coefficients u_α of the solution u, as functions of the representative $\alpha_{\varepsilon^{(r)}}$. Let $\alpha \in \mathcal{I}$, $|\alpha| > 0$ be given by (3.1). Condition (3.14) implies $u_\alpha = \frac{1}{\alpha_r}\, g_{\alpha_{\varepsilon^{(r)}}}\, c_r$. The proof of convergence of the solution (3.16) in $X \otimes (S)_{-1}$ follows in the same way as in the previous theorem. $\square$

Especially, if we choose $c = \mathfrak{e}_i$, for fixed $i \in \mathbb{N}$, then equation (3.15) transforms into

$$\begin{aligned}\mathbb{D}u &= \mathfrak{e}_i \otimes g, \quad g \in X \otimes (S)_{-1}\\ Eu &= \widetilde{u}_0, \quad \widetilde{u}_0 \in X\end{aligned} \tag{3.18}$$

THEOREM 3.3. *Let $g \in X \otimes (S)_{-1}$. Then* (3.18) *has a unique solution in $X \otimes (S)_{-1}$ of the form*

$$u = \widetilde{u}_0 + \sum_{n\in\mathbb{N}} \frac{1}{n}\, g_{(n-1)\varepsilon^{(i)}} \otimes K_{n\varepsilon^{(i)}}, \tag{3.19}$$

if and only if g is of the form

$$g = \sum_{n=0}^{\infty} g_{n\varepsilon^{(i)}} \otimes K_{n\varepsilon^{(i)}} = \sum_{n=0}^{\infty} g_{n\varepsilon^{(i)}} \left(I(\mathfrak{e}_i)\right)^{\diamond\, n}, \tag{3.20}$$

where $I(\cdot)$ represents the Itô integral.

Proof. Let $u \in X \otimes (S)_{-1}$ be a process of the form (3.19). Then, $u \in \mathrm{Dom}(\mathbb{D})$ and from

$$\mathbb{D}u = \sum_{n=1}^{\infty} \frac{1}{n}\, g_{(n-1)\varepsilon^{(i)}} \otimes n\, K_{(n-1)\varepsilon^{(i)}} \otimes \mathfrak{e}_n = \sum_{n=1}^{\infty} g_{n\varepsilon^{(i)}} \otimes K_{n\varepsilon^{(i)}} \otimes \mathfrak{e}_n$$

follows that it is a solution to (3.18).

Conversely, let a process $g \in X \otimes (S)_{-1}$ be of the form (3.20). Then, following the notation of Theorem 3.2, $c = \mathfrak{e}_i$ has the expansion $c = \sum_{k=1}^{\infty} c_k\, \mathfrak{e}_k$, where $c_k = 1$ for $k = i$ and $c_k = 0$ for $k \neq i$, $k \in \mathbb{N}$. The family of equations (3.17) transforms to the family of deterministic equations

$$\begin{array}{rl} (\alpha_i + 1)\, u_{\alpha+\varepsilon^{(i)}} = g_\alpha, & g_\alpha \in X \\ u_{\alpha+\varepsilon^{(k)}} = 0, & k = 1, 2, 3 \ldots, k \neq i \end{array}, \quad \alpha \in \mathcal{I}. \tag{3.21}$$

If (3.20) holds, then for fixed $i \in \mathbb{N}$, $g_\alpha = 0$, for all $\alpha \neq n\varepsilon^{(i)}$, and from (3.21) similarly as in Theorem 3.2 the coefficients are obtained by induction on $|\alpha|$,

$$u_\alpha = \begin{cases} \frac{1}{n} g_{(n-1)\varepsilon^{(i)}}, & \alpha = n\varepsilon^{(i)} \\ 0, & \alpha \neq n\varepsilon^{(i)} \end{cases}, \; n \in \mathbb{N}.$$

The chaos expansion of the solution is

$$u = \widetilde{u}_0 + \sum_{n\in\mathbb{N}} \frac{1}{n}\, g_{(n-1)\varepsilon^{(i)}} \otimes K_{n\varepsilon^{(i)}} = \widetilde{u}_0 + \sum_{n\in\mathbb{N}} \frac{1}{n}\, g_{(n-1)\varepsilon^{(i)}} \otimes (I(\mathfrak{e}_k))^{\diamond n}.$$

Convergence in $X \otimes (S)_{-1}$ can be proven by the same method as in Theorem 3.2. Clearly, there exists $p \in \mathbb{N}$, such that

$$\begin{aligned} \|u\|^2_{X\otimes(S)_{-1,-p}} &= \|\widetilde{u}_0\|^2_X + \sum_{n=1}^{\infty} \frac{1}{n^2}\, \|g_{(n-1)\varepsilon^{(i)}}\|^2_X (2\mathbb{N})^{-pn\varepsilon^{(i)}} \\ &\leqslant \|\widetilde{u}_0\|^2_X + \sum_{n=1}^{\infty} \|g_{(n-1)\varepsilon^{(i)}}\|^2_X (2\mathbb{N})^{-p(n-1)\varepsilon^{(i)}} \\ &= \|\widetilde{u}_0\|^2_X + \sum_{n=0}^{\infty} \|g_{n\varepsilon^{(i)}}\|^2_X (2\mathbb{N})^{-pn\varepsilon^{(i)}} < \infty. \end{aligned}$$

□

3.3. An eigenvalue problem. Using the same method as in the previous cases, in [**12**] we solved an eigenvalue problem of the form

$$\begin{cases} \mathbb{D}u = C \otimes u, & C \in \mathcal{S}'(\mathbb{R}) \\ Eu = \widetilde{u}_0, & \widetilde{u}_0 \in X. \end{cases} \tag{3.22}$$

In the special case, for $C = \mathfrak{e}_i$, $i \in \mathbb{N}$ fixed, we obtained that a unique solution u of the equation (3.22) belongs to $X \otimes (S)_{-1}$ and is of the form $u = \widetilde{u}_0 \otimes \exp\{I(\mathfrak{e}_i)\}$.

3.4. An equation involving the exponential of the Ornstein–Uhlenbeck operator. Consider now a stochastic differential equation of the form

$$e^{c\mathcal{R}}u = h, \tag{3.23}$$

where $e^{c\mathcal{R}} = \sum_{k=0}^{\infty} \frac{c^k \mathcal{R}^k}{k!}$, $c \in \mathbb{R}$ and $h \in X \otimes \exp(S)_{-1,-p}$ is a generalized stochastic process.

THEOREM 3.4. *Let $h \in X \otimes \exp(S)_{-1,-p}$, for some $p > 0$. Then, there exists $q > 0$ such that equation* (3.23) *has a unique generalized solution in $X \otimes \exp(S)_{-1,-q}$ given by the form*

$$u = \sum_{\alpha \in \mathcal{I}} e^{-c|\alpha|} h_\alpha \otimes K_\alpha. \tag{3.24}$$

PROOF. Assume $u \in X \otimes \exp(S)_{-1,-p}$ is a generalized stochastic process of the form (2.1), satisfying condition (2.2) with $q_\alpha^{-p} = e^{-p(2\mathbb{N})^\alpha}$. Note that the differential operator $e^{c\mathcal{R}}$ satisfies the identity

$$e^{c\mathcal{R}} K_\alpha = \sum_{k=0}^{\infty} \frac{c^k \mathcal{R}^k K_\alpha}{k!} = \sum_{k=0}^{\infty} \frac{c^k |\alpha|^k}{k!} K_\alpha = e^{c|\alpha|} K_\alpha,\ \alpha \in \mathcal{I}.$$

Then

$$e^{c\mathcal{R}} u = \sum_{\alpha \in \mathcal{I}} e^{c|\alpha|} u_\alpha \otimes H_\alpha, \quad u_\alpha \in X. \tag{3.25}$$

For $c > 0$ the operator $e^{c\mathcal{R}}$ is a continuous and bounded mapping from $X \otimes \exp(S)_{-1,-p}$ into $X \otimes \exp(S)_{-1,-q}$, for some $q > p + 2c$. From $e^{|\alpha|} \leqslant e^{(2\mathbb{N})^\alpha}$, $\alpha \in \mathcal{I}$ it follows

$$\begin{aligned}
\|e^{c\mathcal{R}} u\|^2_{X \otimes \exp(S)_{-1,-q}} &= \sum_{\alpha \in \mathcal{I}} e^{2c|\alpha|} \|u_\alpha\|_X^2 e^{-q(2\mathbb{N})^\alpha} \\
&\leqslant \sum_{\alpha \in \mathcal{I}} e^{2c|\alpha|}\, e^{-p(2\mathbb{N})^\alpha} \|u_\alpha\|_X^2 e^{-(q-p)(2\mathbb{N})^\alpha} \\
&\leqslant \Big(\sum_{\alpha \in \mathcal{I}} e^{2c|\alpha|} e^{-(q-p)(2\mathbb{N})^\alpha} \Big) \Big(\sum_{\alpha \in \mathcal{I}} \|u_\alpha\|_X^2 e^{-p(2\mathbb{N})^\alpha} \Big) \\
&\leqslant \Big(\sum_{\alpha \in \mathcal{I}} e^{-(q-p-2c)(2\mathbb{N})^\alpha} \Big) \|u\|^2_{X \otimes \exp(S)_{-1,-p}} < \infty,
\end{aligned}$$

for $q > p + 2c$.

If $c \leqslant 0$, then the operator $e^{c\mathcal{R}}$ is a continuous and bounded mapping from $X \otimes \exp(S)_{-1,-p}$ into $X \otimes \exp(S)_{-1,-q}$, for $q > p$:

$$\begin{aligned}
\|e^{c\mathcal{R}} u\|^2_{X \otimes \exp(S)_{-1,-q}} &= \sum_{\alpha \in \mathcal{I}} e^{2c|\alpha|} \|u_\alpha\|_X^2 e^{-q(2\mathbb{N})^\alpha} \\
&\leqslant \Big(\sum_{\alpha \in \mathcal{I}} e^{-(q-p)(2\mathbb{N})^\alpha} \Big) \|u\|^2_{X \otimes \exp(S)_{-1,-p}} < \infty.
\end{aligned}$$

Let $h \in X \otimes \exp(S)_{-1,-p}$ be of the form $h = \sum_{\alpha\in\mathcal{I}} h_\alpha \otimes H_\alpha$ such that $h_\alpha \in X$ and

$$\sum_{\alpha\in\mathcal{I}} \|h_\alpha\|_X^2 \, e^{-p(2\mathbb{N})^\alpha} < \infty. \tag{3.26}$$

We are looking for the solution u of (3.23) in the form (2.1), where $u_\alpha \in X$ are the coefficients to be determined. We apply (3.25) to transform equation (3.23) into the system of deterministic equations $e^{c|\alpha|}u_\alpha = h_\alpha$, $\alpha \in \mathcal{I}$. Thus $u_\alpha = e^{-c|\alpha|}h_\alpha$ and we obtain a unique solution of equation (3.23) in the form (3.24).

Finally, the convergence of the solution in $X \otimes \exp(S)_{-1,-p}$, in the case of $c > 0$, follows directly from (3.26). But, in the case of $c \leqslant 0$ the solution converges in the space $X \otimes \exp(S)_{-1,-q}$, for some $q > p - 2c$, i.e.,

$$\begin{aligned}
\|u\|^2_{X\otimes\exp(S)_{-1,-q}} &= \sum_{\alpha\in\mathcal{I}} e^{-2c|\alpha|}\|h_\alpha\|_X^2 e^{-q(2\mathbb{N})^\alpha} \\
&\leqslant \Big(\sum_{\alpha\in\mathcal{I}} e^{-2c|\alpha|} e^{-(q-p)(2\mathbb{N})^\alpha}\Big)\Big(\sum_{\alpha\in\mathcal{I}} \|h_\alpha\|_X^2 e^{-p(2\mathbb{N})^\alpha}\Big) \\
&\leqslant M\|h\|^2_{X\otimes\exp(S)_{-1,-p}} < \infty,
\end{aligned}$$

where $M = \sum_{\alpha\in\mathcal{I}} e^{-(q-p+2c)(2\mathbb{N})^\alpha} < \infty$ for $q > p - 2c$. □

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(Received 10 01 2011)

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