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GRÖBNER BASES FOR COMPLEX GRASSMANN MANIFOLDS

Branislav I. Prvulović

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Abstract. By Borel's description, integral cohomology of the complex Grassmann manifold $G_{k,n}$ is a polynomial algebra modulo a well-known ideal. A strong Gröbner basis for this ideal is obtained when $k = 2$ and $k = 3$.

1. Introduction

Integral cohomology of complex Grassmannian $G_{k,n} = U(n+k)/U(n) \times U(k)$ is isomorphic to the polynomial algebra on the Chern classes $c_1, c_2, \ldots, c_k$ of the canonical complex vector bundle γ_k over $G_{k,n}$ modulo the ideal $I_{k,n}$ generated by the dual classes $\overline{c}_{n+1}, \overline{c}_{n+2}, \ldots, \overline{c}_{n+k}$. Unfortunately, this description does not provide an efficient algorithm for determining whether a certain cohomology class is zero or not. But, if one has a Gröbner basis for $I_{k,n}$, this task is less demanding. In [**5**] and [**6**], the analogous problem for the mod 2 cohomology of real Grassmannians was considered and Gröbner bases for the corresponding ideals (in the cases $k = 2$ and $k = 3$) were presented. The theory of Gröbner bases over rings has complications that do not appear in the theory over fields. Nevertheless, for principal ideal domains, the generalization is good enough for our purposes.

In this paper, we show that calculations with $\mathbb{Z}$ coefficients, similar to those with $\mathbb{Z}_2$ coefficients in [**5**] and [**6**], provide strong Gröbner bases for the ideals $I_{k,n}$ in $\mathbb{Z}[c_1, c_2, \ldots, c_k]$ for $k = 2, 3$ and all $n \geqslant k$. These results are stated in Theorem 4.1 and Theorem 5.1. As a consequence of Theorem 4.1 (Corollary 4.2), we get the result of Hoggar (obtained in [**3**] by a calculation in terms of K-theory) concerning the structure of $H^*(G_{2,n};\mathbb{Z})$ as an abelian group. In Corollary 5.1 we establish the analogous result for $H^*(G_{3,n};\mathbb{Z})$.

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2. Background on Gröbner bases

In this paper, we denote by $\mathbb{N}_0$ the set of all nonnegative integers and the set of all positive integers is denoted by $\mathbb{N}$.

Let R be a principal ideal domain (PID) and $R[x_1, x_2, \dots, x_k]$ the polynomial algebra over R on k variables. Different authors define monomials and terms in $R[x_1, x_2, \dots, x_k]$ in various ways. Our terminology will be as follows. A *monomial* on variables $x_1, x_2, \dots, x_k$ is a power product $x_1^{a_1} x_2^{a_2} \dots x_k^{a_k} \in R[x_1, x_2, \dots, x_k]$, where $a_1, a_2, \dots, a_k \in \mathbb{N}_0$. The set of all monomials in $R[x_1, x_2, \dots, x_k]$ will be denoted by M. A *term* in $R[x_1, x_2, \dots, x_k]$ is a product αm of a coefficient $\alpha \in R$ and a monomial $m \in M$.

Let $\preceq$ be a fixed well ordering on M (a total ordering such that every nonempty subset of M has a least element) with the property that $m_1 \preceq m_2$ implies $m_1 m_3 \preceq m_2 m_3$ for all $m_1, m_2, m_3 \in M$.

For a polynomial $f = \sum_{i=1}^{r} \alpha_i m_i \in R[x_1, x_2, \dots, x_k]$, where m_i are pairwise different monomials and $\alpha_i \in R \smallsetminus \{0\}$, let $M(f) := \{m_i \mid 1 \leqslant i \leqslant r\}$. We define the *leading monomial of* f, denoted by $\mathrm{LM}(f)$, as $\max M(f)$ with respect to $\preceq$. The *leading coefficient of* f, denoted by $\mathrm{LC}(f)$, is the coefficient of $\mathrm{LM}(f)$ and the *leading term of* f is $\mathrm{LT}(f) := \mathrm{LC}(f) \cdot \mathrm{LM}(f)$.

The notion of a strong Gröbner basis (in [**2**], Becker and Weispfenning use the phrase D-Gröbner basis) for a given ideal I in $R[x_1, x_2, \dots, x_k]$ can be defined in a number of equivalent ways. We have chosen the following one, which avoids the notion of reduction (see [**2**, p. 455] and [**1**, p. 251]).

Definition 2.1. Let $G \subset R[x_1, x_2, \dots, x_k]$ be a finite set of nonzero polynomials and $I_G = (G)$ the ideal in $R[x_1, x_2, \dots, x_k]$ generated by G. We say that G is a *strong Gröbner basis* for I_G (with respect to $\preceq$) if for each $f \in I_G \smallsetminus \{0\}$ there exists $g \in G$ such that $\mathrm{LT}(g) \mid \mathrm{LT}(f)$ (meaning, as usual, that $\mathrm{LT}(f) = t \cdot \mathrm{LT}(g)$ for some term t).

Remark 2.1. If G is a strong Gröbner basis for I_G and $f \notin I_G$, then there may still exist $g \in G$ such that $\mathrm{LT}(g)$ divides $\mathrm{LT}(f)$, but one can see that $f \equiv f_1$ modulo I_G for some polynomial f_1 with the property that $\mathrm{LT}(f_1)$ is not divisible by any of $\mathrm{LT}(g)$, $g \in G$. Namely, if some $\mathrm{LT}(g)$ divides $\mathrm{LT}(f)$, say $\mathrm{LT}(f) = t \cdot \mathrm{LT}(g)$, then the polynomial $f_1 := f - t \cdot g$ is $\equiv f$ modulo I_G and $\mathrm{LM}(f_1) \prec \mathrm{LM}(f)$. If f_1 does not have the desired property, we continue this process. Since $\preceq$ is a well ordering, the process must terminate.

Let G be an arbitrary finite subset of $R[x_1, x_2, \dots, x_k] \smallsetminus \{0\}$ and I_G the ideal generated by G. We now want to formulate a sufficient condition for G to be a strong Gröbner basis. If $m \in M$ is a fixed monomial and if for $f \in R[x_1, x_2, \dots, x_k]$ we have $f = \sum_{i=1}^{s} t_i g_i$, where t_i are some terms and g_i some (not necessarily pairwise different) elements of G such that $\max_{1 \leqslant i \leqslant s} \mathrm{LM}(t_i g_i) \preceq m$, we say that $\sum_{i=1}^{s} t_i g_i$ is an *m-representation of f with respect to G*. An $\mathrm{LM}(f)$-representation of f w.r.t. G is called a *standard representation of f w.r.t. G*.

We shall need the following lemma from [**2**]. We denote by $\mathrm{lcm}(a, b)$ and $\gcd(a, b)$ respectively the least common multiple and the greatest common divisor of a and b, where a and b are either monomials or elements of R.

Lemma 2.1. [**2**, p. 456] *Let G be a finite set of nonzero polynomials from $R[x_1, x_2, \dots, x_k]$ satisfying the following two conditions:*

(i) *For all $g_1, g_2 \in G$ there exists $h \in G$ (which depends on g_1 and g_2) such that* $\mathrm{LM}(h) \mid \mathrm{lcm}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$ *and* $\mathrm{LC}(h) \mid \gcd(\mathrm{LC}(g_1), \mathrm{LC}(g_2))$*;*

(ii) *Every nonzero $f \in I_G$ has a standard representation w.r.t. G.*

Then G is a strong Gröbner basis.

Note that if $\mathrm{LC}(g) = 1$ for all $g \in G$, then the condition (i) from Lemma 2.1 is certainly satisfied. Namely, one can take h to be g_1. Then, $\mathrm{LM}(h) = \mathrm{LM}(g_1)$, so $\mathrm{LM}(h) \mid \mathrm{lcm}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$. The other condition is clearly satisfied.

In order to formulate an important theorem, we need the following definition. Recall that we have a fixed ordering $\preceq$ on the monomials.

Definition 2.2. The *S-polynomial* of polynomials $f, g \in R[x_1, x_2, \dots, x_k]$ is given by

$$S(f,g) := \frac{l}{\mathrm{LC}(f)} \cdot \frac{L}{\mathrm{LM}(f)} \cdot f - \frac{l}{\mathrm{LC}(g)} \cdot \frac{L}{\mathrm{LM}(g)} \cdot g,$$

where $l = \mathrm{lcm}(\mathrm{LC}(f), \mathrm{LC}(g))$ and $L = \mathrm{lcm}(\mathrm{LM}(f), \mathrm{LM}(g))$.

Let us note that since $\mathrm{lcm}(\mathrm{LC}(f), \mathrm{LC}(g))$ is not uniquely determined in a PID, there is some indeterminacy in Definition 2.2, but any two least common multiples of the same pair of elements are associates and so, this indeterminacy makes no harm to the following theory. Nevertheless, we shall make the S-polynomial unique when $R = \mathbb{Z}$, by requiring that $\mathrm{lcm}(\mathrm{LC}(f), \mathrm{LC}(g)) > 0$. With this convention in mind, we see that (for $R = \mathbb{Z}$), S-polynomial is antisymmetric, $S(g, f) = -S(f, g)$.

We are now able to formulate the announced theorem.

Theorem 2.1. [**2**, p. 457] *Let G be a finite subset of $R[x_1, x_2, \dots, x_k]$, $0 \notin G$, and let I_G be the ideal in $R[x_1, x_2, \dots, x_k]$ generated by G. If condition (i) from Lemma 2.1 holds and for all $g_1, g_2 \in G$, $S(g_1, g_2)$ either equals zero or has a standard representation with respect to G, then every nonzero $f \in I_G$ has a standard representation w.r.t. G.*

Remark 2.2. In the statement of this theorem in [**2**], Becker and Weispfenning reformulated the condition (i) from Lemma 2.1 in terms of G-polynomial of g_1 and g_2, but we do not need this reformulation.

It is obvious from Definition 2.2 that $\mathrm{LM}(S(g_1, g_2)) \prec \mathrm{lcm}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$ since $\mathrm{lcm}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$ cancels out in the upper expression. This means that if we have a standard representation of $S(g_1, g_2)$ w.r.t. G, then we have an m-representation of $S(g_1, g_2)$ w.r.t. G for a monomial $m \prec \mathrm{lcm}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$. By a careful analysis of the proof of Theorem 2.1 in [**2**], one observes that the authors use only this weaker assumption (that $S(g_1, g_2)$ has an m-representation w.r.t. G for some monomial $m \prec \mathrm{lcm}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$). Moreover, the corresponding theorem when R is a field (see [**2**, p. 219]) was given in this form.

By summarizing the preceding discussion, we obtain sufficient conditions for a set $G \subset R[x_1, x_2, \dots, x_k]$ to be a strong Gröbner basis. These are stated in the following theorem which will be the crucial tool in proving our main results.

Theorem 2.2. *Let G be a finite subset of $R[x_1, x_2, \dots, x_k]$, $0 \notin G$, and I_G the ideal in $R[x_1, x_2, \dots, x_k]$ generated by G. If for all $g \in G$, $\mathrm{LC}(g) = 1$ and for all $g_1, g_2 \in G$, $S(g_1, g_2)$ either equals zero or has an m-representation with respect to G for some $m \prec \mathrm{lcm}(\mathrm{LM}(g_1), \mathrm{LM}(g_2))$, then G is a strong Gröbner basis for I_G.*

In the rest of the paper, we use the grlex ordering $\preceq$ on the monomials in $R[x_1, x_2, \dots, x_k]$ with $x_1 > x_2 > \cdots > x_k$. It is defined as follows. The monomials are compared by the sum of the exponents and if these are equal for the two monomials, they are compared lexicographically from the left. That is, we shall write $x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} \prec x_1^{b_1} x_2^{b_2} \cdots x_k^{b_k}$ if either $a_1 + a_2 + \cdots + a_k < b_1 + b_2 + \cdots + b_k$ or else $a_1 + a_2 + \cdots + a_k = b_1 + b_2 + \cdots + b_k$ and $a_s < b_s$, where $s = \min\{i \mid a_i \neq b_i\}$.

3. Cohomology of $G_{k,n}$

Let $G_{k,n}$ be the complex Grassmann manifold of k-dimensional complex vector subspaces in $\mathbb{C}^{n+k}$ and let $c_1, c_2, \dots, c_k$ be the Chern classes of the canonical bundle γ_k over $G_{k,n}$. It is known that the cohomology algebra $H^*(G_{k,n}; \mathbb{Z})$ is isomorphic to the quotient $\mathbb{Z}[c_1, c_2, \dots, c_k]/I_{k,n}$ of the polynomial algebra $\mathbb{Z}[c_1, c_2, \dots, c_k]$ by the ideal $I_{k,n}$ generated by polynomials $\overline{c}_{n+1}, \overline{c}_{n+2}, \dots, \overline{c}_{n+k}$. These are obtained from the equation $(1 + c_1 + c_2 + \cdots + c_k)(1 + \overline{c}_1 + \overline{c}_2 + \cdots) = 1$, that is

$$
\begin{aligned}
(3.1)\quad 1 + \overline{c}_1 + \overline{c}_2 + \cdots &= \frac{1}{1 + c_1 + c_2 + \cdots + c_k} = \sum_{t \geqslant 0} (-1)^t (c_1 + c_2 + \cdots + c_k)^t, \\
&= \sum_{t \geqslant 0} \sum_{a_1 + \cdots + a_k = t} (-1)^t [a_1, a_2, \dots, a_k] c_1^{a_1} c_2^{a_2} \cdots c_k^{a_k} \\
&= \sum_{a_1, \dots, a_k \geqslant 0} (-1)^{a_1 + \cdots + a_k} [a_1, a_2, \dots, a_k] c_1^{a_1} c_2^{a_2} \cdots c_k^{a_k}
\end{aligned}
$$

where $[a_1, a_2, \dots, a_k]$ ($a_j \in \mathbb{N}_0$) denotes the multinomial coefficient,

$$
\begin{aligned}
[a_1, a_2, \dots, a_k] &= \frac{(a_1 + a_2 + \cdots + a_k)!}{a_1! \cdot a_2! \cdot \cdots \cdot a_k!} \\
&= \binom{a_1 + a_2 + \cdots + a_k}{a_1} \binom{a_2 + \cdots + a_k}{a_2} \cdots \binom{a_{k-1} + a_k}{a_{k-1}}.
\end{aligned}
$$

By identifying the homogenous parts of (cohomological) degree $2r$ in formula (3.1), we obtain the following proposition.

Proposition 3.1. *For $r \in \mathbb{N}$,*

$$
\overline{c}_r = \sum_{a_1 + 2a_2 + \cdots + k a_k = r} (-1)^{a_1 + \cdots + a_k} [a_1, a_2, \dots, a_k] c_1^{a_1} c_2^{a_2} \cdots c_k^{a_k}.
$$

It is understood that $a_1, a_2, \dots, a_k \in \mathbb{N}_0$.

Let us add here that $H^*(G_{k,n}; \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \dots, c_k]/I_{k,n}$ is a free (graduated) abelian group. Namely, the manifold $G_{k,n}$ has a cell subdivision with no cells in odd dimensions (see [**4**, Problem 14-D]). Therefore, the (co)boundary operators in the cochain complex $C^*(G_{k,n}; \mathbb{Z})$ are all trivial and $H^*(G_{k,n}; \mathbb{Z}) \cong C^*(G_{k,n}; \mathbb{Z})$ is free. Furthermore, the number of $2i$-cells in this CW-decomposition is $p_{k,n}(i)$, where

$p_{k,n}(i)$ is the number of partitions of integer i into at most n nonnegative integers each of which is $\leqslant k$, and so the rank of the group $H^*(G_{k,n};\mathbb{Z})$ is $\sum_{i=0}^{nk} p_{k,n}(i)$.

4. Gröbner basis for $I_{2,n}$

The binomial coefficient for $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{N}$ is defined by $\binom{\alpha}{\beta} := \frac{\alpha(\alpha-1)\cdots(\alpha-\beta+1)}{\beta!}$. Also, $\binom{\alpha}{0} := 1$. If β is a negative integer, we define $\binom{\alpha}{\beta}$ to be equal to zero. Then it is easy to see that the well known formula

$$\binom{\alpha}{\beta} = \binom{\alpha-1}{\beta} + \binom{\alpha-1}{\beta-1} \tag{4.1}$$

is valid for all $\alpha, \beta \in \mathbb{Z}$.

For $k = 2$, Proposition 3.1 gives us: $\overline{c}_r = \sum_{a+2b=r}(-1)^{a+b}\binom{a+b}{a}c_1^a c_2^b$.

Let $n \geqslant 2$ be a fixed integer. In order to find a Gröbner basis for $I_{2,n} = (\overline{c}_{n+1}, \overline{c}_{n+2})$, we define polynomials g_m $(m \geqslant 0)$.

Definition 4.1. For $m \in \mathbb{N}_0$, let

$$g_m := \sum_{a+2b=n+1+m} (-1)^{n+1+a+b}\binom{a+b-m}{a}c_1^a c_2^b.$$

As before, it is understood that $a, b \geqslant 0$.

By comparing this definition with the above expression for $\overline{c}_r$, one observes that $g_0 = (-1)^{n+1}\overline{c}_{n+1}$. Also,

$$\begin{aligned} c_2\overline{c}_n &= \sum_{a+2b=n}(-1)^{a+b}\binom{a+b}{a}c_1^a c_2^{b+1} = \sum_{a+2b=n+2}(-1)^{a+b-1}\binom{a+b-1}{a}c_1^a c_2^b \\ &= (-1)^n \sum_{a+2b=n+2}(-1)^{n+1+a+b}\binom{a+b-1}{a}c_1^a c_2^b = (-1)^n g_1. \end{aligned}$$

The change of variable $b \mapsto b-1$ does not affect the requirement that $b \geqslant 0$ since for $b = 0$ the binomial coefficient $\binom{a+b-1}{a} = \binom{n+1}{n+2}$ is equal to 0.

From the defining formula for g_m, one can see that if $m \leqslant n+2$, then b must be such that $m \leqslant b \leqslant \frac{n+1+m}{2}$. Namely, $a+b-m$ cannot be negative since $a+b-m \leqslant -1$ implies $a+2b \leqslant 2(a+b) \leqslant 2m-2 \leqslant n+m$ contradicting the requirement that $a+2b = n+1+m$. Now, $a+b-m$ must be $\geqslant a$ in order for $\binom{a+b-m}{a}$ to be nonzero and we conclude that $b \geqslant m$. The second inequality comes from the condition $a+2b = n+1+m$. In particular, $g_{n+2} = 0$ and for $0 \leqslant m \leqslant n+1$ we have

$$g_m = \sum_{b=m}^{[\frac{n+1+m}{2}]} (-1)^{b-m}\binom{n+1-b}{b-m}c_1^{n+1+m-2b}c_2^b. \tag{4.2}$$

Let $G := \{g_0, g_1, \ldots, g_{n+1}\}$. We shall prove that, with respect to the grlex ordering, G is a strong Gröbner basis for $I_{2,n}$. It is obvious that the summand in (4.2) obtained for $b = m$ provides the leading monomial $\mathrm{LM}(g_m) = \mathrm{LT}(g_m) =$

$c_1^{n+1-m}c_2^m$. From this it will follow that an additive basis for $H^*(G_{2,n};\mathbb{Z})$ is the set of all monomials $c_1^a c_2^b$ such that $a+b\leqslant n$.

In order to show that G is a strong Gröbner basis for $I_{2,n}$, we define the ideal $I_G := (G) = (g_0, g_1, \dots, g_{n+1})$ in $\mathbb{Z}[c_1,c_2]$. As we have already noticed, $\overline{c}_{n+1} = (-1)^{n+1}g_0 \in I_G$, $\overline{c}_{n+2} = -c_1\overline{c}_{n+1} - c_2\overline{c}_n = (-1)^n c_1 g_0 + (-1)^{n+1}g_1 \in I_G$, so $I_{2,n}\subseteq I_G$.

It remains to prove that $I_G\subseteq I_{2,n}$ and that G is a strong Gröbner basis. It turns out that the following proposition plays the crucial role in proving these facts.

Proposition 4.1. *For each* $m\in\mathbb{N}_0$, $c_2 g_m - c_1 g_{m+1} = -g_{m+2}$.

Proof. We proceed directly to the calculation.

$$\begin{aligned}
c_2 g_m - c_1 g_{m+1} &= \sum_{a+2b=n+1+m} (-1)^{n+1+a+b}\binom{a+b-m}{a} c_1^a c_2^{b+1}\\
&\quad - \sum_{a+2b=n+m+2} (-1)^{n+1+a+b}\binom{a+b-m-1}{a} c_1^{a+1} c_2^{b}\\
&= \sum_{a+2b=n+m+3} (-1)^{n+a+b}\binom{a+b-m-1}{a} c_1^a c_2^{b}\\
&\quad - \sum_{a+2b=n+m+3} (-1)^{n+a+b}\binom{a+b-m-2}{a-1} c_1^a c_2^{b}\\
&= \sum_{a+2b=n+m+3} (-1)^{n+a+b}\binom{a+b-m-2}{a} c_1^a c_2^{b} = -g_{m+2},
\end{aligned}$$

by equality (4.1). We note that, for the similar reasons as above, the change of variable $b\mapsto b-1$ ($a\mapsto a-1$) does not affect the requirement that $b\geqslant 0$ ($a\geqslant 0$). The proposition follows. □

Corollary 4.1. $I_G\subseteq I_{2,n}$.

Proof. We already know that $g_0 = (-1)^{n+1}\overline{c}_{n+1}\in I_{2,n}$ and $g_1 = (-1)^n c_2\overline{c}_n = (-1)^{n+1}(c_1\overline{c}_{n+1} + \overline{c}_{n+2})\in I_{2,n}$. Proposition 4.1 applies and by induction on m we have $g_m\in I_{2,n}$ $(0\leqslant m\leqslant n+1)$. The Corollary follows. □

Therefore G is a basis for $I_{2,n}$ and we wish to prove that it is a strong Gröbner basis. We need the following lemma.

Lemma 4.1. *For* $0\leqslant m < m+s\leqslant n+1$, *we have*

$$S(g_m, g_{m+s}) = -\sum_{i=0}^{s-1} c_1^i c_2^{s-1-i} g_{m+2+i}.$$

Proof. We have that

$$\mathrm{lcm}(\mathrm{LM}(g_m), \mathrm{LM}(g_{m+s})) = \mathrm{lcm}(c_1^{n+1-m}c_2^m, c_1^{n+1-m-s}c_2^{m+s}) = c_1^{n+1-m}c_2^{m+s}$$

and so $S(g_m, g_{m+s}) = c_2^s g_m - c_1^s g_{m+s}$.

We proceed by induction on s. For $s=1$, we need to prove that $S(g_m, g_{m+1}) = -g_{m+2}$. We calculate: $S(g_m, g_{m+1}) = c_2 g_m - c_1 g_{m+1} = -g_{m+2}$, by Proposition 4.1. For the inductive step, we have

$$\begin{aligned} S(g_m, g_{m+s}) &= c_2^s g_m - c_1^s g_{m+s} = c_2^s g_m - c_2 c_1^{s-1} g_{m+s-1} + c_2 c_1^{s-1} g_{m+s-1} - c_1^s g_{m+s} \\ &= c_2 S(g_m, g_{m+s-1}) + c_1^{s-1} S(g_{m+s-1}, g_{m+s}) \\ &= -c_2 \sum_{i=0}^{s-2} c_1^i c_2^{s-2-i} g_{m+2+i} - c_1^{s-1} g_{m+s+1} \\ &= -\sum_{i=0}^{s-2} c_1^i c_2^{s-1-i} g_{m+2+i} - c_1^{s-1} g_{m+s+1} = -\sum_{i=0}^{s-1} c_1^i c_2^{s-1-i} g_{m+2+i}, \end{aligned}$$

again by Proposition 4.1 and the induction hypothesis. □

THEOREM 4.1. *Let $n \geqslant 2$. Then $G = \{g_0, g_1, \ldots, g_{n+1}\}$ defined above is a strong Gröbner basis for the ideal $I_{2,n}$ in $\mathbb{Z}[c_1, c_2]$ with respect to the grlex ordering $\preceq$.*

PROOF. We have already shown that G is a basis for $I_{2,n}$. We want to apply Theorem 2.2. It is immediate from (4.2) that $\mathrm{LC}(g) = 1$ for all $g \in G$. Let g_m and g_{m+s} $(0 \leqslant m < m+s \leqslant n+1)$ be two arbitrary elements of G. Since S-polynomial is antisymmetric, it suffices to show that $S(g_m, g_{m+s})$ has required representation.

If $m = n$, then $m+s$ must be $n+1$ and, using Proposition 4.1, one obtains $S(g_m, g_{m+s}) = S(g_n, g_{n+1}) = c_2 g_n - c_1 g_{n+1} = -g_{n+2} = 0$.

If $m \leqslant n-1$, according to Lemma 4.1, $S(g_m, g_{m+s}) = -\sum_{i=0}^{s-1} c_1^i c_2^{s-1-i} g_{m+2+i}$. Note that for $i \in \{0, 1, \ldots, s-1\}$, $m+2+i \leqslant m+s+1 \leqslant n+2$ and so, either $g_{m+2+i} \in G$ (if $m+2+i \leqslant n+1$) or $g_{m+2+i} = 0$ (if $m+2+i = n+2$). Define $t = t(m,s) := c_1^{n-1-m} c_2^{m+s+1}$. First of all, observe that

$$t \prec c_1^{n+1-m} c_2^{m+s} = \mathrm{lcm}(\mathrm{LM}(g_m), \mathrm{LM}(g_{m+s})).$$

Now, for all $i \in \{0, 1, \ldots, s-1\}$,

$$\begin{aligned} \mathrm{LM}(c_1^i c_2^{s-1-i} g_{m+2+i}) = c_1^i c_2^{s-1-i}\, \mathrm{LM}(g_{m+2+i}) &= c_1^i c_2^{s-1-i} c_1^{n+1-m-2-i} c_2^{m+2+i} \\ &= c_1^{n-1-m} c_2^{m+s+1} = t. \end{aligned}$$

Theorem 2.2 applies and we conclude that G is a strong Gröbner basis for $I_{2,n}$. □

COROLLARY 4.2. *Let $n \geqslant 2$. If c_i is the i-th Chern class of the canonical complex vector bundle γ_2 over $G_{2,n}$, then the set $\{c_1^a c_2^b \mid a+b \leqslant n\}$ is an additive basis for the free abelian group $H^*(G_{2,n}; \mathbb{Z})$.*

PROOF. A monomial $c_1^a c_2^b$ corresponds to the partition of the (nonnegative) integer $a + 2b$:

$$\underbrace{1+1+\cdots+1}_{a} + \underbrace{2+2+\cdots+2}_{b}.$$

Furthermore, it is obvious that this produces a one-to-one correspondence between the set $\{c_1^a c_2^b \mid a+b \leqslant n\}$ and the set of all partitions of the nonnegative integers $\leqslant 2n$ into at most n integers each of which is $\leqslant 2$. This means that the cardinality of the set $\{c_1^a c_2^b \mid a+b \leqslant n\}$ is equal to $\operatorname{rank}(H^*(G_{2,n};\mathbb{Z})) = \sum_{i=0}^{2n} p_{2,n}(i)$. Hence, it suffices to show that the set $\{c_1^a c_2^b \mid a+b \leqslant n\}$ generates $H^*(G_{2,n};\mathbb{Z})$.

Let $\sigma \in H^*(G_{2,n};\mathbb{Z}) \cong \mathbb{Z}[c_1,c_2]/I_{2,n}$ be a nonzero class and $f \in \mathbb{Z}[c_1,c_2]$ its representative. Since G is a strong Gröbner basis for $I_{2,n}$ and $f \notin I_{2,n}$, by Remark 2.1 we have that $f \equiv f_1$ modulo $I_{2,n}$ for some $f_1 \in \mathbb{Z}[c_1,c_2]$ such that $\mathrm{LT}(f_1)$ is not divisible by any of $\mathrm{LT}(g) = \mathrm{LM}(g)$, $g \in G$. Observe that $\{\mathrm{LM}(g) \mid g \in G\}$ is the set of all monomials $c_1^a c_2^b$ such that $a+b = n+1$. This means that the sum of the exponents in $\mathrm{LM}(f_1)$ and so, in every monomial from $M(f_1)$, is $\leqslant n$. Since f_1 also represents σ, this concludes the proof. □

Let us remark that our strong Gröbner basis G has some additional nice properties. It is minimal in the sense of Definition 4.5.6. from [**1**, p. 251] since no $\mathrm{LT}(g_i)$ divides $\mathrm{LT}(g_j)$ for $i \neq j$. Moreover, it is reduced in the sense of the definition of this notion for Gröbner bases over fields, meaning that all leading coefficients in G are equal to 1 and no term of g_i is divisible by $\mathrm{LT}(g_j)$ for $i \neq j$. This follows from formula (4.2) by observation that all monomials in $M(g_i)$ except the leading one have the sum of the exponents $< n+1$. Finally, one can verify that, since $\mathbb{Z}[c_1,c_2]/I_{2,n}$ is free, G produces unique normal forms (remainders).

Let us now calculate a few elements of the strong Gröbner basis G. By formula (4.2), $g_{n+1} = c_2^{n+1}$ and $g_n = c_1 c_2^n$. Using this and Proposition 4.1, we obtain $c_2 g_{n-1} = c_1 g_n - g_{n+1} = c_1^2 c_2^n - c_2^{n+1} = c_2(c_1^2 c_2^{n-1} - c_2^n)$ and so we deduce that $g_{n-1} = c_1^2 c_2^{n-1} - c_2^n$. Continuing in the same manner, one gets:

$$\begin{aligned}
g_{n-2} &= c_1^3 c_2^{n-2} - 2c_1 c_2^{n-1};\\
g_{n-3} &= c_1^4 c_2^{n-3} - 3c_1^2 c_2^{n-2} + c_2^{n-1};\\
g_{n-4} &= c_1^5 c_2^{n-4} - 4c_1^3 c_2^{n-3} + 3c_1 c_2^{n-2};\\
g_{n-5} &= c_1^6 c_2^{n-5} - 5c_1^4 c_2^{n-4} + 6c_1^2 c_2^{n-3} - c_2^{n-2}, \quad \text{etc.}
\end{aligned}$$

5. Gröbner basis for $I_{3,n}$

We now focus on the case $k=3$ and our goal is to find a strong Gröbner basis for the ideal $I_{3,n}$ in $\mathbb{Z}[c_1,c_2,c_3]$ (for all $n \geqslant 3$) and obtain some new information concerning the cohomology algebra $H^*(G_{3,n};\mathbb{Z})$.

In the case $k=3$, Proposition 3.1 gives us

$$\overline{c}_r = \sum_{a+2b+3c=r} (-1)^{a+b+c} \binom{a+b+c}{a}\binom{b+c}{b} c_1^a c_2^b c_3^c, \qquad r \in \mathbb{N}.$$

Let $\preceq$ be the grlex ordering on the monomials in $\mathbb{Z}[c_1,c_2,c_3]$ (with $c_1 > c_2 > c_3$) and let $n \geqslant 3$ be a fixed integer. In order to find a strong Gröbner basis for the ideal $I_{3,n} = (\overline{c}_{n+1}, \overline{c}_{n+2}, \overline{c}_{n+3})$, we define the polynomials $g_{m,l} \in \mathbb{Z}[c_1,c_2,c_3]$, $m,l \in \mathbb{N}_0$.

Definition 5.1. For $m,l \in \mathbb{N}_0$, let

$$g_{m,l} := \sum_{a+2b+3c=n+1+m+2l} (-1)^{n+1+a+b+c} \binom{a+b+c-m-l}{a}\binom{b+c-l}{b} c_1^a c_2^b c_3^c.$$

As before, it is understood that $a, b, c \in \mathbb{N}_0$.

Let us remark first that $g_{0,0} = (-1)^{n+1}\overline{c}_{n+1}$. Although in the expression for $\overline{c}_r$ the product of binomial coefficients reduces to a trinomial coefficient, this is not the case for polynomials $g_{m,l}$ for $m > 0$. Therefore, we are not able to use trinomial coefficients and their properties in the upcoming calculations with these polynomials.

In addition to that, we note that the coefficient $\binom{a+b+c-m-l}{a}\binom{b+c-l}{b}$ may be nonzero when $a+b+c-m-l < 0$ (or $b+c-l < 0$). For example, if $n = 4$ we have

$$\begin{aligned} g_{5,0} &= \sum_{a+2b+3c=10} (-1)^{1+a+b+c}\binom{a+b+c-5}{a}\binom{b+c}{b} c_1^a c_2^b c_3^c \\ &= -\binom{-1}{0}\binom{4}{2} c_2^2 c_3^2 + \binom{0}{0}\binom{5}{5} c_2^5 - \binom{-1}{1}\binom{3}{0} c_1 c_3^3 = c_2^5 - 6c_2^2 c_3^2 + c_1 c_3^3. \end{aligned}$$

However, we can prove the following lemma.

Lemma 5.1. *Let a, b, c, m, l be nonnegative integers. Then the following implication holds:*

$$\begin{aligned} &\binom{a+b+c-m-l}{a}\binom{b+c-l}{b} \neq 0 \\ &\qquad \Longrightarrow \quad a+b+c < m+l \quad \text{or} \quad (b+c \geqslant m+l \quad \text{and} \quad c \geqslant l). \end{aligned}$$

Proof. Assume that $\binom{a+b+c-m-l}{a}\binom{b+c-l}{b} \neq 0$ and $a+b+c \geqslant m+l$. Then we have that $\binom{a+b+c-m-l}{a} \neq 0$ and since both $a+b+c-m-l$ and a are nonnegative we conclude that $a+b+c-m-l \geqslant a$, i.e., $b+c \geqslant m+l$.

If $c < l$, then $b+c-l < b$ and since $\binom{b+c-l}{b} \neq 0$ it must be $b+c-l < 0$. From this we have $0 \leqslant a+b+c-m-l < a-m \leqslant a$, but this implies that $\binom{a+b+c-m-l}{a} = 0$ contradicting the assumption $\binom{a+b+c-m-l}{a}\binom{b+c-l}{b} \neq 0$. This contradiction proves that $c \geqslant l$. □

Finally, we define the set $G \subseteq \mathbb{Z}[c_1, c_2, c_3]$, our candidate for the strong Gröbner basis.

Definition 5.2. $G := \{g_{m,l} \mid m+l \leqslant n+1, m, l \in \mathbb{N}_0\}$.

We now prove an important property of G.

Proposition 5.1. *For $m, l \in \mathbb{N}_0$ such that $m+l \leqslant n+1$, we have that the leading monomial* $\mathrm{LM}(g_{m,l}) = \mathrm{LT}(g_{m,l}) = c_1^{n+1-m-l} c_2^m c_3^l$ *and all other monomials in $M(g_{m,l})$ have the sum of the exponents $< n+1$.*

Proof. Obviously, the (nonnegative) integers $a := n+1-m-l$, $b := m$, $c := l$ satisfy the conditions $a+2b+3c = n+1+m+2l$ and the coefficient $(-1)^{n+1+a+b+c}\binom{a+b+c-m-l}{a}\binom{b+c-l}{b} = \binom{a}{a}\binom{b}{b} = 1$. So, the monomial $c_1^{n+1-m-l} c_2^m c_3^l$ does appear in $g_{m,l}$ with coefficient 1.

Now, it suffices to prove the inequality $a+b+c < n+1$ for all other monomials $c_1^a c_2^b c_3^c$ appearing in $g_{m,l}$ with nonzero coefficient. If $c_1^a c_2^b c_3^c$ is such a monomial, then $a+2b+3c = n+1+m+2l$ (i.e., $a = n+1+m+2l-2b-3c$) and

$\binom{a+b+c-m-l}{a}\binom{b+c-l}{b} \neq 0$. According to Lemma 5.1, $a+b+c < m+l$ or $b+c \geqslant m+l$ and $c \geqslant l$.

In the first case $a+b+c < m+l \leqslant n+1$ and we are done.

Otherwise, $b+c \geqslant m+l$ and $c \geqslant l$ give us that $b+2c \geqslant m+2l$, where the equality holds only if $c = l$ and $b = m$. But then $a = n+1+m+2l-2b-3c = n+1-m-l$ and since $c_1^a c_2^b c_3^c \neq c_1^{n+1-m-l} c_2^m c_3^l$, we actually have $b+2c > m+2l$. This implies that $a+b+c = n+1+m+2l-b-2c < n+1$. □

Let I_G be the ideal in $\mathbb{Z}[c_1, c_2, c_3]$ generated by G. Eventually, we shall prove that $I_G = I_{3,n} = (\overline{c}_{n+1}, \overline{c}_{n+2}, \overline{c}_{n+3})$, but for the moment we prove that I_G contains $I_{3,n}$.

Proposition 5.2. $I_{3,n} \subseteq I_G$.

Proof. As we have already noticed, $\overline{c}_{n+1} = (-1)^{n+1} g_{0,0} \in I_G$. Since

$$\begin{aligned}
-c_1 g_{0,0} + g_{1,0} = {} & -c_1 \sum_{a+2b+3c=n+1} (-1)^{n+1+a+b+c} \binom{a+b+c}{a}\binom{b+c}{b} c_1^a c_2^b c_3^c \\
& + \sum_{a+2b+3c=n+2} (-1)^{n+1+a+b+c} \binom{a+b+c-1}{a}\binom{b+c}{b} c_1^a c_2^b c_3^c \\
= {} & \sum_{a+2b+3c=n+1} (-1)^{n+2+a+b+c} \binom{a+b+c}{a}\binom{b+c}{b} c_1^{a+1} c_2^b c_3^c \\
& + \sum_{a+2b+3c=n+2} (-1)^{n+1+a+b+c} \binom{a+b+c-1}{a}\binom{b+c}{b} c_1^a c_2^b c_3^c \\
= {} & \sum_{a+2b+3c=n+2} (-1)^{n+1+a+b+c} \binom{a+b+c-1}{a-1}\binom{b+c}{b} c_1^a c_2^b c_3^c \\
& + \sum_{a+2b+3c=n+2} (-1)^{n+1+a+b+c} \binom{a+b+c-1}{a}\binom{b+c}{b} c_1^a c_2^b c_3^c \\
= {} & \sum_{a+2b+3c=n+2} (-1)^{n+1+a+b+c} \binom{a+b+c}{a}\binom{b+c}{b} c_1^a c_2^b c_3^c = (-1)^{n+1}\overline{c}_{n+2},
\end{aligned}$$

we conclude that $\overline{c}_{n+2} = (-1)^n c_1 g_{0,0} + (-1)^{n+1} g_{1,0} \in I_G$.

In order to show that $\overline{c}_{n+3} \in I_G$ we calculate:

$$\begin{aligned}
c_1^2 g_{0,0} - 2c_1 g_{1,0} + g_{2,0} = {} & \sum_{a+2b+3c=n+1} (-1)^{n+1+a+b+c} \binom{a+b+c}{a}\binom{b+c}{b} c_1^{a+2} c_2^b c_3^c \\
& -2 \sum_{a+2b+3c=n+2} (-1)^{n+1+a+b+c} \binom{a+b+c-1}{a}\binom{b+c}{b} c_1^{a+1} c_2^b c_3^c \\
& + \sum_{a+2b+3c=n+3} (-1)^{n+1+a+b+c} \binom{a+b+c-2}{a}\binom{b+c}{b} c_1^a c_2^b c_3^c \\
= {} & \sum_{a+2b+3c=n+3} (-1)^{n+1+a+b+c} \binom{a+b+c-2}{a-2}\binom{b+c}{b} c_1^a c_2^b c_3^c
\end{aligned}$$

$$+\sum_{a+2b+3c=n+3}(-1)^{n+1+a+b+c}\cdot 2\cdot\binom{a+b+c-2}{a-1}\binom{b+c}{b}c_1^ac_2^bc_3^c$$
$$+\sum_{a+2b+3c=n+3}(-1)^{n+1+a+b+c}\binom{a+b+c-2}{a}\binom{b+c}{b}c_1^ac_2^bc_3^c.$$

First, we note that the change of variable $a\mapsto a-2$ in the first sum does not affect the requirement that a runs through $\mathbb{N}_0$ since for $a=0$ and $a=1$ the binomial coefficient $\binom{a+b+c-2}{a-2}$ is equal to zero. Similarly for the second sum. From formula (4.1) we deduce directly that $\binom{\alpha}{\beta}=\binom{\alpha-2}{\beta}+2\binom{\alpha-2}{\beta-1}+\binom{\alpha-2}{\beta-2}$ for all $\alpha,\beta\in\mathbb{Z}$, so we have

$$\begin{aligned}c_1^2g_{0,0}-2c_1g_{1,0}+g_{2,0}&=\sum_{a+2b+3c=n+3}(-1)^{n+1+a+b+c}\binom{a+b+c}{a}\binom{b+c}{b}c_1^ac_2^bc_3^c\\&=(-1)^{n+1}\overline{c}_{n+3}\end{aligned}$$

and the proposition is proved. □

In the subsequent calculations, the polynomials $g_{m,l}$ with $m+l=n+2$ will take part. We note that these polynomials are not necessarily elements of G, but, as Proposition 5.3 below states, they can be written as linear combinations of some elements of G.

In order to achieve this kind of presentation for $g_{m,l}$ $(m+l=n+2)$, we prove the crucial fact which is stated in the following lemma. (We recall that the integer $n\geqslant 3$ is fixed.)

Lemma 5.2. *Let m,l,a,b,c be nonnegative integers such that $m+l=n+2$ and $a+2b+3c=n+1+m+2l$. Then the following formula is true.*

$$\sum_{j=0}^{[\frac{m}{2}]}(-1)^j\binom{m-j}{j}\binom{a+b+c-n-2+j}{a}\binom{b+c-l-j}{b}=0,$$

or, singling out the summand for $j=0$,

$$\begin{aligned}&\binom{a+b+c-n-2}{a}\binom{b+c-l}{b}\\&\qquad=\sum_{j=1}^{[\frac{m}{2}]}(-1)^{j-1}\binom{m-j}{j}\binom{a+b+c-n-2+j}{a}\binom{b+c-l-j}{b}.\end{aligned}$$

Proof. We prove the lemma by induction on m. Let

$$S(m,l,a,b,c):=\sum_{j=0}^{[\frac{m}{2}]}(-1)^j\binom{m-j}{j}\binom{a+b+c-n-2+j}{a}\binom{b+c-l-j}{b}$$

The induction base will consist of three parts: $m=0$, $m=1$ and $m=2$.

Take $m=0$ and nonnegative integers l,a,b,c such that $l=n+2$ and $a+2b+3c=n+1+2l$. The statement of the lemma in this case simplifies to:

$$S(0,l,a,b,c)=\binom{a+b+c-n-2}{a}\binom{b+c-n-2}{b}=0.$$

Since $a+2b+3c = n+1+2l = 3n+5$, we have that $3c \leqslant a+2b+3c = 3n+5$, so $c \leqslant n+\frac{5}{3} < n+2$, i.e., $b+c-n-2 < b$.

If $b+c-n-2 \geqslant 0$, then $\binom{b+c-n-2}{b} = 0$ and we are done.

If $b+c-n-2 < 0$, then $a+b+c-n-2 < a$. Also, $3(a+b+c) \geqslant a+2b+3c = 3n+5$ implying $a+b+c \geqslant n+\frac{5}{3}$. But since $a+b+c$ is an integer, we actually have that $a+b+c \geqslant n+2$. So, $0 \leqslant a+b+c-n-2 < a$, and we conclude that $\binom{a+b+c-n-2}{a} = 0$.

For $m = 1$, take $l := n+1$ and $a, b, c \geqslant 0$ such that $a+2b+3c = n+1+1+2l = 3n+4$. In this case we need to prove

$$S(1,l,a,b,c) = \binom{a+b+c-n-2}{a}\binom{b+c-n-1}{b} = 0.$$

As in the case $m = 0$, we obtain that $a+b+c \geqslant n+2$ and $c \leqslant n+1$. If $c < n+1$, the proof is analogous to that of the first case. If $c = n+1$, then, since $a+2b+3c = 3n+4$, a must be 1 and b must be 0 and we obtain $S(1,l,a,b,c) = \binom{0}{1}\binom{0}{0} = 0$.

If $m = 2$, then $l = n$ and let a, b, c be nonnegative integers such that $a+2b+3c = n+1+2+2l = 3n+3$. Here, the statement of the lemma reduces to

$$\binom{a+b+c-n-2}{a}\binom{b+c-n}{b} - \binom{a+b+c-n-1}{a}\binom{b+c-n-1}{b} = 0,$$

since the left-hand side of this equality is $S(2,l,a,b,c)$. In this case, from the condition $a+2b+3c = 3n+3$ we can deduce that $a+b+c \geqslant n+1$ and $c \leqslant n+1$.

If $c = n+1$, then necessarily $a = b = 0$, and we have

$$S(2,l,a,b,c) = \binom{-1}{0}\binom{1}{0} - \binom{0}{0}\binom{0}{0} = 1-1 = 0.$$

If $a+b+c = n+1$, since $0 \leqslant c \leqslant b+c \leqslant a+b+c$ and $c+(b+c)+(a+b+c) = 3(n+1)$, we conclude that c must be $n+1$ and this case reduces to the previous one.

Suppose now that $a+b+c \geqslant n+2$ and $c \leqslant n$. If $c < n$, then by the method of the case $m = 0$ one proves that both summands must be zero. If $c = n$, then there are two possibilities for the pair (a, b) such that the condition $a+2b+3c = 3n+3$ is satisfied. First, if $a = 3$ and $b = 0$, we have

$$S(2,l,a,b,c) = \binom{1}{3}\binom{0}{0} - \binom{2}{3}\binom{-1}{0} = 0-0 = 0.$$

Finally, if $a = b = 1$, we obtain

$$S(2,l,a,b,c) = \binom{0}{1}\binom{1}{1} - \binom{1}{1}\binom{0}{1} = 0-0 = 0$$

and the basis for the induction is completed.

For the induction step take $m \geqslant 3$, nonnegative integers l, a, b, c such that $m+l = n+2$ and $a+2b+3c = n+1+m+2l$ and suppose that the statement of the lemma is true for all nonnegative integers $< m$. We need to prove that

$S(m,l,a,b,c)$ is zero. Since $\binom{m-j}{j}=\binom{m-1-j}{j}+\binom{m-1-j}{j-1}$, we have:

$$S(m,l,a,b,c)=\sum_{j=0}^{[\frac{m}{2}]}(-1)^j\binom{m-1-j}{j}\binom{a+b+c-n-2+j}{a}\binom{b+c-l-j}{b}$$
$$+\sum_{j=0}^{[\frac{m}{2}]}(-1)^j\binom{m-1-j}{j-1}\binom{a+b+c-n-2+j}{a}\binom{b+c-l-j}{b}.$$

Denoting these two sums by S_1 and S_2, respectively, we have $S(m,l,a,b,c)=S_1+S_2$. Since $\binom{b+c-l-j}{b}=\binom{b+c-l-j-1}{b}+\binom{b+c-l-j-1}{b-1}$, we obtain:

$$S_1=\sum_{j=0}^{[\frac{m}{2}]}(-1)^j\binom{m-1-j}{j}\binom{a+b+c-n-2+j}{a}\binom{b+c-l-j-1}{b}$$
$$+\sum_{j=0}^{[\frac{m}{2}]}(-1)^j\binom{m-1-j}{j}\binom{a+b+c-n-2+j}{a}\binom{b+c-l-j-1}{b-1}.$$

We now denote these two sums by S_3 and S_4 respectively and obtain $S_1=S_3+S_4$ implying $S(m,l,a,b,c)=S_2+S_3+S_4$.

First, we consider the sum S_4. If m is odd, then $[\frac{m}{2}]=[\frac{m-1}{2}]$ and if m is even, say $m=2r$ $(r\geqslant 2)$, then the first binomial coefficient in the last summand of the sum S_4 (for $j=[\frac{m}{2}]=r$) is $\binom{r-1}{r}=0$, so in either case

$$S_4=\sum_{j=0}^{[\frac{m-1}{2}]}(-1)^j\binom{m-1-j}{j}\binom{a+b+c-n-2+j}{a}\binom{b+c-l-j-1}{b-1}$$
$$=S(m-1,l+1,a,b-1,c+1)=0,$$

by the induction hypothesis if $b>0$ and if $b=0$ it is obvious that $S_4=0$.

Now, we have $S(m,l,a,b,c)=S_2+S_3$ and we consider the sum S_3. Since $\binom{m-1-j}{j}=\binom{m-2-j}{j}+\binom{m-2-j}{j-1}$, this sum can be written as:

$$S_3=\sum_{j=0}^{[\frac{m}{2}]}(-1)^j\binom{m-2-j}{j}\binom{a+b+c-n-2+j}{a}\binom{b+c-l-j-1}{b}$$
$$+\sum_{j=0}^{[\frac{m}{2}]}(-1)^j\binom{m-2-j}{j-1}\binom{a+b+c-n-2+j}{a}\binom{b+c-l-j-1}{b}.$$

As before, we denote these two sums by S_5 and S_6 respectively and we have the equality $S(m,l,a,b,c)=S_2+S_5+S_6$.

Consider the sum S_5 and its summand for $j=[\frac{m}{2}]$. The first binomial coefficient in this summand is $\binom{m-2-[m/2]}{[m/2]}$. If $m=3$, this binomial coefficient equals $\binom{0}{1}=0$. If $m\geqslant 4$, we have that $m-2-[\frac{m}{2}]\geqslant[\frac{m}{2}]-2\geqslant 0$. Also, $\frac{m}{2}-1<[\frac{m}{2}]$ implying $m-2-[\frac{m}{2}]<[\frac{m}{2}]$. We conclude that $\binom{m-2-[m/2]}{[m/2]}=0$, i.e., the summand obtained for $j = [\frac{m}{2}]$ is zero and so

$$S_5 = \sum_{j=0}^{[\frac{m}{2}]-1} (-1)^j \binom{m-2-j}{j} \binom{a+b+c-n-2+j}{a} \binom{b+c-l-j-1}{b}$$
$$= \sum_{j=0}^{[\frac{m-2}{2}]} (-1)^j \binom{m-2-j}{j} \binom{a+b+c-n-2+j}{a} \binom{b+c-l-j-1}{b}.$$

By looking at the sum S_2 one easily sees that the first summand (for $j = 0$) equals zero (since $\binom{m-1}{-1} = 0$). This means that

$$S_2 = \sum_{j=1}^{[\frac{m}{2}]} (-1)^j \binom{m-1-j}{j-1} \binom{a+b+c-n-2+j}{a} \binom{b+c-l-j}{b}$$
$$= \sum_{j=0}^{[\frac{m}{2}]-1} (-1)^{j+1} \binom{m-1-j-1}{j} \binom{a+b+c-n-2+j+1}{a} \binom{b+c-l-j-1}{b}$$
$$= -\sum_{j=0}^{[\frac{m-2}{2}]} (-1)^j \binom{m-2-j}{j} \binom{a+b+c-n-1+j}{a} \binom{b+c-l-j-1}{b}.$$

Now the sums S_2 and S_5 are similar and since $\binom{a+b+c-n-2+j}{a} - \binom{a+b+c-n-1+j}{a} = -\binom{a+b+c-n-2+j}{a-1}$, we have that

$$S_5 + S_2 = -\sum_{j=0}^{[\frac{m-2}{2}]} (-1)^j \binom{m-2-j}{j} \binom{a+b+c-n-2+j}{a-1} \binom{b+c-l-j-1}{b}$$
$$= -S(m-2, l+2, a-1, b, c+1) = 0.$$

Again, we note that the upper sum is zero if $a = 0$ and if $a > 0$ we apply the induction hypothesis and obtain the latter equality.

We have reduced the sum $S(m, l, a, b, c)$ to S_6. Finally, by considering the sum S_6 we see that the summand for $j = 0$ is zero and so

$$S_6 = \sum_{j=1}^{[\frac{m}{2}]} (-1)^j \binom{m-2-j}{j-1} \binom{a+b+c-n-2+j}{a} \binom{b+c-l-j-1}{b}$$
$$= -\sum_{j=0}^{[\frac{m-2}{2}]} (-1)^j \binom{m-3-j}{j} \binom{a+b+c-n-1+j}{a} \binom{b+c-l-j-2}{b}.$$

If $m-2$ is odd, then $[\frac{m-2}{2}] = [\frac{m-3}{2}]$. If $m-2$ is even, then $[\frac{m-2}{2}] = [\frac{m-3}{2}] + 1$, but, as in the case of the sum S_4, for $m-2 = 2r$ ($r \geqslant 1$ since $m \geqslant 3$) the first binomial coefficient in the summand obtained for $j = [\frac{m-2}{2}] = r$ equals $\binom{r-1}{r} = 0$. We conclude that S_6 is equal to the sum

$$- \sum_{j=0}^{[\frac{m-3}{2}]} (-1)^j \binom{m-3-j}{j} \binom{a+b+c-n-1+j}{a} \binom{b+c-l-j-2}{b}$$

$$= -S(m-3, l+3, a, b, c+1) = 0,$$

by the induction hypothesis. Hence, $S(m, l, a, b, c) = 0$ and the proof of the Lemma 5.2 is completed. ☐

PROPOSITION 5.3. *Let $m, l \in \mathbb{N}_0$ such that $m + l = n + 2$. Then*

$$g_{m,l} = \sum_{j=1}^{[\frac{m}{2}]} (-1)^{j-1} \binom{m-j}{j} g_{m-2j,l+j}.$$

PROOF. In a simplified notation, the product $\binom{a+b+c-n-2+j}{a}\binom{b+c-l-j}{b}$ will be denoted by $\lambda_j(a, b, c)$. Now, the previous lemma asserts that $\lambda_0(a, b, c) = \sum_{j=1}^{[\frac{m}{2}]} (-1)^{j-1}\binom{m-j}{j}\lambda_j(a, b, c)$ if $a, b, c \geqslant 0$ are such that $a+2b+3c = n+1+m+2l$.

Using the assumption $m + l = n + 2$ and Lemma 5.2, we have

$$\begin{aligned}
g_{m,l} &= \sum_{a+2b+3c=n+1+m+2l} (-1)^{n+1+a+b+c} \binom{a+b+c-m-l}{a} \binom{b+c-l}{b} c_1^a c_2^b c_3^c \\
&= \sum_{a+2b+3c=n+1+m+2l} (-1)^{n+1+a+b+c} \lambda_0(a, b, c) c_1^a c_2^b c_3^c \\
&= \sum_{a+2b+3c=n+1+m+2l} (-1)^{n+1+a+b+c} \sum_{j=1}^{[\frac{m}{2}]} (-1)^{j-1} \binom{m-j}{j} \lambda_j(a, b, c) c_1^a c_2^b c_3^c \\
&= \sum_{j=1}^{[\frac{m}{2}]} (-1)^{j-1} \binom{m-j}{j} \sum_{a+2b+3c=n+1+m+2l} (-1)^{n+1+a+b+c} \lambda_j(a, b, c) c_1^a c_2^b c_3^c.
\end{aligned}$$

It remains to prove that $\sum_{a+2b+3c=n+1+m+2l} (-1)^{n+1+a+b+c} \lambda_j(a, b, c) c_1^a c_2^b c_3^c$ is equal to $g_{m-2j,l+j}$. But,

$$\begin{aligned}
\lambda_j(a, b, c) &= \binom{a+b+c-n-2+j}{a} \binom{b+c-l-j}{b} \\
&= \binom{a+b+c-(m-2j)-(l+j)}{a} \binom{b+c-(l+j)}{b}
\end{aligned}$$

and the proposition follows by Definition 5.1. ☐

In the following three propositions (5.4, 5.5 and 5.6) we give some convenient presentations for S-polynomials of elements of G. For the first one we need a few lemmas.

LEMMA 5.3. *For any integers $\alpha, \beta, \gamma, \delta$ we have*

$$\binom{\alpha}{\beta}\binom{\gamma-1}{\delta-1} - \binom{\alpha-1}{\beta-1}\binom{\gamma}{\delta} = \binom{\alpha-1}{\beta}\binom{\gamma}{\delta} - \binom{\alpha}{\beta}\binom{\gamma-1}{\delta}.$$

Proof. We calculate

$$\binom{\alpha}{\beta}\binom{\gamma-1}{\delta-1}-\binom{\alpha-1}{\beta-1}\binom{\gamma}{\delta}=\binom{\alpha}{\beta}\binom{\gamma-1}{\delta-1}-\binom{\alpha}{\beta}\binom{\gamma}{\delta}+\binom{\alpha}{\beta}\binom{\gamma}{\delta}$$
$$-\binom{\alpha-1}{\beta-1}\binom{\gamma}{\delta}=-\binom{\alpha}{\beta}\binom{\gamma-1}{\delta}+\binom{\alpha-1}{\beta}\binom{\gamma}{\delta},$$

by formula 4.1. □

Lemma 5.4. *Let $m,l\in\mathbb{N}_0$, $r\in\mathbb{N}$ and $m+l<m+r+l\leqslant n+1$. Then*

$$S(g_{m,l},g_{m+r,l})=\sum_{i=0}^{r-1}c_1^ic_2^{r-1-i}(g_{m+i,l+1}-g_{m+2+i,l}).$$

Proof. First, we observe that, according to Proposition 5.1, $\mathrm{LM}(g_{m,l})=c_1^{n+1-m-l}c_2^mc_3^l$ and $\mathrm{LM}(g_{m+r,l})=c_1^{n+1-m-r-l}c_2^{m+r}c_3^l$, so we have

$$\mathrm{lcm}(\mathrm{LM}(g_{m,l}),\mathrm{LM}(g_{m+r,l}))=c_1^{n+1-m-l}c_2^{m+r}c_3^l,$$

and since $\mathrm{LC}(g_{m,l})=\mathrm{LC}(g_{m+r,l})=1$ (Proposition 5.1), we obtain that

$$S(g_{m,l},g_{m+r,l})=c_2^rg_{m,l}-c_1^rg_{m+r,l}.$$

We prove the lemma by induction on r. For $r=1$, we need to verify the equality $S(g_{m,l},g_{m+1,l})=g_{m,l+1}-g_{m+2,l}$. We have

$$\begin{aligned}S(g_{m,l},g_{m+1,l})&=c_2g_{m,l}-c_1g_{m+1,l}\\
&=\sum_{a+2b+3c=n+1+m+2l}(-1)^{n+1+a+b+c}\binom{a+b+c-m-l}{a}\binom{b+c-l}{b}c_1^ac_2^{b+1}c_3^c\\
&-\sum_{a+2b+3c=n+1+m+1+2l}(-1)^{n+1+a+b+c}\binom{a+b+c-m-1-l}{a}\binom{b+c-l}{b}c_1^{a+1}c_2^bc_3^c\\
&=\sum_{a+2b+3c=n+m+2l+3}(-1)^{n+a+b+c}\binom{a+b+c-m-l-1}{a}\binom{b+c-l-1}{b-1}c_1^ac_2^bc_3^c\\
&-\sum_{a+2b+3c=n+m+2l+3}(-1)^{n+a+b+c}\binom{a+b+c-m-l-2}{a-1}\binom{b+c-l}{b}c_1^ac_2^bc_3^c.\end{aligned}$$

By the previous lemma

$$\begin{aligned}&\binom{a+b+c-m-l-1}{a}\binom{b+c-l-1}{b-1}-\binom{a+b+c-m-l-2}{a-1}\binom{b+c-l}{b}\\
&=\binom{a+b+c-m-l-2}{a}\binom{b+c-l}{b}-\binom{a+b+c-m-l-1}{a}\binom{b+c-l-1}{b}\end{aligned}$$

and we obtain:

$$\begin{aligned}&S(g_{m,l},g_{m+1,l})\\
&=\sum_{a+2b+3c=n+m+2l+3}(-1)^{n+a+b+c}\binom{a+b+c-m-l-2}{a}\binom{b+c-l}{b}c_1^ac_2^bc_3^c\\
&-\sum_{a+2b+3c=n+m+2l+3}(-1)^{n+a+b+c}\binom{a+b+c-m-l-1}{a}\binom{b+c-l-1}{b}c_1^ac_2^bc_3^c\end{aligned}$$

$$= -(g_{m+2,l} - g_{m,l+1}) = g_{m,l+1} - g_{m+2,l}.$$

For the induction step we take $r \geqslant 2$ and calculate:

$$\begin{aligned}
S(g_{m,l}, g_{m+r,l}) &= c_2^r g_{m,l} - c_1^r g_{m+r,l} \\
&= c_2^r g_{m,l} - c_1^{r-1} c_2 g_{m+r-1,l} + c_1^{r-1} c_2 g_{m+r-1,l} - c_1^r g_{m+r,l} \\
&= c_2 S(g_{m,l}, g_{m+r-1,l}) + c_1^{r-1} S(g_{m+r-1,l}, g_{m+r,l}) \\
&= c_2 \sum_{i=0}^{r-2} c_1^i c_2^{r-2-i} (g_{m+i,l+1} - g_{m+2+i,l}) + c_1^{r-1}(g_{m+r-1,l+1} - g_{m+r+1,l}) \\
&= \sum_{i=0}^{r-1} c_1^i c_2^{r-1-i} (g_{m+i,l+1} - g_{m+2+i,l}),
\end{aligned}$$

by the induction hypothesis. □

LEMMA 5.5. *Let $m, l \in \mathbb{N}_0$, $s \in \mathbb{N}$ and $m + l < m + l + s \leqslant n + 1$. Then*

$$S(g_{m,l}, g_{m,l+s}) = -\sum_{j=0}^{s-1} c_1^j c_3^{s-1-j} g_{m+1,l+1+j}.$$

PROOF. Again using Proposition 5.1, we obtain

$$S(g_{m,l}, g_{m,l+s}) = c_3^s g_{m,l} - c_1^s g_{m,l+s}.$$

We proceed by induction on s. For $s = 1$, the statement of the lemma reduces to $S(g_{m,l}, g_{m,l+1}) = -g_{m+1,l+1}$. We have

$$\begin{aligned}
&S(g_{m,l}, g_{m,l+1}) = c_3 g_{m,l} - c_1 g_{m,l+1} \\
&= \sum_{a+2b+3c=n+1+m+2l} (-1)^{n+1+a+b+c} \binom{a+b+c-m-l}{a} \binom{b+c-l}{b} c_1^a c_2^b c_3^{c+1} \\
&- \sum_{a+2b+3c=n+m+2l+3} (-1)^{n+1+a+b+c} \binom{a+b+c-m-l-1}{a} \binom{b+c-l-1}{b} c_1^{a+1} c_2^b c_3^c \\
&= \sum_{a+2b+3c=n+m+2l+4} (-1)^{n+a+b+c} \binom{a+b+c-m-l-1}{a} \binom{b+c-l-1}{b} c_1^a c_2^b c_3^c \\
&- \sum_{a+2b+3c=n+m+2l+4} (-1)^{n+a+b+c} \binom{a+b+c-m-l-2}{a-1} \binom{b+c-l-1}{b} c_1^a c_2^b c_3^c.
\end{aligned}$$

As in some previous proofs, the change of variable $a \mapsto a-1$ in the second sum does not affect the requirement that a runs through $\mathbb{N}_0$ since $\binom{a+b+c-m-l-2}{a-1} = 0$ for $a = 0$. In the first sum, the change of variable $c \mapsto c-1$ was made and we need to prove that for $c = 0$, the coefficient $\binom{a+b-m-l-1}{a}\binom{b-l-1}{b} = 0$, provided that $a + 2b = n + m + 2l + 4$.

If $b \geqslant l+1$, then $0 \leqslant b-l-1 < b$ and the second factor equals zero. If $b \leqslant l$, then $a+b-m-l-1 \leqslant a-m-1 < a$, so $\binom{a+b-m-l-1}{a} \neq 0$ only if $a+b-m-l-1 < 0$, i.e., $a+b < m+l+1$. But then $a+2b = a+b+b < m+l+1+l = m+2l+1 < n+m+2l+4$ contradicting the fact that $a+2b = n+m+2l+4$. Hence, $\binom{a+b-m-l-1}{a}\binom{b-l-1}{b} = 0$.

Finally, since $\binom{a+b+c-m-l-1}{a} - \binom{a+b+c-m-l-2}{a-1} = \binom{a+b+c-m-l-2}{a}$, we get

$$\begin{aligned} &S(g_{m,l}, g_{m,l+1}) \\ &= \sum_{a+2b+3c=n+m+2l+4} (-1)^{n+a+b+c} \binom{a+b+c-m-l-2}{a} \binom{b+c-l-1}{b} c_1^a c_2^b c_3^c \\ &\qquad = -g_{m+1,l+1} \end{aligned}$$

and the induction base is completed.

Passing to the induction step, for $s \geqslant 2$ we have

$$\begin{aligned} S(g_{m,l}, g_{m,l+s}) &= c_3^s g_{m,l} - c_1^s g_{m,l+s} \\ &= c_3^s g_{m,l} - c_1^{s-1} c_3 g_{m,l+s-1} + c_1^{s-1} c_3 g_{m,l+s-1} - c_1^s g_{m,l+s} \\ &= c_3 S(g_{m,l}, g_{m,l+s-1}) + c_1^{s-1} S(g_{m,l+s-1}, g_{m,l+s}) \\ &= -c_3 \sum_{j=0}^{s-2} c_1^j c_3^{s-2-j} g_{m+1,l+1+j} - c_1^{s-1} g_{m+1,l+s} = -\sum_{j=0}^{s-1} c_1^j c_3^{s-1-j} g_{m+1,l+1+j} \end{aligned}$$

and we are done. □

Note that lemmas 5.4 and 5.5 hold also when $r = 0$ ($s = 0$) since by definition $S(f,f) = 0$ and the sums on the right-hand side of the equalities are empty.

Now we generalize both Lemma 5.4 and Lemma 5.5.

PROPOSITION 5.4. *Let $m, l, r, s \in \mathbb{N}_0$ such that $m + l < m + l + r + s \leqslant n + 1$. Then*

$$\begin{aligned} &S(g_{m,l}, g_{m+r,l+s}) \\ &\quad = \sum_{i=0}^{r-1} c_1^{s+i} c_2^{r-1-i} (g_{m+i,l+s+1} - g_{m+2+i,l+s}) - \sum_{j=0}^{s-1} c_1^j c_2^r c_3^{s-1-j} g_{m+1,l+1+j}. \end{aligned}$$

PROOF. Using Proposition 5.1, we easily obtain that

$$\operatorname{lcm}(\operatorname{LM}(g_{m,l}), \operatorname{LM}(g_{m+r,l+s})) = c_1^{n+1-m-l} c_2^{m+r} c_3^{l+s},$$

and so

$$S(g_{m,l}, g_{m+r,l+s}) = c_2^r c_3^s g_{m,l} - c_1^{r+s} g_{m+r,l+s}.$$

Moving on, we have

$$\begin{aligned} S(g_{m,l}, g_{m+r,l+s}) &= c_2^r c_3^s g_{m,l} - c_2^r c_1^s g_{m,l+s} + c_2^r c_1^s g_{m,l+s} - c_1^{r+s} g_{m+r,l+s} \\ &= c_2^r S(g_{m,l}, g_{m,l+s}) + c_1^s S(g_{m,l+s}, g_{m+r,l+s}) \\ &= -\sum_{j=0}^{s-1} c_1^j c_2^r c_3^{s-1-j} g_{m+1,l+1+j} + \sum_{i=0}^{r-1} c_1^{s+i} c_2^{r-1-i} (g_{m+i,l+s+1} - g_{m+2+i,l+s}), \end{aligned}$$

by lemmas 5.4 and 5.5. □

LEMMA 5.6. *Let $m, l \in \mathbb{N}_0$, $s \in \mathbb{N}$, $m \geqslant s$ and $m+l \leqslant n+1$. Then*

$$S(g_{m,l}, g_{m-s,l+s}) = -\sum_{j=0}^{s-1} c_2^j c_3^{s-1-j} g_{m-1-j,l+2+j}.$$

PROOF. It is easy to see that by definition

$$S(g_{m,l}, g_{m-s,l+s}) = c_3^s g_{m,l} - c_2^s g_{m-s,l+s}.$$

The proof is by induction on s. For the induction base, we want to show that $S(g_{m,l}, g_{m-1,l+1}) = -g_{m-1,l+2}$. We have

$$\begin{aligned}
&S(g_{m,l}, g_{m-1,l+1}) = c_3 g_{m,l} - c_2 g_{m-1,l+1}\\
&= \sum_{a+2b+3c=n+1+m+2l} (-1)^{n+1+a+b+c} \binom{a+b+c-m-l}{a} \binom{b+c-l}{b} c_1^a c_2^b c_3^{c+1}\\
&- \sum_{a+2b+3c=n+m+2l+2} (-1)^{n+1+a+b+c} \binom{a+b+c-m-l}{a} \binom{b+c-l-1}{b} c_1^a c_2^{b+1} c_3^c\\
&= \sum_{a+2b+3c=n+m+2l+4} (-1)^{n+a+b+c} \binom{a+b+c-m-l-1}{a} \binom{b+c-l-1}{b} c_1^a c_2^b c_3^c\\
&- \sum_{a+2b+3c=n+m+2l+4} (-1)^{n+a+b+c} \binom{a+b+c-m-l-1}{a} \binom{b+c-l-2}{b-1} c_1^a c_2^b c_3^c.
\end{aligned}$$

For the same reasons as in the proof of Lemma 5.5, we may assume that a, b and c still run through $\mathbb{N}_0$. Finally, since $\binom{b+c-l-1}{b} - \binom{b+c-l-2}{b-1} = \binom{b+c-l-2}{b}$, we obtain

$$\begin{aligned}
&S(g_{m,l}, g_{m-1,l+1})\\
&= \sum_{a+2b+3c=n+m+2l+4} (-1)^{n+a+b+c} \binom{a+b+c-m-l-1}{a} \binom{b+c-l-2}{b} c_1^a c_2^b c_3^c\\
&= -g_{m-1,l+2}.
\end{aligned}$$

Now let $s \geqslant 2$ and if the lemma holds for all positive integers $< s$, then

$$\begin{aligned}
S(g_{m,l}, g_{m-s,l+s}) &= c_3^s g_{m,l} - c_2^s g_{m-s,l+s}\\
&= c_3^s g_{m,l} - c_2^{s-1} c_3 g_{m-s+1,l+s-1} + c_2^{s-1} c_3 g_{m-s+1,l+s-1} - c_2^s g_{m-s,l+s}\\
&= c_3 S(g_{m,l}, g_{m-s+1,l+s-1}) + c_2^{s-1} S(g_{m-s+1,l+s-1}, g_{m-s,l+s})\\
&= -c_3 \sum_{j=0}^{s-2} c_2^j c_3^{s-2-j} g_{m-1-j,l+2+j} - c_2^{s-1} g_{m-s,l+s+1} = -\sum_{j=0}^{s-1} c_2^j c_3^{s-1-j} g_{m-1-j,l+2+j}
\end{aligned}$$

and the lemma follows. ☐

We finally use Lemma 5.6 to obtain two additional propositions concerning S-polynomials of the elements of G.

PROPOSITION 5.5. *Let $m, l, \in \mathbb{N}_0$, $r, s \in \mathbb{N}$, $l \geqslant s$, $r \geqslant s$ and $m+r+l-s \leqslant n+1$. Then*

$$S(g_{m,l}, g_{m+r,l-s}) = \sum_{i=0}^{r-s-1} c_1^i c_2^{r-1-i}(g_{m+i,l+1} - g_{m+2+i,l}) - \sum_{j=0}^{s-1} c_1^{r-s} c_2^j c_3^{s-1-j} g_{m+r-1-j,l-s+2+j}.$$

PROOF. By Proposition 5.1,

$$\mathrm{lcm}(\mathrm{LM}(g_{m,l}), \mathrm{LM}(g_{m+r,l-s})) = c_1^{n+1-m-l} c_2^{m+r} c_3^l,$$

implying

$$\begin{aligned} S(g_{m,l}, g_{m+r,l-s}) &= c_2^r g_{m,l} - c_1^{r-s} c_3^s g_{m+r,l-s} \\ &= c_2^r g_{m,l} - c_1^{r-s} c_2^s g_{m+r-s,l} + c_1^{r-s} c_2^s g_{m+r-s,l} - c_1^{r-s} c_3^s g_{m+r,l-s} \\ &= c_2^s S(g_{m,l}, g_{m+r-s,l}) + c_1^{r-s} S(g_{m+r,l-s}, g_{m+r-s,l}) \\ &= \sum_{i=0}^{r-s-1} c_1^i c_2^{r-1-i}(g_{m+i,l+1} - g_{m+2+i,l}) - \sum_{j=0}^{s-1} c_1^{r-s} c_2^j c_3^{s-1-j} g_{m+r-1-j,l-s+2+j}, \end{aligned}$$

by lemmas 5.4 and 5.6. □

PROPOSITION 5.6. *Let* $m, l, \in \mathbb{N}_0$, $r, s \in \mathbb{N}$, $l \geqslant s$, $r < s$ *and* $m + l \leqslant n + 1$. *Then*

$$S(g_{m,l}, g_{m+r,l-s}) = -\sum_{i=0}^{s-r-1} c_1^i c_2^r c_3^{s-r-1-i} g_{m+1,l-s+r+1+i} - \sum_{j=0}^{r-1} c_2^j c_3^{s-1-j} g_{m+r-1-j,l-s+2+j}.$$

PROOF. In this case ($r < s$) Proposition 5.1 tells us that

$$\mathrm{lcm}(\mathrm{LM}(g_{m,l}), \mathrm{LM}(g_{m+r,l-s})) = c_1^{n+1-m-l+s-r} c_2^{m+r} c_3^l,$$

and we conclude

$$\begin{aligned} S(g_{m,l}, g_{m+r,l-s}) &= c_1^{s-r} c_2^r g_{m,l} - c_3^s g_{m+r,l-s} \\ &= c_1^{s-r} c_2^r g_{m,l} - c_2^r c_3^{s-r} g_{m,l-s+r} + c_2^r c_3^{s-r} g_{m,l-s+r} - c_3^s g_{m+r,l-s} \\ &= c_2^r S(g_{m,l-s+r}, g_{m,l}) + c_3^{s-r} S(g_{m+r,l-s}, g_{m,l-s+r}) \\ &= -\sum_{i=0}^{s-r-1} c_1^i c_2^r c_3^{s-r-1-i} g_{m+1,l-s+r+1+i} - \sum_{j=0}^{r-1} c_2^j c_3^{s-1-j} g_{m+r-1-j,l-s+2+j}, \end{aligned}$$

by lemmas 5.5 and 5.6. □

Observe that in the previous three propositions the S-polynomials of the elements of G are presented as some functions of polynomials $g_{m,l}$, where $m+l \leqslant n+2$. Those for which $m+l \leqslant n+1$ are the elements of G and those for which $m+l = n+2$ are either zero (for $m = 0$ and $m = 1$) or some linear combination of elements of G according to Proposition 5.3.

In order to prove that G is a basis for the ideal $I_{3,n}$, i.e., $I_G = I_{3,n}$, we list the following equalities:

$$g_{m+2,l} = g_{m,l+1} - c_2 g_{m,l} + c_1 g_{m+1,l}, \tag{5.1}$$

$$g_{m+1,l+1} = c_1 g_{m,l+1} - c_3 g_{m,l}, \tag{5.2}$$

$$g_{m-1,l+2} = c_2 g_{m-1,l+1} - c_3 g_{m,l}, \tag{5.3}$$

which are obtained in the proofs of lemmas 5.4, 5.5 and 5.6 respectively as the induction bases.

Proposition 5.7. $I_G = I_{3,n}$.

Proof. According to Proposition 5.2, $I_{3,n} \subseteq I_G$, so it remains to prove that $g \in I_{3,n}$ for all $g \in G$, i.e., $g_{m,l} \in I_{3,n}$ for all $m, l \in \mathbb{N}_0$ such that $m + l \leqslant n + 1$. The proof is by induction on $m + l$. We already have that $g_{0,0} = (-1)^{n+1}\overline{c}_{n+1} \in I_{3,n}$. Also, in the proof of Proposition 5.2 we established that

$$g_{1,0} = c_1 g_{0,0} + (-1)^{n+1}\overline{c}_{n+2} = (-1)^{n+1} c_1 \overline{c}_{n+1} + (-1)^{n+1}\overline{c}_{n+2} \in I_{3,n}$$

and that $g_{2,0} = -c_1^2 g_{0,0} + 2c_1 g_{1,0} + (-1)^{n+1}\overline{c}_{n+3} \in I_{3,n}$. By formula (5.1), $g_{2,0} = g_{0,1} - c_2 g_{0,0} + c_1 g_{1,0}$ and so

$$g_{0,1} = g_{2,0} + c_2 g_{0,0} - c_1 g_{1,0} \in I_{3,n}.$$

Therefore, $g_{m,l} \in I_{3,n}$ if $m + l \leqslant 1$.

Now, take $g_{m,l} \in G$ such that $m + l \geqslant 2$ and assume that $g_{\widetilde{m},\widetilde{l}} \in I_{3,n}$ if $\widetilde{m} + \widetilde{l} < m + l$. If $l = 0$, then $m \geqslant 2$ and by formula (5.1) we have

$$g_{m,0} = g_{m-2,1} - c_2 g_{m-2,0} + c_1 g_{m-1,0} \in I_{3,n}.$$

If $l = 1$, formula (5.2) gives us

$$g_{m,1} = c_1 g_{m-1,1} - c_3 g_{m-1,0} \in I_{3,n}.$$

Finally, if $l \geqslant 2$, we use formula (5.3) and obtain

$$g_{m,l} = c_2 g_{m,l-1} - c_3 g_{m+1,l-2} \in I_{3,n},$$

by the induction hypothesis. □

We are left to prove that G is a strong Gröbner basis for $I_{3,n}$. We are going to do that by showing that G satisfies sufficient conditions for being a strong Gröbner basis stated in Theorem 2.2.

Theorem 5.1. *Let $n \geqslant 3$. The set G (see definitions 5.1 and 5.2) is a strong Gröbner basis for the ideal $I_{3,n}$ in $\mathbb{Z}[c_1, c_2, c_3]$ with respect to the grlex ordering $\preceq$.*

Proof. In order to apply Theorem 2.2, we first accord to Proposition 5.1 for the fact that $\mathrm{LC}(g) = 1$ for all $g \in G$. Then, we take two arbitrary elements of G, say $g_{m,l}$ and $g_{\widetilde{m},\widetilde{l}}$ $(g_{m,l} \neq g_{\widetilde{m},\widetilde{l}})$. Since S-polynomials are antisymmetric, without loss of generality we may assume that either (a) $m < \widetilde{m}$ or else (b) $m = \widetilde{m}$ and $l < \widetilde{l}$. We distinguish three cases.

1° If condition (b) holds or if $m < \widetilde{m}$ and $l \leqslant \widetilde{l}$, writing $\widetilde{m} = m + r$, $\widetilde{l} = l + s$, $r, s \in \mathbb{N}_0$, we have $m + l < m + l + r + s \leqslant n + 1$, so the conditions of Proposition 5.4 are satisfied implying

$$\begin{aligned} S(g_{m,l}, g_{\widetilde{m},\widetilde{l}}) &= S(g_{m,l}, g_{m+r,l+s}) \\ &= \sum_{i=0}^{r-1} c_1^{s+i} c_2^{r-1-i} (g_{m+i,l+s+1} - g_{m+2+i,l+s}) - \sum_{j=0}^{s-1} c_1^j c_2^r c_3^{s-1-j} g_{m+1,l+1+j}. \end{aligned}$$

If $m + l + r + s < n + 1$, then all polynomials $g_{m,l}$ appearing in this expression are elements of G. If $m + l + r + s = n + 1$, then $g_{m+r+1,l+s}$ and eventually $g_{m+1,l+s}$ (if $r = 0$) are not in G. But, according to Proposition 5.3, these two can be written as linear combinations of the elements of G and henceforth we consider these polynomials as the appropriate linear combinations.

By Proposition 5.1 the leading monomials of the elements of G all have the sum of the exponents equal to $n+1$. Therefore, the leading monomials of the summands in the first sum all have the sum of the exponents $s+i+r-1-i+n+1 = n+r+s$ and in the second $j+r+s-1-j+n+1 = n+r+s$ too. We define $t = t(m, l, \widetilde{m}, \widetilde{l})$ to be the maximum (with respect to $\preceq$) of all these leading monomials. Hence, the above expression is a t-representation of $S(g_{m,l}, g_{\widetilde{m},\widetilde{l}})$ w.r.t. G, t has the sum of the exponents equal to $n + r + s$ and so

$$t \prec c_1^{n+1-m-l} c_2^{m+r} c_3^{l+s} = \mathrm{lcm}(\mathrm{LM}(g_{m,l}), \mathrm{LM}(g_{\widetilde{m},\widetilde{l}})).$$

2° If $m < \widetilde{m}$, $l > \widetilde{l}$ and $\widetilde{m} - m \geqslant l - \widetilde{l}$, writing $\widetilde{m} = m + r$, $\widetilde{l} = l - s$, $r, s \in \mathbb{N}$, we have $l \geqslant s$, $r \geqslant s$ and $m + r + l - s \leqslant n + 1$, i.e., the conditions of Proposition 5.5 are satisfied and by that proposition

$$\begin{aligned} S(g_{m,l}, g_{\widetilde{m},\widetilde{l}}) &= S(g_{m,l}, g_{m+r,l-s}) \\ &= \sum_{i=0}^{r-s-1} c_1^i c_2^{r-1-i} (g_{m+i,l+1} - g_{m+2+i,l}) - \sum_{j=0}^{s-1} c_1^{r-s} c_2^j c_3^{s-1-j} g_{m+r-1-j,l-s+2+j}. \end{aligned}$$

As in the previous case, for $m + r + l - s = n + 1$ the polynomials $g_{m+r-s+1,l}$ and $g_{m+r-1-j,l-s+2+j}$ $(j = \overline{0, s-1})$ are treated as linear combinations of elements of G (obtained in Proposition 5.3).

Again, we define t to be the maximum of all leading monomials in this expression and so we have a t-representation of $S(g_{m,l}, g_{\widetilde{m},\widetilde{l}})$ w.r.t. G. Since the sum of the exponents in the leading monomials is equal to $i + r - 1 - i + n + 1 = n + r$, i.e., $r - s + j + s - 1 - j + n + 1 = n + r$, we have

$$t \prec c_1^{n+1-m-l} c_2^{m+r} c_3^{l} = \mathrm{lcm}(\mathrm{LM}(g_{m,l}), \mathrm{LM}(g_{\widetilde{m},\widetilde{l}})).$$

3° Finally, if $m < \widetilde{m}$, $l > \widetilde{l}$ and $\widetilde{m} - m < l - \widetilde{l}$, again we put $\widetilde{m} = m + r$, $\widetilde{l} = l - s$ $(r, s \in \mathbb{N})$. In this case, $l \geqslant s$, $r < s$ and $m + l \leqslant n + 1$, hence we may apply Proposition 5.6 and obtain

$$S(g_{m,l}, g_{\widetilde{m},\widetilde{l}}) = S(g_{m,l}, g_{m+r,l-s})$$

$$= -\sum_{i=0}^{s-r-1} c_1^i c_2^r c_3^{s-r-1-i} g_{m+1,l-s+r+1+i} - \sum_{j=0}^{r-1} c_2^j c_3^{s-1-j} g_{m+r-1-j,l-s+2+j}.$$

Considering this case as the previous two, we observe that the sum of the exponents in the leading monomials is $i+r+s-r-1-i+n+1 = n+s$, i.e., $j+s-1-j+n+1 = n+s$. Defining t as before, we have

$$t \prec c_1^{n+1-m-l+s-r} c_2^{m+r} c_3^l = \operatorname{lcm}(\mathrm{LM}(g_{m,l}), \mathrm{LM}(g_{\widetilde{m},\widetilde{l}})).$$

Therefore, by Theorem 2.2 we conclude that G is a strong Gröbner basis. □

Since $\mathrm{LM}(g_{m,l}) = \mathrm{LT}(g_{m,l}) = c_1^{n+1-m-l} c_2^m c_3^l$ ($m, l \in \mathbb{N}_0$, $m+l \leqslant n+1$), we see that the set of all leading monomials in G is the set of all monomials with the sum of the exponents equal to $n+1$. Therefore, a monomial $c_1^a c_2^b c_3^c \in \mathbb{Z}[c_1, c_2, c_3]$ is not divisible by any of these leading monomials if and only if $a+b+c \leqslant n$. Now, the proof of the following corollary is completely analogous to that of Corollary 4.2.

Corollary 5.1. *Let $n \geqslant 3$. If c_i is the i-th Chern class of the canonical complex vector bundle γ_3 over $G_{3,n}$, then the set $\{c_1^a c_2^b c_3^c \mid a+b+c \leqslant n\}$ is a basis for the free abelian group $H^*(G_{3,n}; \mathbb{Z})$.*

As in the case $k = 2$, one can verify that the strong Gröbner basis G from Theorem 5.1 is minimal, reduced and produces unique normal forms.

Let us now calculate a few elements of the strong Gröbner basis G. By Proposition 5.1, excluding the leading monomial $\mathrm{LM}(g_{m,l}) = c_1^{n+1-m-l} c_2^m c_3^l$, the monomial $c_1^a c_2^b c_3^c$ appears in $g_{m,l}$ with nonzero coefficient only if $a+b+c < n+1$, so then we have $c \leqslant b+c \leqslant a+b+c \leqslant n$ and we conclude that $a+2b+3c \leqslant 3n$. Since $a+2b+3c$ must be equal to $n+1+m+2l$, we see that if $n+1+m+2l > 3n$ (i.e., $m+2l > 2n-1$) then $g_{m,l} = \mathrm{LT}(g_{m,l}) = \mathrm{LM}(g_{m,l}) = c_1^{n+1-m-l} c_2^m c_3^l$. In particular, we have the equalities:

$$g_{0,n+1} = c_3^{n+1}; \qquad g_{0,n} = c_1 c_3^n; \qquad g_{1,n} = c_2 c_3^n.$$

Starting from these three, we can calculate the polynomials $g_{m,n-1}$, $g_{m,n-2}$, $g_{m,n-3}$ etc. in the following way. From formula (5.2), we have $c_3 g_{0,n-1} = c_1 g_{0,n} - g_{1,n} = c_1^2 c_3^n - c_2 c_3^n$, so

$$g_{0,n-1} = c_1^2 c_3^{n-1} - c_2 c_3^{n-1}.$$

Using formula (5.3), one obtains $c_3 g_{1,n-1} = c_2 g_{0,n} - g_{0,n+1} = c_1 c_2 c_3^n - c_3^{n+1}$, implying:

$$g_{1,n-1} = c_1 c_2 c_3^{n-1} - c_3^n.$$

Applying formula (5.1), we have

$$\begin{aligned} g_{2,n-1} &= g_{0,n} - c_2 g_{0,n-1} + c_1 g_{1,n-1} \\ &= c_1 c_3^n - c_1^2 c_2 c_3^{n-1} + c_2^2 c_3^{n-1} + c_1^2 c_2 c_3^{n-1} - c_1 c_3^n = c_2^2 c_3^{n-1}. \end{aligned}$$

Obviously, continuing in the same manner, one can compute $g_{m,l} \in G$ when l is close to n.

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Faculty of Mathematics
University of Belgrade
Belgrade
Serbia
bane@matf.bg.ac.rs

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