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# ON THE SELBERG INTEGRAL OF THE $k$-DIVISOR FUNCTION AND THE $2k$-TH MOMENT OF THE RIEMANN ZETA-FUNCTION

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Abstract. In the literature one can find links between the $2k$-th moment of the Riemann zeta-function and averages involving $d_k(n)$, the divisor function generated by $\zeta^k(s)$. There are, in fact, two bounds: one for the $2k$-th moment of $\zeta(s)$ coming from a simple average of correlations of the $d_k$; and the other, which is a more recent approach, for the Selberg integral involving $d_k(n)$, applying known bounds for the $2k$-th moment of the zeta-function. Building on the former work, we apply an elementary approach (based on arithmetic averages) in order to get the reverse link to the second work; i.e., we obtain (conditional) bounds for the $2k$-th moment of the zeta-function from the Selberg integral bounds involving $d_k(n)$.

## 1. Introduction and statement of the result

We shall link the $2k$-th moment of the Riemann $\zeta$-function $\zeta(s)$ on the (critical) line ($\sigma = \operatorname{Re} s = \frac{1}{2}$), see [**10**]:

$$I_k(T) \stackrel{\text{def}}{=} \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt$$

(which we shall abbreviate with $I_k$, not to be confused with the similar $2k$-th moment off the line, i.e.,

$$I_k(\sigma,T) \stackrel{\text{def}}{=} \int_0^T |\zeta(\sigma+it)|^{2k} dt \qquad (\tfrac{1}{2}<\sigma<1),$$

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compare with [**10**]), with the Selberg integral of the $k$-divisor function, $d_k(n)$ (having the generating Dirichlet series $\zeta^k(s)$)

$$J_k(x,h) \stackrel{\mathrm{def}}{=} \int_{hx^\varepsilon}^{x} \Big| \sum_{t<n\leqslant t+h} d_k(n) - M_k(t,h) \Big|^2 dt,$$

(compare with [**1**]; henceforth we abbreviate $J_k(x,h)$ by $J_k$) where, say, $M_k(t,h)$ is the "expected value" of the (inner) sum.

This gives $d_k$ over the (say) "short interval" $[t,t+h]$ (as $h=o(t)$ $\forall t\in[hx^\varepsilon,x]$); here and in the sequel $\varepsilon>0$ may be arbitrarily small, but not necessarily the same at each occurrence.

Actually, Ivić gave in [**12**] a nontrivial bound for $J_k(x,h)$ when the width of the s.i. (abbreviation for short interval), namely $\theta := \frac{\log h}{\log xn}$ is greater than $\theta_k \stackrel{\mathrm{def}}{=} 2\sigma_k - 1$ (with $\sigma_k$ the Carlson's abscissa, i.e., $\inf\{\sigma\in\,]1/2,1[\,: I_k(\sigma,T)\ll T\}$, here):

$$\theta>\theta_k \;\Rightarrow\; \exists\delta=\delta(k)>0 : J_k(x,h) \ll \frac{xh^2}{x^\delta}$$

(with the trivial bound: $J_k(x,h)\ll xh^2(\log x)^c$, where $c=c(k)>0$, see the following).

This result clearly gives nontrivial bounds for $J_k$, using the information for zeta-moments off the critical line. For example, $\theta_3=\frac16$, $\theta_4=\frac14$, $\theta_5=\frac{11}{30},\dots$ (from the known values of $\sigma_k$).

Thus the knowledge of the moments of $\zeta(s)$ provides information on $d_k(n)$ in almost all short intervals (a.a.s.i.).

$$\Big(\text{See: } J_k \text{ nontrivial } \Rightarrow \sum_{t<n\leqslant t+h} d_k(n) \sim M_k(t,h), \quad \text{a.a.s.i.}\Big)$$

However, we can also go in the opposite direction: if we have some kind of nontrivial information about the distribution of $d_k(n)$, we can improve our knowledge (at least, on the $2k$-th moments) of the Riemann $\zeta$-function. Actually, this idea is due to Ivić, who linked $I_k$ to the "(auto-)correlation" of $d_k$ with "shift-parameter" $a$, i.e.,

$$\mathcal{C}_k(a) \stackrel{\mathrm{def}}{=} \sum_{n\leqslant x} d_k(n)d_k(n+a), \quad a\in\mathbb{N} \;\; (\text{here } x\in\mathbb{N},\ x\to\infty)$$

(the *shift* is a positive integer: $\mathcal{C}_k(-a)$ is close enough to $\mathcal{C}_k(a)$ and $\mathcal{C}_k(0)$ is relatively easy to compute).

Here it comes into play the idea of Ivić (see [**11**]) of *linking* the estimate of the *$2k$-th moment*, $I_k(T)$, *to a sum of correlations* $\mathcal{C}_k(a)$ performed *over $a$* (*the shift*), up to (roughly, we avoid technicalities), say, $h := \frac{x}{T}$ (the s.i. comes in!), where $x \ll T^{k/2}$.

In order to be more precise, we need to abbreviate (with $x, X$ or even $T$ our *main variables*, all independent and $\to\infty$):

$$A \lll B \stackrel{\mathrm{def}}{\Longleftrightarrow} \forall\varepsilon>0\ A \ll_\varepsilon x^\varepsilon B$$

i.e., the *modified Vinogradov notation* $\lll$ allows us to ignore all the arbitrarily small powers. We shall also say that the *arithmetic function* (a.f.) $f:\mathbb{N}\to\mathbb{R}$ is *essentially bounded*, write $f \lll 1$, when $\forall\varepsilon > 0$ $f(n) \ll_\varepsilon n^\varepsilon$ (as $n \to \infty$). For example, all the $d_k$ ($\forall k \in \mathbb{N}$) are essentially bounded:

$$\forall k \in \mathbb{N} \quad d_k \lll 1,$$

whence they contribute individually small powers, which for our purposes may be ignored. Shiu [**19**] obtains (see $J_k$ for trivial estimate above) a kind of Brun–Titchmarsh estimate for (suitable multiplicative a.f., like) $d_k$, and these give, on average over (all) s.i., powers of log. By the way,

$$L := \log T \quad (\text{or } \ L := \log x)$$

is the abbreviation for the logarithm of our main variable.

We quote the formula (proved $\forall k \leqslant 2$, see §3) for the $d_k$ correlations:

$$\mathcal{C}_k(a) = xP_{2k-2}(\log x) + \Delta_k(x,a), \ \ \Delta_k(x,a) = o(x); \tag{1.1}$$

here, the (conjectured, $\forall k > 2$) *main term* is $xP_{2k-2}(\log x) \ll_k xL^{2k-2} \lll x$ (since $P_{2k-2}(z)$ is a polynomial of degree $2k-2$ in $z$, see the following).

Here, it seems that the first to propose explicitly this form for (1.1) is Ivić, who also gave explicitly the polynomial $P_{2k-2}$, that is essentially bounded (w.r.t. $x$). However, as we shall see in a moment, it depends, also, on the shift $a > 0$.

We shall sketch now, avoiding technicalities, Ivić' s argument. After some work (expand the square & mollify, take relevant ranges, ...) he gets that $I_k(T)$ is

$$I_k(T) = I_k''(T) + O_\varepsilon(T^\varepsilon T)$$

with

$$I_k''(T) \stackrel{\text{def}}{=} \frac{1}{M} \sum_{a \leqslant h} \sum_{M < n \leqslant M'} d_k(n) d_k(n+a) \int_{T/2}^{2T} \phi(t) e^{ita/n} dt,$$

where $M < M' \leqslant 2M$, with $M \lll T^{k/2}$, say $h \lll M/T$, the smooth (i.e., $\in C^\infty$) test-function $\phi$ has support in $]T/2, 2T[$, $\phi([3T/4, 4T/3]) \equiv 1$, and has good decay

$$\phi^{(R)}(t) \ll_R T^{-R}, \ \ \forall R \in \mathbb{N}.$$

From now on (see the reason in next section) we can ignore (in bounds for $I_k$) all terms which are $\lll T$.

We give an idea of the shape of $P_{2k-2}$, given by Ivić, before we proceed. It is (see [**11**] for details)

$$P_{2k-2}(\log x) \stackrel{\text{def}}{=} \frac{1}{x} \int_0^x \sum_{q=1}^{\infty} \frac{c_q(a)}{q^2} R_k^2(\log t) dt,$$

with, say,

$$R_k(\log t) \stackrel{\text{def}}{=} \frac{C_{-k}(q)}{(k-1)!} \log^{k-1} t + \frac{C_{1-k}(q)}{(k-2)!} \log^{k-2} t + \cdots + \frac{C_{-2}(q)}{1!} \log t + C_{-1}(q)$$

depending on $q$, but not on $a$ (this is vital); also, w.r.t. $x$, $R_k(\log t) \lll 1$ and this is very useful. We shall see in a moment that the shape of these $C_j(q)$ is important only in the case $q=1$. By the way, here $c_q(a)$ is the *Ramanujan sum*, defined as ($\sum^*$ denotes summation over $j$ ($<q$) coprime with $q$)

$$c_q(a) \stackrel{\mathrm{def}}{=} \sum_{j \,(\mathrm{mod}\, q)}^{*} e_q(ja) = \sum_{\substack{d|q \\ d|a}} d\mu\Big(\frac{q}{d}\Big).$$

Hence, say, $S(a) \stackrel{\mathrm{def}}{=} \max(0, h-|a|)$ ($\hat{S}$ is Fejér's kernel) gives ($J_k$ is coming soon)

$$\widehat{S}\Big(\frac{j}{q}\Big) \stackrel{\mathrm{def}}{=} \sum_a S(a)e_q(ja) \geqslant 0 \Rightarrow \sum_a S(a)c_q(a) \geqslant 0$$

and from the elementary identity, $\forall d \in \mathbb{N}$, (hereon $n \equiv r(q)$ is $n \equiv r \pmod q$ abbreviation)

$$\sum_{\substack{a \\ a\equiv 0\,(d)}} S(a) = h + 2\sum_{b\leqslant h/d}(h-db) = \frac{h^2}{d} + d\Big\{\frac{h}{d}\Big\}\Big(1-\Big\{\frac{h}{d}\Big\}\Big),$$

we get (apply $c_q(a)$, above), writing $\mathbf{1}_\wp = 1$ if $\wp$ holds, $=0$ else:

$$\sum_a S(a)c_q(a) = \mathbf{1}_{q=1}h^2 + \sum_{d|q} d^2\mu\Big(\frac{q}{d}\Big)\Big\{\frac{h}{d}\Big\}\Big(1-\Big\{\frac{h}{d}\Big\}\Big).$$

(It is here evident that $q=1$ has a greater importance.) Thus, (see Ivić [**11**] and compare [**1**]):

$$\sum_a S(a)xP_{2k-2}(\log x) = h^2\int_{hx^\varepsilon}^{x} R_k^2(1,\log t)dt + \text{tails}, \tag{1.2}$$

where we mean, by "tails", remainders which are $\lll h^3$. Here, the part of $R_k(\log t)$ term with $q=1$ is, say,

$$R_k(1,\log t) \stackrel{\mathrm{def}}{=} \frac{C_{-k}(1)}{(k-1)!}\log^{k-1} t + \frac{C_{1-k}(1)}{(k-2)!}\log^{k-2} t + \cdots + \frac{C_{-2}(1)}{1!}\log t + C_{-1}(1)$$

and gives (see the above) the term $M_k(\log t)$ into the Selberg integral; as it should be, since (from an elementary version of Linnik's Dispersion method, compare [**1**, Lemmas]), assuming (1.1) with this $P_{2k-2}$, we get

$$J_k(x,h) \sim \sum_a S(a)\mathfrak{C}_k(a) - h^2\int_{hx^\varepsilon}^{x} M_k^2(\log t)\,dt \sim \sum_a S(a)\Delta_k(x,a), \tag{1.3}$$

where $\sim$ means ignoring "tails" (see above) and "diagonals" i.e., remainders $\lll xh$. We remark that *both these errors are negligible* (at least, for $k=3,4$, see Section 5), since they both contribute $\lll T$ to $I_k(T)$.

Then, due to the expression for $I_k''$, Ivić [**11**] made a hypothesis about (avoiding technicalities) the sums of $\Delta_k(x,a)$ (remainders into (1.1) above), performed over the shift $a$, say $G_k$, which implies the bound $I_k(T) \lll T$ (for the same $k>2$). Henceforth we assume that $k>2$.

Of course, he does not need (1.1) to hold *individually* $\forall a$ ($\leqslant h$, here), but he observes that he is summing up, into $G_k$, *without* the modulus over the remainder, $\Delta_k(x,a)$, so some $a$-cancellation can take place.

So far, he passes from an asymptotic formula (1.1) to an $a$-averaged form of it, which is easier to prove (however, nobody has done it yet!).

Here, with applications in mind, *we pass from a single average to a double average.*

Building on his expression for $I_k''$, it is possible to make a less stringent hypothesis, to *have a more flexible procedure* for the remainders $\Delta_k(x,a)$.

We use, also, our previous work on the Selberg integral of the a.f. $f$ (*essentially bounded and real*), compare [**1**], in order to let the Selberg integral of $d_k$, i.e., $J_k(x,h)$, come into play. (It is a kind of "double average" of $\Delta_k(x,a)$.)

Unfortunately, due to an exponential factor multiplying $d_k(n)d_k(n+a)$ into $\mathcal{C}_k(a)$ we cannot get a link with $I_k(T)$ using only $J_k(x,h)$ (*with* $h \lll \frac{x}{T}$, $x \lll T^{k/2}$), but we also need to make a hypothesis on another double average of remainders $\Delta_k(x,a)$. Furthermore, our bounds are also affected by the limitation $k \leqslant 4$, due to some error terms arising from the Linnik method (compare (5.1) proof in Section 5).

We shall formulate our original Theorem in Section 4, and then present its proof.

There is a way to improve the result (with a different proof, see Section 5) as follows.

THEOREM 1.1. *Let $T \to \infty$ and, $\forall \varepsilon > 0$, $T^{1+\varepsilon} \ll M \ll T^{k/2}$. Then $\forall k > 2$ we have*

$$I_k(T) \lll T \left( 1 + \max_{T \lll M \ll T^{k/2}} \frac{T}{M^2} \max_{0<h\ll M/T} J_k(M,h) \right).$$

In the next two sections we shall briefly mention some history of $I_k$ and the (related) additive divisor problems. Then, we shall prove our result in the subsequent section, before some remarks, and in the concluding section we shall prove Theorem 1.1.

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## 2. A concise history of moments of the Riemann zeta-function

We should keep in mind, here, that for fixed $k \in \mathbb{N}$ we seek the bound

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt \lll T \quad (2k\text{-th moment problem})$$

(it is "on the line", since $\sigma = \frac{1}{2}$ is the critical line: "off the line" means with $\frac{1}{2} < \sigma \leqslant 1$) that (for $k > 2$) is our aim; in fact, the English school gave first 2 cases: first Hardy and Littlewood [**5**] in 1916 gave asymptotics for $k = 1$ (not too difficult!) and then Ingham [**8**] in 1927 for $k = 2$ (actually, for both $k = 1, 2$ he gave only $P_{2k-2}$ leading term, hence error $\log x$ better than main term); then, Heath-Brown in 1979 [**7**] gave, for $k = 2$ (solved 2-add.div.pbm., i.e., the binary additive divisor problem, see Section 3, using Weil's bound for the Kloosterman sums), $P_6$ (not explicitly!) plus error $E_2 \lll T^{7/8}$.

In a series of papers from the early '90's, Ivić and Motohashi (applying considerations from $SL(\mathbb{Z},2)$ for the binary add.div.pbm.) obtained $E_2 \ll T^{2/3}L^c$ and, in the mean-square, even $E_2(T) \ll \sqrt{T}\log^C T$. Ivić explicited $P_{2k-2}$ (when $k=2$). Like the binary additive divisor problem, this is not the whole story.

The case $k=3$ (recall $\mathcal{C}_3(a)$ problem), is again unsolved.

The bound $I_3(T) \lll T$ is called the "sixth moment" problem (actually, this is the *weak version* and has a *link* (see Ivić [**13**]) *with the ternary additive divisor problem.*

Another interesting result on higher moments is (Heath-Brown [**6**]): $I_6(T) \ll T^2L^c$.

One glimpse at the *higher moments* (for $k \leqslant 2$, see [**14**]): the *predicted asymptotics* is $I_k \sim C(k)TL^{k^2}$, $\forall k \geqslant 1$ by applying the methods from Random Matrix Theory in the 2000's seminal works (early 2000's) of Keating and Snaith [**15**] (at Bristol); many others (Conrey, Ghosh [**2**] to name two).

## 3. A history of some additive divisor problems

The problem of proving (1.1) (at least *fixed* $a > 0$) is called the *k-ary additive divisor problem*: trivial case $k = 1$ ($\mathcal{C}_1(a) = x\ \forall a \in \mathbb{Z}$) and the *binary additive divisor problem*, $k = 2$, are the only solved problems.

The case $k = 3$ is the *ternary additive divisor problem* (sometimes called *Linnik problem*): some time ago, Vinogradov and Takhtadzhjan (see below $k = 2$) announced its solution but with, as yet, unfilled holes in their (very technical) "proof". Their approach still suffers from our lack of information about $SL(\mathbb{Z},3)$; while our (enough good) state of the art about, instead, $SL(\mathbb{Z},2)$ (actually, through the application of the Kuznetsov trace formula, see [**20**]) allowed (starting from [**18**] approach) Ivić, Motohashi and Jutila to solve satisfactorily, see especially [**14**] (and recent Meurman's [**16**]), the binary additive divisor problem (different approaches work, with weaker remainders). We mention (still $k = 2$), in passing, Kloosterman sums bounds (like Weil's) in the $\delta$-method of Duke–Friedlander–Iwaniec for "determinantal equations" (especially [**4**]). An even more general problem than this last one has been solved by Ismoilov [**9**].

Thus, so far, no one has proved (for $k > 2$), given $a \in \mathbb{N}$,

$$\mathcal{C}_k(a) = xP_{2k-2}(\log x) + \Delta_k(x,a), \quad \Delta_k(x,a) = o(x),$$

as $x \to \infty$ (the $k$-ary additive divisor problem), not even for a single shift $a > 0$ (already $k = 2$ has delicate "uniformity" issues: see [**14**]).

## 4. Statement and proof of the original theorem

We state here our original theorem, together with its proof.

Theorem 4.1. *Let $M < M' \leqslant 2M$, $T^{1+\varepsilon} \leqslant M \ll T^{k/2}$ and $H = M^{1+\varepsilon}/T$, with double average $\widetilde{G}_k = \widetilde{G}_k(M,T)$ defined as*

$$\widetilde{G}_k \stackrel{\text{def}}{=} \sup_{M \leqslant x \leqslant M',\, t \leqslant H} \left( \frac{1}{t} J_k(x,t) + \frac{1}{t} \left| \sum_{h \leqslant t} \sum_{h < a \leqslant t} \Delta_k(x,a) \right| \right).$$

*Then for $k=3,4$ we have*

$$I_k(T) \lll T\Big(1+\sup_{T\lll M\ll T^{k/2}} \widetilde{G}_k(M,T)/M\Big).$$

PROOF. First of all, the main terms in (1.2), with $P_{2k-2}$, are treated like Ivić does [**11**]; actually, since his $G_k$ is the supremum (see [**11**, Theorem 1]) of (the absolute value of)

$$\sum_{a\leqslant t}\Delta_k(x,a),$$

we split it as

$$\frac{1}{t}\sum_{h\leqslant t}\sum_{a\leqslant h}\Delta_k(x,a)+\frac{1}{t}\sum_{h\leqslant t}\sum_{h<a\leqslant t}\Delta_k(x,a),$$

whence

$$\Big|\sum_{a\leqslant t}\Delta_k(x,a)\Big| \leqslant \Big|\frac{1}{t}\sum_{h\leqslant t}\sum_{a\leqslant h}\Delta_k(x,a)\Big| + \Big|\frac{1}{t}\sum_{h\leqslant t}\sum_{h<a\leqslant t}\Delta_k(x,a)\Big|,$$

where the second (double) sum is in our $\widetilde{G_k}$; in the sequel, we shall let the Selberg integral appear from the other term; that is, in fact, the *arithmetic mean*

$$\frac{1}{t}\sum_{h\leqslant t}\sum_{a\leqslant h}\Delta_k(x,a)$$

(a kind of average, something like the $C^1$ process in Fourier series) and can be expressed as (on exchanging summations)

$$\frac{1}{t}\sum_{a\leqslant t}(t-a+1)\Delta_k(x,a)=\frac{1}{t}\sum_{a\leqslant t}(t-a)\Delta_k(x,a)+\frac{1}{t}\sum_{a\leqslant t}\Delta_k(x,a).$$

Before we proceed further, we need to express how the *diagonal* remainders into the estimate of $J_k(x,t)$, i.e., the terms $\lll xt$, and the *tails*, i.e., $\lll t^3$, appear in our final estimate for $I_k$: compare the calculations soon after this proof. This last equation especially has a term

$$\frac{1}{t}\sum_{a\leqslant t}\Delta_k(x,a)\lll x,$$

from the trivial estimate $\Delta_k(x,a)\lll x$, giving *diagonal* terms; and the easily proved relation $\Delta_k(x,-a)=\Delta_k(x,a)+O_\varepsilon(x^\varepsilon a)$ gives *tails* (another *diagonal*: $\Delta_k(x,0)\lll x$):

$$\frac{1}{t}\sum_{a\leqslant t}(t-a)\Delta_k(x,a)\sim\frac{1}{2t}\sum_{0\leqslant|a|\leqslant t}(t-|a|)\Delta_k(x,a)$$

(the $\sim$ means "+ diagonals and tails"); in all, the term in $\widetilde{G_k}$ with $J_k$ is

$$\frac{1}{t}\sum_{h\leqslant t}\sum_{a\leqslant h}\Delta_k(x,a)\sim\frac{1}{2t}\sum_{0\leqslant|a|\leqslant t}(t-|a|)\Delta_k(x,a)$$

and, since $S(a)=\max(t-|a|,0)$, $\forall 0\leqslant |a|\leqslant t$, applying (1.3) (compare [**1**]),

$$\sum_{0\leqslant |a|\leqslant t}(t-|a|)\Delta_k(x,a)\sim J_k(x,t),$$

we obtain the desired bound with the Selberg integral (and the double average). $\square$

We remark that, in spite of the fact that our "new" theorem holds $\forall k>2$ (integer), we have here some trouble in handling Selberg's integral *tails*, since they contribute to $\widetilde{G}_k$ as (in the sup above)

$$\lll \frac{1}{t}t^3 \lll H^2 \lll \frac{M^2}{T^2}$$

which gives to $I_k(T)$ a contribution (the other sup above)

$$\lll T\Big(\frac{M^2}{T^2}M^{-1}\Big) \lll \frac{M}{T} \lll T^{k/2-1}.$$

This is $\lll T$ *only* when $k/2\leqslant 2$, i.e., $k\leqslant 4$ (diagonals give $\lll T$, good $\forall k$.)

The *tails* arise naturally when applying the Linnik method and even a more careful analysis will almost surely not eliminate them. While they are negligible for the Selberg integral, they are not negligible for our present approach.

We remark, in passing, that the "additional" double average, in this approach, cannot be dispensed with.

## 5. Proof of Theorem 1.1

First of all, we may restrict ourselves to the following definitions of $I_k$, $J_k$ :

$$I_k(T):=\int_{T/2}^{T}|\zeta(\frac{1}{2}+it)|^{2k}dt,\quad J_k(M,h):=\int_{M/2}^{3M}\Big|\sum_{t<n\leqslant t+h}d_k(n)-M_k(t,h)\Big|^2dt.$$

In fact, we can combine a dissection argument (for $I_k$) and (for $J_k$) a positivity and monotonicity argument to reduce these integrals to the ones defined in the introduction. The logarithmic factor(s) will be in the $\lll$ (let us fix $\varepsilon>0$). In order to keep the exposition clearer, we write $\lll$ even when it is only $\ll$, in the sequel.

We start by choosing a Dirichlet series ($C_j$ is not to be confused with $C_j(q)$ above)

$$f_k(s)=\sum_{n=1}^{\infty}a_k(n)n^{-s}\stackrel{\text{def}}{=}\sum_{j=0}^{k-1}C_j\zeta^{(j)}(s),\ \text{with } \zeta^k(s)-f_k(s) \text{ holomorphic at } s=1.$$

This is done, in order to give the expected main term of the short sum $\sum_n d_k(n)$, that comes from the residue of $\frac{h}{x}\zeta^k(s)\frac{x^s}{s}$ at $s=1$; like (we shall see in a moment) for the other short sum $\sum_n a_k(n)$, which is this main therm, together with $\lll 1$, remainder terms. Here we obtain the coefficients $a_k(n)=\sum_j C_j(-\log n)^j$, since we recall

$$\zeta^{(j)}(s)=\sum_{n=1}^{\infty}(-\log n)^j n^{-s}$$

and we still have to calculate the coefficients $C_j$ . This can be done using the Laurent expansion at $s = 1$ of the Riemann zeta-function:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty} \gamma_n (s-1)^n$$

and (taking $k$-th powers) we get, from this, that

$$C_0\zeta(s) + C_1\zeta'(s) + \cdots + C_{k-1}\zeta^{(k-1)}(s),$$

here, has to have the same principal at $s = 1$ as $\zeta^k(s)$. This is accomplished by studying a linear system in $k$ equations and unknowns $C_0, \ldots, C_{k-1}$, which has a unique solution.

Then we isolate the mean-square of $f_k$:

$$I_k(T) \leqslant \int_{T/2}^{T} |\zeta(\tfrac{1}{2}+it)^k - f_k(\tfrac{1}{2}+it)|^2 dt + \int_{T/2}^{T} |f_k(\tfrac{1}{2}+it)|^2 dt,$$

since it is relatively easy to bound (analogously as the mean square of the zeta-function, compare with Section 2):

$$\int_{T/2}^{T} |f_k(\tfrac{1}{2}+it)|^2 dt \lll T$$

and, in the following, we shall ignore (see why in Sections 1 and 2) all such kind of remainders. We wish to express $\zeta^k - f_k$ at $s = 1/2 + it$, with $t \asymp T$ (i.e., $T \ll t \ll T$ now on), as ( $\ll \log T$) smoothed Dirichlet polynomials such that there holds (leaving $\lll T$)

$$(5.1)\quad \int_{T/2}^{T} \Big|\zeta^k(\tfrac{1}{2}+it) - f_k(\tfrac{1}{2}+it)\Big|^2 dt \lll \int_{T/2}^{T} \Bigg|\sum_{M\leqslant n\leqslant 2M} w(n)\frac{d_k(n)-a_k(n)}{n^{1/2+it}}\Bigg|^2 dt,$$

where $T^{1+\varepsilon} \ll M \ll T^{k/2}$ and $w$ here is a $C^1$ weight supported in $[M, 2M]$, with

$$w(x) \ll 1, \quad w'(x) \lll \frac{1}{M} \quad \forall x \in [M, 2M].$$

This can be done applying the functional equation ($\forall k > 2$) for $\zeta^k$, as in [**10**]; plug $t \asymp T$, $\alpha = \frac{1}{2} + 1/\log T$, $Y \asymp M \asymp T^{k/2}$, $h = \log^2 T$ (do not confuse with our $h$ at last), into [**10**, Theorem 4.4] to obtain

$$\zeta^k(\tfrac{1}{2}+it) = \sum_{n\leqslant 2Y} d_k(n) e^{-(n/Y)\log^2 T} n^{-1/2-it} + \chi^k(\tfrac{1}{2}+it) \sum_{n\leqslant M} d_k(n) n^{-1/2+it}$$
$$-\frac{1}{2\pi i}\int_{\substack{\mathrm{Re}(w)=1/\log T\\ |\mathrm{Im}(w)|\leqslant \log^4 T}} \chi^k(\tfrac{1}{2}+it+w) \sum_{n\leqslant M} d_k(n) n^{w-1/2+it} Y^w \Gamma\Big(1+\frac{w}{\log^2 T}\Big)\frac{dw}{w} + o(1).$$

Subtracting $f_k(1/2+it)$ from the left-hand side and inserting $a_k(n)$ into $n-$sums we get that

$$\zeta^k(\tfrac{1}{2}+it) - f_k(\tfrac{1}{2}+it)$$

can be expressed (apart from the $\lll 1$-terms) by sums of the following types:

$$\sum_{n\leqslant 2Y} a_k(n)e^{-(n/Y)\log^2 T}n^{-1/2-it} - f_k(\tfrac{1}{2}+it), \tag{5.2}$$

$$\sum_{n\leqslant M} a_k(n)n^{-1/2+it}, \tag{5.3}$$

$$\sum_{n\leqslant M} a_k(n)n^{\frac{1}{\log T}+iu-1/2+it}, \quad (|u|\leqslant \log^4 T), \tag{5.4}$$

$$\sum_{n\leqslant 2Y}(d_k(n)-a_k(n))e^{-(n/Y)}\log^2 Tn^{-1/2-it}, \ \sum_{n\leqslant M}(d_k(n)-a_k(n))n^{-1/2+it}, \tag{5.5}$$

$$\sum_{n\leqslant M}(d_k(n)-a_k(n))n^{\frac{1}{\log T}+iu-1/2+it} \qquad (|u|\leqslant \log^4 T). \tag{5.6}$$

For our purposes, the bounds $\chi^k(\frac{1}{2}+it)\ll 1$ and $\chi^k(\frac{1}{2}+\frac{1}{\log T}+it)\ll 1$ suffice (see e.g., [**10**]), with uniform constants $\forall k$. Also, we can transform the $n$-sums into integrals, which (apart from these sums) give an essentially bounded contribution. We first treat the terms (5.2), (5.3), (5.4) like remainders, while (5.5) and (5.6) will be into our final (5.1). Recall that there will be $\ll \log T$ (i.e., $\lll 1$) of such sums. Analogously as in Theorem 4.11 of Titchmarsh's book [**21**]

$$\sum_{n\leqslant M} a_k(n)n^{-1/2+it'} = \operatorname{Res}\Big(f_k(s+\tfrac{1}{2}-it')\frac{M^s}{s}, \frac{1}{2}+it'\Big) + f_k(\tfrac{1}{2}-it') + O_\varepsilon(M^\varepsilon);$$

here $t'=t$ or $t'=t+(1/\log T)+u$, with $|u|\leqslant \log^4 T$, an eventual perturbation to $w$ of the kind $n^{1/\log T+iu}$, which gives $w'\ll(\log T)^4/M$, into our $w'\lll 1/M$.

We perform $k$ times an $M$-average over these (5.3), (5.4) terms to let the residue term $O_k(1)$, while (see above) $f_k$ mean-square is negligible, like $O_\varepsilon(M^\varepsilon)\lll 1$. The other terms are managed like these, with a ($k$ times) $M$-average ($Y\asymp M$), on writing

$$\begin{aligned}\sum_{n\leqslant 2Y} a_k(n)e^{-(n/Y)\log^2 T}n^{-1/2-it} - f_k(\tfrac{1}{2}+it) &= \sum_{n\leqslant A} a_k(n)n^{-1/2-it} - f_k(\tfrac{1}{2}+it)\\ &+ \sum_{n\leqslant A} a_k(n)O\Big(\frac{n}{Y\log^2 T}\Big)n^{-1/2-it} + \sum_{A<n\leqslant 2Y} a_k(n)e^{-(n/Y)\log^2 T}n^{-1/2-it},\end{aligned}$$

where the difference on the right-hand side is treated as above; also,

$$\sum_{n\leqslant A} a_k(n)O\Big(\frac{n}{Y\log^2 T}\Big)n^{-1/2-it} \lll \sum_{n\leqslant A}\frac{\sqrt{n}}{Y\log^2 T} \lll \frac{A\sqrt{A}}{Y},$$

while partial summation allows to isolate the exponential factor, in order to average (like before) the inner sums, and, with the choice $A\asymp Y^{2/3}$, we get $\lll 1$, so that even (5.2) can be neglected. Next, we $M$-average and use a dyadic dissection (to arrive to $[M,2M]$, there) for the remaining sums. (In the other proof, not-averaged residue terms in (5.2)–(5.4) correspond to tails, giving the limit $k\leqslant 4$, see Section 4.) Collecting all these sums and estimates, we get (5.1).

Now we apply Gallagher's Lemma (see e.g., Lemma 1.10 in [**17**]) to the sum $\sum_n w(n)(d_k(n) - a_k(n))\, n^{-1/2-it}$, to obtain

$$\int_{-T}^{T} \Big| \sum_n \frac{w(n)}{\sqrt{n}} (d_k(n) - a_k(n)) n^{-it} \Big|^2 dt \ll T^2 \int_0^\infty \Big| \sum_y^{\tau y} \frac{w(n)}{\sqrt{n}} (d_k(n) - a_k(n)) \Big|^2 \frac{dy}{y},$$

with $\tau := \exp(1/T)$ (see that $\tau - 1 \asymp 1/T$). Here, since $w$ is supported in $[M, 2M]$,

$$I_k(T) \lll \max_{T^{1+\varepsilon} \ll M \ll T^{k/2}} \frac{T^2}{M} \int_{M/2}^{3M} \Big| \sum_{y < n \leqslant \tau y} \frac{w(n)}{\sqrt{n}} (d_k(n) - a_k(n)) \Big|^2 dy \tag{5.7}$$

(again, leaving $\lll T$). Let us now denote $\rho := \tau - 1 \sim 1/T$.

In order to avoid the inner dependence on $y$, applying the partial summation we have

$$\sum_{y < n \leqslant y + \rho y} c(n) f(n) \ll \max_{h \ll \max(\rho y)} \Big| \sum_{y < n \leqslant y + h} c(n) \Big| \left( |f(y + \rho y)| + \int_y^{y + \rho y} |f'(v)| dv \right)$$

(where we can assume $y$ integer, the difference giving $\lll 1$ in the sum), with the (local) definitions $c(n) := d_k(n) - a_k(n)$ and $f(v) := w(v)/\sqrt{v}$, where

$$|f(v)| \ll \frac{1}{\sqrt{v}} \text{ and } |f'(v)| \lll \frac{1}{v\sqrt{v}} + \frac{1}{M\sqrt{v}} \quad \forall v \in [y, y + \rho y] \ \ (M < y \leqslant 2M)$$

we have

$$\Big| \sum_{y < n \leqslant y + \rho y} w(n) n^{-1/2} (d_k(n) - a_k(n)) \Big|^2 \lll \max_{h \ll M/T} \frac{1}{M} \Big| \sum_{y < n \leqslant y + h} (d_k(n) - a_k(n)) \Big|^2.$$

Finally, inserting this in (5.7) we obtain

$$I_k(T) \lll T \max_{T^{1+\varepsilon} \ll M \ll T^{k/2}} \max_{0 < h \ll M/T} \frac{T}{M^2} \int_{M/2}^{3M} \Big| \sum_{x < n \leqslant x + h} d_k(n) - M_k(x, h) \Big|^2 dx,$$

where it is evident that the main term $M_k(x,h)$ is the $a_k(n)$−short sum in $]x, x+h]$.

## References


1. G. Coppola, *On the Correlations, Selberg integral and symmetry of sieve functions in short intervals*, http://arxiv.org/abs/0709.3648v3; to appear in J. Comb. Number Theory **2**(2) (2010).
2. J. B. Conrey, A. Ghosh, *A conjecture for the sixth power moment of the Riemann zeta-function*, Internat. Math. Res. Notices **15** (1998), 775–780.
3. H. Davenport, *Multiplicative Number Theory*, Third edition, Revised and with a preface by Hugh L. Montgomery, Grad. Texts Math. **74**, Springer-Verlag, New York, 2000.
4. W. Duke, J. Friedlander, H. Iwaniec, *A quadratic divisor problem*, Invent. Math. **115** (1994), no. 2, 209–217.
5. G. H. Hardy, J. E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes*, Acta Math. **41**(1) (1916), 119–196.
6. D. R. Heath-Brown, *The twelfth power moment of the Riemann-function*, Q. J. Math. Oxf. II. Ser. **29** (1978), 443–462.
7. D. R. Heath-Brown, *The fourth power moment of the Riemann zeta function*, Proc. London Math. Soc. (3) **38** (1979), 385–422.

8. A. E. Ingham, *Mean-value theorems in the theory of the Riemann Zeta-function*, Proc. London Math. Soc. (2) **27** (1928), 273–300.
9. D. Ismoilov, *An asymptotic formula in additive number theory*, Math. Notes **39** (1986), 257–266. Translation from Mat. Zametki **39** (1986), 465–482.
10. A. Ivić, *The Riemann Zeta Function, The theory of the Riemann zeta-function with applications* Wiley, New York, 1985.
11. A. Ivić, *The general additive divisor problem and moments of the zeta-function*, in: *New trends in probability and statistics*, Vol. 4 (Palanga, 1996), 69–89, VSP, Utrecht, 1997.
12. A. Ivić, *On the mean square of the divisor function in short intervals*, J. Théor. Nombres Bordx. **21**(2) (2009), 251–261.
13. A. Ivić, *On the ternary additive divisor problem and the sixth moment of the zeta-function*; in: *Sieve methods, exponential sums, and their applications in number theory* (Cardiff, 1995), 205–243, London Math. Soc. Lecture Note Ser. 237, Cambridge Univ. Press, Cambridge, 1997.
14. A. Ivić, Y. Motohashi, *On some estimates involving the binary additive divisor problem*, Q. J. Math. Oxf. II. Ser. **46** (1995), 471–483.
15. J. P. Keating, N. Snaith, *Random matrix theory and* $\zeta(1/2+it)$, Comm. Math. Phys. **214**(1) (2000), 91–110.
16. T. Meurman, *On the binary additive divisor problem*; in: *Number Theory* (Turku, 1999), 223–246, de Gruyter, Berlin, 2001.
17. H. L. Montgomery, *Topics in multiplicative number theory*, Lect. Notes Math. 227, Springer-Verlag, Berlin–New York, 1971.
18. Y. Motohashi, *The binary additive divisor problem*, Ann. Sci. École Norm. Sup. (4) **27**(5) (1994), 529–572.
19. P. Shiu, *A Brun-Titchmarsh theorem for multiplicative functions*, J. Reine Angew. Math. **313** (1980), 161–170.
20. L. A. Takhtadzhjan, A. I. Vinogradov, *The zeta-function of the additive divisor problem and the spectral expansion of the automorphic Laplacian* (Russian), Automorphic functions and number theory, II. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 134 (1984), 84–116.
21. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Second edition. Edited and with a preface by D. R. Heath-Brown, Clarendon Press, Oxford University Press, New York, 1986.

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