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# CHARACTERIZATION OF THE PSEUDO-SYMMETRIES OF IDEAL WINTGEN SUBMANIFOLDS OF DIMENSION 3

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Abstract. Recently, Choi and Lu proved that the Wintgen inequality $\rho \leqslant H^2 - \rho^\perp + k$, (where $\rho$ is the normalized scalar curvature and $H^2$, respectively $\rho^\perp$, are the squared mean curvature and the normalized scalar normal curvature) holds on any 3-dimensional submanifold $M^3$ with arbitrary codimension $m$ in any real space form $\widetilde{M}^{3+m}(k)$ of curvature $k$. For a given Riemannian manifold $M^3$, this inequality can be interpreted as follows: for all possible isometric immersions of $M^3$ in space forms $\widetilde{M}^{3+m}(k)$, the value of the intrinsic curvature $\rho$ of $M$ puts a lower bound to all possible values of the extrinsic curvature $H^2 - \rho^\perp + k$ that $M$ in any case can not avoid to "undergo" as a submanifold of $\tilde{M}$. From this point of view, $M$ is called a Wintgen ideal submanifold of $\widetilde{M}$ when this extrinsic curvature $H^2 - \rho^\perp + k$ actually assumes its theoretically smallest possible value, as given by its intrinsic curvature $\rho$, at all points of $M$. We show that the pseudo-symmetry or, equivalently, the property to be quasi-Einstein of such 3-dimensional Wintgen ideal submanifolds $M^3$ of $\widetilde{M}^{3+m}(k)$ can be characterized in terms of the intrinsic minimal values of the Ricci curvatures and of the Riemannian sectional curvatures of $M$ and of the extrinsic notions of the umbilicity, the minimality and the pseudo-umbilicity of $M$ in $\widetilde{M}$.

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## 1. Wintgen ideal submanifolds

For surfaces $M^2$ in the Euclidean space $E^3$, the *Euler inequality* $K \leqslant H^2$, where $K$ is the (intrinsic) *Gauss curvature* of $M^2$ and $H^2$ is the (extrinsic) *squared mean curvature* of $M^2$ in $E^3$, at once follows from the fact that $K = k_1k_2$ and $H = \frac{1}{2}(k_1 + k_2)$ where $k_1$ and $k_2$ are the *principal curvatures* of $M^2$ in $E^3$, and, obviously, $K = H^2$ everywhere on $M^2$ if and only if the surface $M^2$ is *totally umbilical* in $E^3$, i.e., $k_1 = k_2$ at all points of $M^2$, or still, by a *theorem of Meusnier*, if and only if $M^2$ is a part of a *plane* $E^2$ or of a *round sphere* $S^2$ in $E^3$.

For surfaces $M^2$ in the 4-dimensional Euclidean space $E^4$, Wintgen proved that the Gauss curvature $K$ and the squared mean curvature $H^2$ and the (extrinsic) *normal curvature* $K^\perp$ always satisfy the inequality $K \leqslant H^2 - K^\perp$, and that actually the equality holds if and only if the *curvature ellipse* of $M^2$ in $E^4$ is a *circle* [36]; (cf. e.g. [6, 7] for studies also on the global differential geometry of submanifolds by a.o. Smale, Lashof, Chern, Chen and Willmore concerning the Euler characteristic of the normal bundle, the number of self-intersections and the total mean curvature). This fundamental inequality between the most important intrinsic and extrinsic scalar valued curvatures of surfaces $M^2$ in $E^4$ was later shown, by Rouxel and by Guadalupe and Rodriguez, to hold more generally for all surfaces $M^2$ of *arbitrary codimension* $m$ in the *real space forms* $\tilde{M}^{2+m}(k)$ of constant sectional curvature $k$, inclusive the characterisation of the equality case [21, 29]. After these extensions of the above *Wintgen inequality* for submanifolds of dimension 2 and of codimension 2 to submanifolds of *dimension* 2 and *arbitrary codimension* $m \geqslant 2$, in 1999 De Smet and Dillen and Vrancken and one of the authors proved the Wintgen inequality $\rho \leqslant H^2 - \rho^\perp + k$, where $\rho$ and $\rho^\perp$ respectively are the (intrinsic) *normalized scalar curvature* and the (extrinsic) *normalized scalar normal curvature*, for 2-*codimensional* submanifolds $M^n$ of *arbitrary dimension* $n \geqslant 2$ in the real space forms $\tilde{M}^{n+2}(k)$, and characterized the equality situation explicitly in terms of the shape operators of $M^n$ in $\tilde{M}^{n+2}(k)$ [12]. Moreover, in [12] it was conjectured that this Wintgen inequality holds for submanifolds $M^n$ of *any dimension* $n \geqslant 2$ and of *any codimension* $m \geqslant 2$ in real space forms $\tilde{M}^{n+m}(k)$, (referring to the initials of the authors of [12], Suceavă recently started to call this "the DDVV conjecture" [31], and was therein followed by others, although the "*conjecture on Wintgen's inequality*" may well be a more appropriate terminology). Recently, Choi and Lu proved that this conjecture is true for all 3-dimensional submanifolds $M^3$ of *arbitrary codimension* $m \geqslant 2$ in $\widetilde{M}^{3+m}(k)$ and obtained characteristic expressions for the shape operators of the submanifolds $M^3$ in $\tilde{M}^{3+m}(k)$ which do realize the equality in this general inequality [8]. Concrete descriptions of some classes of 3-dimensional *Wintgen ideal submanifolds* were given by Bryant, Dillen, Fastenakels and Van der Veken [4, 18].

At this stage we would like further finally to mention that De Smet, Dillen, Fastenakels, Van der Veken, Vrancken and one of the authors studied the Wintgen inequality for *invariant submanifolds in Kaehler, nearly Kaehler and Sasakian spaces* [11, 17], and that Gmira, Haesen, Dillen and two of the authors studied this inequality for *submanifolds in semi-Riemannian spaces* [19, 20].

## 2. Pseudo-symmetric spaces

Let $M^n$ be an $n$-dimensional *Riemannian manifold* with *metric* $(0,2)$ *tensor* $g$ and *Levi-Civita connection* $\nabla$. Let $R$ denote the $(0,4)$ *Riemann–Christoffel curvature tensor* of $M$ as well as the *curvature* (1,1) *operator* $R(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, thus having

$$R(X,Y,Z,W) := g(R(X,Y)Z,W), \tag{1}$$

where $X, Y$, etc. denote arbitrary vector fields on $M$ and $[.,.]$ stands for the Lie bracket. By the action of the curvature operator working as a derivation on the curvature tensor $R$, the following $(0,6)$ tensor $R\cdot R$ is obtained:

$$\begin{aligned}(R\cdot R)(X_1,X_2,X_3,X_4;X,Y) &:= (R(X,Y)\cdot R)(X_1,X_2,X_3,X_4)\\ &= -R(R(X,Y)X_1,X_2,X_3,X_4) - R(X_1,R(X,Y)X_2,X_3,X_4)\\ &\quad - R(X_1,X_2,R(X,Y)X_3,X_4) - R(X_1,X_2,X_3,R(X,Y)X_4).\end{aligned}$$

As was recently shown by Haesen and one of the authors [22], the tensor $R\cdot R$ can be geometrically interpreted as giving *the second-order measure of the change of the sectional curvatures $K(p,\pi)$ for tangent 2-planes $\pi$ at points $p$ after the parallel transport of $\pi$ around infinitesimal co-ordinate parallelograms in $M$ cornered at $p$.* Thus, the *semi-symmetric* or *Szabó symmetric spaces* [32, 33], i.e., the manifolds $M$ for which $R\cdot R = 0$, are those Riemannian manifolds for which all sectional curvatures remain preserved after the parallel transport of their planes around all infinitesimal co-ordinate parallelograms. The *locally symmetric* or *Cartan symmetric spaces*, i.e., the manifolds $M$ for which $\nabla R = 0$, constitute a proper subclass of the class of the Szabó symmetric spaces.

We recall that the definition (1) of the curvature tensor goes back to Schouten's geometrical interpretation of $R$ as *the second order measure of the change of the direction of vector fields after their parallel transport around closed infinitesimal curves on $M$* [30]. Then the *locally flat* or *locally Euclidean* spaces, thus the manifolds $M$ for which $R = 0$, are those Riemannian manifolds for which all directions remain preserved after parallel transport around all closed infinitesimal curves. The simplest nonflat Riemannian manifolds $M$ are the *spaces of constant curvature* $K = k$, i.e., the spaces whose function $K$ is *isotropic* (meaning that, at each point $p$, the Gauss curvature $K(p,\pi)$ at $p$ of the local surface formed by the geodesics of $M$ which pass through $p$ and whose tangent vector at $p$ lies in $\pi$, has the same value for all choices of planes $\pi$ at $p$, thus $K$ becoming a real function on $M$, which by the lemma of Schur, for $n > 2$, then necessarily has to be constant). These real *space forms* $M^n(k)$, by a theorem of Beltrami, can be obtained from the locally Euclidean spaces by *projective transformations* and their class is closed under such transformations. Further, we also recall that the knowledge of the curvature tensor $R$ is *equivalent* to the knowledge of the sectional or Riemannian curvatures $K$, as was shown by Cartan. Finally, as is well known, the curvature tensor $R$ of a space of constant curvature $k$ is given by

$$R(X,Y,Z,W) = k\, g((X\wedge_g Y)Z,W), \tag{2}$$

where the $\wedge_g$ stands for the *metrical endomorphism* $(X \wedge_g Y)Z := g(Y,Z)X - g(X,Z)Y$. Thus for the real space forms $M^n(k)$, $n > 2$, there exists a *real valued* function $K$ on $M$ such that $R(X,Y,Z,W) = K\,G(X,Y,Z,W)$, where the $(0,4)$-tensor $G$ is defined by $G(X,Y,Z,W) := g((X \wedge_g Y)Z, W)$.

A main interest of Riemann, Helmholtz, Lie, Klein,... in the spaces of constant curvature was related to the fact that these are precisely the Riemannian manifolds which satisfy the *axiom of free mobility.*

Now, similarly as proceeding from the locally Euclidean spaces to the real space forms, one can proceed from the Szabó symmetric spaces to the *pseudo-symmetric* or *Deszcz symmetric spaces* [1, 13, 22, 35]. The pseudo-symmetric spaces were defined as the manifolds $M$ for which the $(0,6)$ tensor $R\cdot R$ and the $(0,6)$ *Tachibana tensor* $Q(g,R) := -\wedge_g \cdot R$, where the metrical endomorphism $\wedge_g$ acts on the $(0,4)$ curvature tensor $R$ as a derivation, are proportional, say $R \cdot R = L(-\wedge_g \cdot R)$ for some real valued function $L$ on $M$;

$$\begin{aligned} Q(g,R)(X_1,X_2,X_3,X_4;X,Y) &:= -((X \wedge_g Y)\cdot R)(X_1,X_2,X_3,X_4) \\ &= R((X \wedge_g Y)X_1,X_2,X_3,X_4) + R(X_1,(X \wedge_g Y)X_2,X_3,X_4) \\ &\quad + R(X_1,X_2,(X \wedge_g Y)X_3,X_4) + R(X_1,X_2,X_3,(X \wedge_g Y)X_4). \end{aligned}$$

A classical result states that the identical vanishing of this Tachibana tensor, $Q(g,R) = 0$, characterizes the real space forms. Further, results of Mikesh, Venzi, Defever and Deszcz learn that pseudo-symmetric spaces are obtained by applying *projective transformations* to the semi-symmetric spaces and that the class of the pseudo-symmetric spaces is closed under such transformations. Two 2-planes $\pi$ and $\bar{\pi}$, spanned by vectors $u,v$ and $x,y$ respectively, at the same point $p$ of $M$, are said to be *curvature dependent* if $Q(g,R)(u,v,v,u;x,y) \neq 0$, which is independent of the choices of bases for $\pi$ and $\bar{\pi}$. For such planes, the *double sectional curvature* or the *sectional curvature of Deszcz* or the *Riemann curvature of Deszcz* $L(p,\pi,\bar{\pi})$ is defined as the real number given by

$$L(p,\pi,\bar{\pi}) := \frac{(R\cdot R)(u,v,v,u;x,y)}{Q(g,R)(u,v,v,u;x,y)},$$

(which is independent of the choices of bases for $\pi$ and $\bar{\pi}$); it is a scalar valued Riemannian invariant. The knowledge of the tensor $R \cdot R$ is *equivalent* to the knowledge of the sectional curvatures $L(p,\pi,\bar{\pi})$ of Deszcz. And just like the geometrical interpretation of the sectional curvatures $K(p,\pi)$ of Riemann in terms of the *parallelogramoïds of Levi-Civita* [27], also the sectional curvatures $L(p,\pi,\bar{\pi})$ of Deszcz can be interpreted in these terms (in this respect, we refer to [23] and [24] where in particular such interpretations are obtained for the sectional curvatures as well as for the *Ricci* and *conformal Weyl curvatures* of *Deszcz* in terms of the *squaroïds* of Levi-Civita). Finally the Deszcz symmetric spaces are characterized by the *isotropy* of the curvatures $L(p,\pi,\bar{\pi})$, i.e., by the property that at every point $p$ of $M$ the scalars $L(p,\pi,\bar{\pi})$ are the same for all possible pairs of curvature dependent tangent planes $\pi$ and $\bar{\pi}$ at $p$. In the present situation however there is no lemma of Schur, which then would further force this real valued function

$L: M \to R$ automatically to be constant; therefore, Kowalski and Sekizawa called the pseudo-symmetric spaces for which *the double sectional curvature* $L$ *is* indeed a *constant*, independent of the planes $\pi$ and $\bar{\pi}$ as well as of the points $p$ of $M$, the *pseudo-symmetric spaces of constant type* $L$ [26]. By way of examples in this respect we would like to mention here that the 3-*dimensional Thurston geometries* [34], which in a kind of axiomatic way originated as natural *anisotropic extensions* of the spaces of constant Riemannian curvature $K$ with their typical free mobility, all do have *constant sectional curvature* $L$ *of Deszcz* (we set $L = 0$ for $E^3$ since $(K = c = 0)$, $S^3$ $(K = c > 0)$, $H^3$ $(K = c < 0)$, $S^2 \times E^1$ and $H^2 \times E^1$; $L = 1$ for $\widetilde{SL(2,R)}$ and for the 3-*dimensional Heisenberg group* $H_3$; and $L = -1$ for the Lie group Sol) [2].

A similar study concerning the geometrical meaning of *Ricci pseudo-symmetry in the sense of Deszcz*, i.e., of the manifolds $M$ satisfying the curvature condition $R \cdot S = L_S Q(g,S) = L_S(- \wedge_g \cdot S)$, where $S$ denotes the $(0,2)$ *Ricci curvature tensor* and $Q(g,S) = - \wedge_g \cdot S$ the *Ricci–Tachibana tensor* of $M$ and $L_S$ is a real valued function on $M$, was carried out by Jahanara, Haesen and two of the authors in [25], (in this respect, see also [9] and [15]). As shown in [16], a 3-dimensional Riemannian manifold $M$ is pseudo-symmetric if and only if it is *quasi-Einstein*, i.e., if its Ricci tensor $S$ has an eigenvalue of multiplicity $\geqslant 2$. The class of the Riemannian manifolds $M$ with pseudo-symmetric Ricci tensor $S$ as such is considerably larger in general than the class of the manifolds $M$ with pseudo-symmetric Riemann–Christoffel tensor $R$, (which it obviously contains as a subclass). However, as shown in [10], for manifolds of dimension 3, these two pseudo-symmetry conditions are equivalent. As is well known, Schouten and Struik showed that the 3-dimensional Riemannian manifolds $M$ are *Einstein* if and only if they have constant curvature $K$. In [28] two of the authors made a study of the *pseudo-symmetry in the sense of Deszcz* of the tensors $R$ and $S$ of the Wintgen ideal submanifolds $M^n$ of *dimension* $n > 3$ and of *codimension* 2 in the real space forms $\widetilde{M}^{n+2}(k)$. In particular, they showed that for those Wintgen ideal submanifolds these two, a priori distinct, curvature conditions are equivalent and occur if and only if those submanifolds are either *totally umbilical* or *minimal.* In comparison, the 3-dimensional case will show to offer two additional kinds of pseudo-symmetric Wintgen ideal submanifolds.

## 3. On the symmetry of ideal submanifolds

Let $M^n$ be a *submanifold* of a *real space form* $\widetilde{M}^{n+m}(k)$ of constant curvature $k$. Let $g$, $\nabla$ and $R$, and, respectively, $\widetilde{g}$, $\widetilde{\nabla}$ and $\widetilde{R}$, denote the *Riemannian metric*, the *Levi-Civita connection* and the *Riemann–Cristoffel* $(0,4)$ *curvature tensor* of $M$ and $\tilde{M}$.

The *formulae of Gauss and Weingarten* then are

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \text{and} \quad \widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{3}$$

where $h$, $A_\xi$ and $\nabla^\perp$ denote the *second fundamental form*, the *shape operator* or the *Weingarten map* with respect to $\xi$ and the *normal connection* of $M$ in $\widetilde{M}$, respectively, systematically using here and hereafter $X, Y$, etc. for *tangent vector* *fields* on $M$ and $\xi$ etc. for *normal vector fields* on $M$ in $\tilde{M}$, (as basic references for Riemannian submanifolds, see [5] and [7]).

From (3) it follows that $\widetilde{g}(h(X,Y),\xi) = g(A_\xi(X),Y)$, such that, for any *orthonormal local normal frame* $\xi_\alpha$ on $M$ in $\widetilde{M}$, ($\alpha$ etc. running from 1 till the codimension $m$),

$$h(X,Y) = \sum_\alpha g(A_\alpha(X),Y)\,\xi_\alpha, \tag{4}$$

where $A_\alpha = A_{\xi_\alpha}$. The *mean curvature vector field* $\vec{H}$ of $M$ in $\tilde{M}$ is defined as $\vec{H} = \frac{1}{n}\operatorname{trace} h$ and its length $H = \|\vec{H}\|$ is the *mean curvature* of $M$ in $\widetilde{M}$. By the *equation of Ricci*, the *normal curvature tensor* $R^\perp$ of $M$ in $\widetilde{M}$ is given as follows:

$$R^\perp(X,Y;\xi,\eta) := \widetilde{g}(R^\perp(X,Y)\xi,\eta) = g([A_\xi,A_\eta](X),Y), \tag{5}$$

where

$$R^\perp(X,Y) := \nabla^\perp_X\nabla^\perp_Y - \nabla^\perp_Y\nabla^\perp_X - \nabla^\perp_{[X,Y]} \text{ and } [A_\xi,A_\eta] := A_\xi A_\eta - A_\eta A_\xi.$$

The *normalized scalar normal curvature* $\rho^\perp$ of $M$ in $\tilde{M}$ is then defined by

$$\rho^\perp := \frac{2}{n(n-1)}\left\{\sum_{i<j}\sum_{\alpha<\beta}\left[R^\perp(E_i,E_j;\xi_\alpha,\xi_\beta)\right]^2\right\}^{1/2},$$

for any normal frame $\xi_\alpha$ and for any *orthonormal local tangent frame* $E_i$ on $M$, ($i$ etc. running from 1 till the dimension $n$).

We remark that $\rho^\perp = 0$ if and only if *the normal connection is flat*, which, as follows from (5) and as was already observed by Cartan, is equivalent to the *simultaneous diagonalizability of all shape operators* $A_\xi$. The *equation of Gauss* of $M$ in $\widetilde{M}$ is given by

$$\begin{aligned} R(X,Y,Z,W) &= \widetilde{g}(h(Y,Z),h(X,W)) - \widetilde{g}(h(X,Z),h(Y,W)) \\ &\quad + k\ \{g(Y,Z)\ g(X,W) - g(X,Z)\ g(Y,W)\}. \end{aligned} \tag{6}$$

Let $S$ be the $(0,2)$-*Ricci tensor* of $M$: $S(X,Y) = \sum_i R(E_i,X,Y,E_i)$. Then from (2), (4) and (6), we obtain that

$$\begin{aligned} S(X,Y) &= (n-1)k\,g(X,Y) + \sum_\alpha \operatorname{trace} A_\alpha\, g(A_\alpha(X),Y) \\ &\quad - \sum_\alpha\sum_i g(A_\alpha(X),E_i)\,g(A_\alpha(Y),E_i), \end{aligned} \tag{7}$$

for any choice of frames $E_i$ and $\xi_\alpha$. And the *normalized scalar curvature* $\rho$ of $M$ is defined by

$$\rho := \frac{2}{n(n-1)}\sum_{i<j} R(E_i,E_j,E_j,E_i).$$

Choi and Lu gave the following *affirmative solution of the conjecture concerning the inequality of Wintgen for the* 3-*dimensional submanifolds of the real space forms.*

THEOREM 1. [8] *For any* $M^3$ *in* $\widetilde{M}^{3+m}(k)$, *with* $m \geqslant 3$*:*

$$\rho \leqslant H^2 - \rho^\perp + k, \tag{8}$$

*and the equality holds if and only if, with respect to suitably chosen local orthonormal tangent and normal frames* $E_1, E_2, E_3$ *and* $\xi_1, \dots \xi_m$, *the shape operators* $A_\alpha$ *of* $M$ *in* $\widetilde{M}$ *take the forms:*

$$A_1 = \begin{pmatrix} c & \mu & 0 \\ \mu & c & 0 \\ 0 & 0 & c \end{pmatrix}, \; A_2 = \begin{pmatrix} b+\mu & 0 & 0 \\ 0 & b-\mu & 0 \\ 0 & 0 & b \end{pmatrix}, \; A_3 = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}, \tag{9}$$

$$A_4 = \cdots = A_m = 0,$$

*for some real valued functions* $a, b, c$ *and* $\mu$ *on* $M$.

On the other hand, from the paper of De Smet, Dillen, Vrancken and one of the authors on Wintgen's inequality, we have the following.

THEOREM 2. [12] *For any* $M^3$ *in* $\widetilde{M}^5(k)$, (8) *is satisfied and the equality holds if and only if, with respect to suitably chosen local orthonormal frames* $E_1, E_2, E_3$ *and* $\xi_1, \xi_2$, *the shape operators* $A_1$ *and* $A_2$ *of* $M$ *in* $\widetilde{M}$ *are given by*

$$A_1 = \begin{pmatrix} c & \mu & 0 \\ \mu & c & 0 \\ 0 & 0 & c \end{pmatrix}, \quad A_2 = \begin{pmatrix} \mu & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, for what follows and which concerns essentially dealing with the above shape operators filled into the Gauss equation of $M^3$ in $\widetilde{M}^{3+m}(k)$, the latter situation ($m = 2$) is in some sense algebraically included in the former one ($m > 2$) by considering $a = b = 0$. And, in the case of codimension 1 there is no question about a normal curvature, we can carry out a general study of the 3-dimensional Wintgen ideal submanifolds $M^3$ in arbitrary space forms $\widetilde{M}^{3+m}(k)$, $m \geqslant 2$, by dealing with the forms of the shape operators as given in (9). Frames $E_1, E_2, E_3$ and $\xi_1, \dots, \xi_m$ for which the corresponding shape operators $A_\alpha$ assume such forms which further on will be called *Choi–Lu frames* of Wintgen ideal $M^3$ in $\tilde{M}^{3+m}$.

From (7) and (9), the $(1,1)$ *Ricci operator* $S$ which is metrically related to the $(0,2)$ Ricci tensor $S$ by $g(S(X), Y) = S(X, Y)$, with respect to Choi–Lu frames $E_i$ and $\xi_\alpha$ is readily found to be given by

$$S = \begin{pmatrix} S_{11} & c\mu & 0 \\ c\mu & S_{22} & 0 \\ 0 & 0 & S_{33} \end{pmatrix}, \tag{10}$$

where

$$\begin{aligned} S_{11} &= 2(a^2 + b^2 + c^2 + k) + \mu(b - 2\mu), \\ S_{22} &= 2(a^2 + b^2 + c^2 + k) - \mu(b + 2\mu), \\ S_{33} &= 2(a^2 + b^2 + c^2 + k). \end{aligned} \tag{11}$$

Hence $E_3$ always determines a *Ricci principal direction*, with a corresponding *Ricci curvature*

$$\rho_3 = 2\,(a^2 + b^2 + c^2 + k). \tag{12}$$

And, from (10) and (11), the other two *Ricci curvatures*, $\rho_1$ and $\rho_2$, corresponding to some orthogonal eigendirections $\widetilde{E}_1$ and $\widetilde{E}_2$ in the plane field $E_1 \wedge E_2$, are easily derived as

$$\begin{aligned} \rho_1 &= \rho_3 - 2\mu^2 + |\mu|\sqrt{b^2 + c^2}, \\ \rho_2 &= \rho_3 - 2\mu^2 - |\mu|\sqrt{b^2 + c^2}. \end{aligned} \tag{13}$$

Since $M^3$ is *pseudo-symmetric* or still, *symmetric in the sense of Deszcz*, if and only if its Ricci tensor has an eigenvalue of multiplicity $\geqslant 2$, from (13) we have the following.

Lemma 1. *A Wintgen ideal* 3*-dimensional submanifold in a real space form is Deszcz symmetric if and only if*

$$\text{(I)}\ \ \mu = 0 \quad \textit{or} \quad \text{(II)}\ \ \mu \neq 0,\ b = c = 0 \quad \textit{or} \quad \text{(III)}\ \ \mu \neq 0,\ b^2 + c^2 = 4\mu^2.$$

Proceeding more straightforwardly, from (6) and (9) the components, with respect to Choi–Lu frames, of the $(0,4)$ *curvature tensor* $R$ of a Wintgen ideal $M^3$ in $\widetilde{M}^{3+m}(k)$ are readily found to be either zero or else to be completely determined, via the algebraic symmetries of $R$, by the following ones:

$$\begin{aligned} \alpha &:= K_{12} = R_{1221} = a^2 + b^2 + c^2 - 2\mu^2 + k, \\ \beta &:= K_{13} = R_{1331} = a^2 + b^2 + c^2 + b\mu + k, \\ \gamma &:= K_{23} = R_{2332} = a^2 + b^2 + c^2 - b\mu + k, \\ \delta &:= R_{1332} = c\mu, \end{aligned} \tag{14}$$

($K_{ij} = K(E_i \wedge E_j)$ are the *sectional curvatures* of $M$ for the 2-planes $E_i \wedge E_j$ determined by a Choi–Lu frame $E_1, E_2, E_3$). And then also the components of the $(0,6)$ tensors $R \cdot R$ and $\wedge_g \cdot R$ are readily computable, and turn out either to be zero "together" or, when at least a priori non-zero, they appear in pairs which are, via algebraic symmetries of both these $(0,6)$ tensors (cf. [22]), completely determined by the following ones:

$$\begin{aligned} (R \cdot R)(E_1, E_3, E_1, E_3; E_1, E_2) &= -2\alpha\delta, \\ (\wedge_g \cdot R)(E_1, E_3, E_1, E_3; E_1, E_2) &= -2\delta; \\ (R \cdot R)(E_1, E_3, E_2, E_3; E_1, E_2) &= \alpha(\beta - \gamma), \\ (\wedge_g \cdot R)(E_1, E_3, E_2, E_3; E_1, E_2) &= \beta - \gamma; \\ (R \cdot R)(E_1, E_2, E_2, E_3; E_1, E_3) &= -\alpha\beta + \beta\gamma - \delta^2, \\ (\wedge_g \cdot R)(E_1, E_2, E_2, E_3; E_1, E_3) &= \gamma - \alpha. \end{aligned} \tag{15}$$

Then, by (14) and (15), the condition for the *pseudo-symmetry of* $R$, i.e., of the existence of a function $L\colon\ M \to R$ for which $R \cdot R = L\,(- \wedge_g \cdot R)$, for Wintgen ideal submanifolds $M^3$ in $\widetilde{M}^{3+m}(k)$, yields, of course, the previous cases (I, II, and III) of Lemma 1, but moreover in each case at the point $p$ gives the *double sectional curvature function* $L$ as stated in the following.

LEMMA 2. *We have*

$$\begin{aligned} L &= 0 && \textit{in case } (\mathrm{I}), \\ L &= a^2 + k && \textit{in case } (\mathrm{II}), \\ L &= a^2 + 2\mu^2 + k && \textit{in case } (\mathrm{III}). \end{aligned}$$

In case (I), $M^3$ is Einstein and thus a real space form and so, in particular, $M^3$ is then semi-symmetric and hence (also without calculations, one could know that) $L = 0$. In cases (II) and (III), according to the general theory in this respect, if $M^3$ is quasi-Einstein and has $\lambda$ as an *eigenvalue of the Ricci tensor $S$ of multiplicity* 1, then $L = \frac{\lambda}{2}$, which allows to obtain Lemma 2 also from the consideration of (12) and (13).

Now we aim to geometrically characterize the cases (I), (II), and (III). Clearly, from (9), we see that (I) corresponds to the *totally umbilical* Wintgen ideal submanifolds $M^3$ in real space forms $\widetilde{M}^{3+m}(k)$; such $M^3$ are intrinsically itself spaces of constant curvature $K = a^2 + b^2 + c^2 + k$. We recall that a submanifold $M^n$ in a Riemannian manifold $\widetilde{M}^{n+m}$ is said to be *pseudo-umbilical* if its *mean curvature vector field* $\vec{H}$ determines an *umbilical* normal direction on $M$ in $\tilde{M}$. When hereafter we call a submanifold pseudo-umbilical we mean it to be *properly* pseudo-umbilical, i.e., we exclude from it the trivial cases when it is *minimal* ($\vec{H} = \vec{0}$), or when it is *totally umbilical* (i.e., when every normal direction $\xi$ is umbilical). The mean curvature vector field $\vec{H}$ for the submanifolds $M^3$ under consideration being given by $\vec{H} = c\xi_1 + b\xi_2 + a\xi_3$, it further follows from (9) that the shape operator of $M$ in $\widetilde{M}$ with respect to $\vec{H}$ is given by

$$A_{\vec{H}} = \begin{pmatrix} a^2 + b^2 + c^2 + b\mu & c\mu & 0 \\ c\mu & a^2 + b^2 + c^2 - b\mu & 0 \\ 0 & 0 & a^2 + b^2 + c^2 \end{pmatrix}. \tag{16}$$

In case $M$ is *not totally umbilical* in $\widetilde{M}$, i.e., in case $\mu \neq 0$, and in case $M$ is *not minimal* in $\widetilde{M}$, i.e., in case not $a = b = c = 0$ (cf. (9)), then (16) shows that $\vec{H}$ determines an *umbilical normal direction* of $M$ in $\widetilde{M}$ if and only if $b = c = 0$. In summary, from the above we know that cases (I) and (II) correspond to the Wintgen ideal submanifolds which are *totally umbilical* or *minimal* or *pseudo-umbilical*. Finally, we next aim for a geometrical characterisation of case (III): $b^2 + c^2 = 4\mu^2$ where $\mu \neq 0$. To simplify a bit the discussion, we will assume from now on that $\mu > 0$, which we can do without loss of generality, being always realizable in view of (9) by eventual changing orientations of $\xi_\alpha$'s and orderings of $E_1$ and $E_2$). From (12) and (13) we recall that the eigenvalues of the Ricci tensor are given by

$$\begin{aligned}\rho_1 &= 2\,(a^2+b^2+c^2+k) - 2\mu^2 + \mu\sqrt{b^2+c^2},\\ \rho_2 &= 2\,(a^2+b^2+c^2+k) - 2\mu^2 - \mu\sqrt{b^2+c^2},\\ \rho_3 &= 2\,(a^2+b^2+c^2+k),\end{aligned} \tag{17}$$

where $\rho_3$ is the one in the $E_3$-direction of $M$ and that $\rho_1$ and $\rho_2$ are the eigenvalues corresponding to certain eigendirections in the plane $\pi = E_1 \wedge E_2$ perpendicular to $E_3$ and of which the special character is reflected obviously, having $\mu \neq 0$, in the form of the shape operators given in (9), and which we further will call the *Choi–Lu plane* of $M^3$. Studying at present $M^3$'s which are not totally umbilical, in particular, these submanifolds are not Einstein, and so, at every point, they have a *smallest* Ricci curvature, which we'll denote by $\inf \mathrm{Ric}$. From (17) it is clear this is $\rho_2$:

$$\inf \mathrm{Ric} = 2(a^2+b^2+c^2+k) - 2\mu^2 - \mu\sqrt{b^2+c^2},$$

attained by a particular direction in the Choi–Lu plane of $M^3$. On the other hand, we recall from (14) that the *sectional curvature* $K_{\text{Choi–Lu}}$ *of the Choi–Lu plane* is given by

$$K_{\text{Choi–Lu}} = K_{12} = a^2+b^2+c^2+k-2\mu^2.$$

Hence $\inf \mathrm{Ric} = 2\,K_{\text{Choi–Lu}}$ if and only if $-2\mu^2 - \mu\sqrt{b^2+c^2} = -4\mu^2$, i.e., if and only if (III) holds, namely when $b^2+c^2 = 4\mu^2$. From (14) we moreover see that, in this case,

$$\begin{aligned}K_{12} &= K_{\text{Choi–Lu}} = a^2+2\mu^2+k,\\ K_{13} &= K_{\text{Choi–Lu}} + \mu(2\mu+b),\\ K_{23} &= K_{\text{Choi–Lu}} + \mu(2\mu-b),\end{aligned}$$

which implies, obviously having also that $b^2 \leqslant 4\mu^2$ and so, since $\mu > 0$, that

$$-2\mu \leqslant b \leqslant 2\mu,$$

and thus that as well $0 \leqslant 2\mu + b$ as $0 \leqslant 2\mu - b$, together with the equation of Gauss and (9), that in case (III) the sectional curvature $K_{12}$ actually equals $\inf K$, the function on $M$ giving the *minimum of the sectional curvatures* $K$ at each point of $M^3$. So, in this situation we can observe on the side that the $\delta(2)$-*curvature of Chen* [7], $\delta(2) := \tau - \inf K$, where $\tau := \sum_{i<j} K_{ij}$ is the *scalar curvature* of $M^3$, is given by $\delta(2) = \tau - K_{12} = K_{13} + K_{23} = \rho_3 = \rho_1$ which, in this case, is $\sup \mathrm{Ric}$, the real valued function on $M$ giving the maximum Ricci curvature at each point of $M^3$. Once more, in accordance with the general theory of Deszcz symmetric 3-dimensional Riemannian manifolds $M^3$, for the properly quasi-Einstein manifolds $M^3$ the sectional curvature $L$ of Deszcz satisfies $L = \frac{\lambda}{2}$, where $\lambda$ is the principal curvature with multiplicity 1, so actually $\lambda = \rho_2 = 2(a^2+2\mu^2+k)$, such that

$$L = K_{\text{Choi–Lu}} = \inf K.$$

In this respect, coming back to case (II), from (14) and taking further into account (9) and the equation of Gauss, it follows that in this case, since $K_{12} = a^2 + k - 2\mu^2$,

$$L = a^2 + k = K_{23} = K_{13} = \sup K,$$

$\sup K$ denoting the *maximum of the sectional curvature function* on $M$, i.e., the function on $M$ which value at each point is the maximum of all the sectional curvatures of $M$ at this point. Taking into account at last also the case (III) of Lemma 2, in summary we can formulate the following.

THEOREM 3. *A Wintgen ideal submanifold $M^3$ in a real space form $\widetilde{M}^{3+m}(k)$ is Deszcz symmetric if and only if* (I) *$M^3$ is a totally umbilical submanifold with Deszcz sectional curvature $L = 0$ ($M^3$ then being a space of constant sectional curvature $K$), or,* (II) *$M^3$ is a minimal submanifold or a pseudo-umbilical submanifold, with Deszcz sectional curvature $L = \sup K$, or else,* (III) *$M^3$ is characterized by the curvature condition* $\inf \mathrm{Ric} = 2\,K_{\text{Choi–Lu}} = 2\,\inf K$, *with Deszcz sectional curvature $L = \inf K$.*

## 4. Further comments and remarks

(1) We would like to refer again to the references mentioned in Section 1 for explicit descriptions of several *examples* of Wintgen ideal 3-dimensional submanifolds.

(2) Refering amongst others to Berger's discussion in his "Panorama" [3] pertaining to the *extremal values of the sectional curvature function $K$* of Riemannian manifolds, we observe from Theorem 3 that for the nontrivial Wintgen ideal submanifolds $M^3$ in $\widetilde{M}^{3+m}(k)$, i.e., the nontotally umbilical ones, *the isotropic Deszcz sectional curvatures $L$* are either given by the maximum or by the minimum values of $K$ at each point. The Deszcz symmetry of those submanifolds $M^3$ being equivalent to being quasi-Einstein, in the above nontrivial case, $L = \sup K$ or $L = \inf K$ according to the geometrical fact that *the eigendirection of the Ricci tensor whose eigenvalue has multiplicity* 1 *is either perpendicular to the plane of Choi–Lu of these Wintgen ideal submanifolds $M^3$ or belongs to this plane.*

(3) Concerning the *origin of the Ricci tensor* of 3-dimensional Riemannian manifolds and some related views on the $\delta$-curvatures of Chen, see [25].

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