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A COUNTEREXAMPLE ON NONTANGENTIAL CONVERGENCE FOR OSCILLATORY INTEGRALS

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Abstract. Consider the solution of the time-dependent Schrödinger equation with initial data f. It is shown by Sjögren and Sjölin (1989) that there exists f in the Sobolev space $H^s(\mathbf{R}^n)$, $s = n/2$ such that tangential convergence can not be widened to convergence regions. In this paper we show that the corresponding result holds when $-\Delta_x$ is replaced by an operator $\varphi(D)$, with special conditions on φ.

1. Introduction

In this paper we establish non-existence results of non-tangential convergence for the solution $u = S^\varphi f$ to the generalized time-dependent Schrödinger equation

$$(\varphi(D) + i\partial_t)u = 0, \tag{1.1}$$

with the initial condition

$$u(x,0) = f(x).$$

Here φ is real-valued, and its radial derivatives of first and second orders ($\varphi' = \varphi'_r$ and $\varphi'' = \varphi''_{rr}$) are continuous, outside a compact set containing origin, and fulfill appropriate growth conditions. In particular $\varphi(\xi) = |\xi|^a$ will satisfy these conditions, for $a > 1$.

For $\varphi(\xi) = |\xi|^2$ we recover Theorem 3 in [**3**], where Sjögren and Sjölin proved that there exists a function f in the Sobolev space $H^{n/2}$ such that near the vertical line $t \mapsto (x,t)$ through an arbitrary point $(x,0)$ there are points accumulating at $(x,0)$ such that the solution of equation (1.1) takes values far from f. This means that the solution of the time-dependent Schrödinger equation with initial condition $u(x,0) = f(x)$ does not converge non-tangentially to f. Therefore we can not

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consider regions of convergence. As a consequence of our results, it follows that Theorem 3 in [**3**] holds for any $a > 1$.

In this paper, we prove that the property which holds for $\varphi(\xi) = |\xi|^2$ also holds for more general functions $\varphi(\xi)$ of the type described above. In the proof we use some ideas by Sjögren and Sjölin from [**3**] in combination with new estimates, to construct a counterexample. Some ideas can also be found in Sjölin [**5, 6**] and Walther [**9, 10**], and some related results are given in Bourgain [**1**], Kenig, Ponce and Vega [**2**], and Sjölin [**4, 7**].

Existence of regions of convergence has been studied before for other equations. For example, Stein and Weiss consider in [**8**, Chapter II Theorem 3.16] Poisson integrals acting on Lebesgue spaces. These operators are related to the operator S^{φ}.

For an appropriate function φ on $\mathbf{R}^n$, let S^{φ} be the operator acting on functions f defined by

$$f \mapsto \mathcal{F}^{-1}(\exp(it\varphi(\xi))\mathcal{F}f), \tag{1.2}$$

where $\mathcal{F}f$ is the Fourier transform of f, which takes the form

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) \equiv \int_{\mathbf{R}^n} e^{-ix\cdot\xi} f(x)\,dx,$$

when $f \in L^1(\mathbf{R}^n)$. This means that, if $\widehat{f}$ is an integrable function, then S^{φ} in (1.2) takes the form

$$S^{\varphi}f(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{ix\cdot\xi}e^{it\varphi(\xi)}\widehat{f}(\xi)\,d\xi, \quad x \in \mathbf{R}^n,\ t \in \mathbf{R}.$$

If $\varphi(\xi) = |\xi|^2$ and f belongs to the Schwartz class $\mathcal{S}(\mathbf{R}^n)$, then $S^{\varphi}f$ is the solution to the time-dependent Schrödinger equation $(-\Delta_x + i\partial_t)u = 0$ with the initial condition $u(x,0) = f(x)$.

For more general appropriate φ, for which the equation (1.1) is well-defined, the expression $S^{\varphi}f$ is the solution to the generalized time-dependent Schrödinger equation (1.1) with the initial condition $u(x,0) = f(x)$. Note here that $S^{\varphi}f$ is well defined for any real-valued measurable φ and $f \in \mathcal{S}$. On the other hand, it might be difficult to interpret (1.1) if for example $\varphi \notin L^1_{\text{loc}}$.

In order to state the main result we need to specify the conditions on φ and give some definitions. The function φ should satisfy the conditions

$$\liminf_{r\to\infty}\big(\inf_{|\omega|=1}|\varphi'(r,\omega)|\big) = \infty, \tag{1.3}$$

$$\sup_{r\geqslant R}\left(\sup_{|\omega|=1}\frac{r^{\beta}|\varphi''(r,\omega)|}{|\varphi'(r,\omega)|^2}\right) < C, \tag{1.4}$$

for some $\beta > 0$ and some constant C. Here $\varphi'(r\omega) = \varphi'(r,\omega)$ denotes the derivative of $\varphi(r,\omega)$ with respect to r, and similarly for higher orders of derivatives.

We let $H^s(\mathbf{R}^n)$ be the Sobolev space of distributions with $s \in \mathbf{R}$ derivatives in L^2. That is $H^s(\mathbf{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbf{R}^n)$ such that

$$\|f\|_{H^s(\mathbf{R}^n)} \equiv \left(\int_{\mathbf{R}^n}(1+|\xi|^2)^s|\widehat{f}(\xi)|^2 d\xi\right)^{1/2} < \infty. \tag{1.5}$$

THEOREM 1.1. *Assume that the function* $\gamma : \mathbf{R}_+ \to \mathbf{R}_+$ *is strictly increasing and continuous such that* $\gamma(0) = 0$. *Let* $R > 0$, *and let* φ *be real-valued function on* $\mathbf{R}^n$ *such that* $\varphi'(r, \omega)$ *and* $\varphi''(r, \omega)$ *are continuous and satisfy* (1.3) *and* (1.4) *when* $r > R$. *Then there exists a function* $f \in H^{n/2}(\mathbf{R}^n)$ *such that* $S^\varphi f$ *is continuous in* $\{(x, t); t > 0\}$ *and*

$$\limsup_{(y,t)\to(x,0)} |S^\varphi f(y, t)| = +\infty \tag{1.6}$$

for all $x \in \mathbf{R}^n$, *where the limit superior is taken over those* (y, t) *for which* $|y - x| < \gamma(t)$ *and* $t > 0$.

Here we recall that $\varphi' = \varphi'_r$ and $\varphi' = \varphi''_{rr}$ are the first and second orders radial derivatives of φ. When $s > n/2$ no counterexample of the form in Theorem 1.1 can be provided, since $S^\varphi f(y, t)$ converges to $f(x)$ as (y, t) approaches $(x, 0)$ non-tangentially when $f \in H^s(\mathbf{R}^n)$. In fact, Hölder's inequality gives

$$(2\pi)^n |S^\varphi f(x, t)| \leqslant \int_{\mathbf{R}^n} |\widehat{f}(\xi)|\, d\xi \leqslant \left(\int_{\mathbf{R}^n} (1 + |\xi|^2)^{-s}\, d\xi \right) \|f\|_{H^s(\mathbf{R}^n)},$$

which is finite when $f \in H^s(\mathbf{R}^n)$, $s > n/2$. Therefore the convergence along vertical lines can be extended to convergence regions when $s > n/2$ and f belongs to $H^s(\mathbf{R}^n)$.

2. Examples and remarks

In this section we give some examples of functions φ for which Theorem 1.1 holds. In the first example we let φ be a positively homogeneous function of order $a > 1$.

EXAMPLE 2.1. Let $a > 1$ and $\varphi(\xi) = |\xi|^a$; then $S^\varphi f(x, t)$ is the solution to the generalized time-dependent Schrödinger equation $((-\Delta_x)^{a/2} + i\partial_t)u = 0$. By change of variables to polar coordinates and derivate with respect to r we see that $\varphi(r, \omega) = r^a$, $\varphi'(r, \omega) = ar^{a-1}$ and $\varphi''(r, \omega) = a(a - 1)r^{a-2}$. We can see that these derivatives satisfy (1.3) and (1.4). In particular for $a = 2$ this is the solution to the time-dependent Schrödinger equation $(-\Delta_x + i\partial_t)u = 0$ and this case is treated in Sjögren and Sjölin [**3**].

In the following example we let φ be a sum of positively homogeneous functions, where $a > 1$ denotes the term of the highest order.

EXAMPLE 2.2. For $a > 1$, let $\varphi(\xi) = \sum_{i=1}^{d} |\xi|^{a_i} \varphi_{a,i}(1, \omega)$, $a_1 < \cdots < a_d = a$, where $\inf_\omega |\varphi_{a,d}(1, \omega)| = h > 0$ and $\|\varphi_{a,i}(1, \cdot)\|_{L^\infty(S^{n-1})} < \infty$ for each $i \in \{1, 2, ..., d\}$. Here S^{n-1} is the $(n-1)$-dimensional unit sphere. By rewriting this into polar coordinates and differentiating with respect to r, we see that in the first derivative the term $\varphi_{a,i}(1, \omega)r^{a-1}$ dominates the sum and that the second derivative can be estimated by Cr^{a-2}, for some constant C. These derivatives satisfy (1.3) and (1.4).

In the examples given above we have used functions φ such that the modulus of the radial derivative is bounded from below by a positive homogeneous function of order $a - 1$ for some $a > 1$. This condition is not necessary. The assumption in the theorem permits a broader class of functions φ. The following example shows that there are functions, which do not grow as fast as a positive homogeneous function of order $a-1$ for any $a>1$, but satisfy the conditions (1.3) and (1.4).

EXAMPLE 2.3. Let $\varphi(\xi)=|\xi|\log|\xi|$, then $\varphi'(r,\omega)=\log r+1$ and $\varphi''(r,\omega)=r^{-1}$ and (1.3) and (1.4) are satisfied.

We also allow the dominant part of the derivative to grow faster than any positively homogeneous function as long as we have some restrictions on the second derivative. The conditions are given explicitly in (1.3) and (1.4). The following example contains such functions.

EXAMPLE 2.4. Let $\varphi(\xi)=\varphi(r,\omega)=e^{\mu(\omega)r^b}$, where $\inf_{|\omega|=1}\mu(\omega)=c>0$ and $0<b\leqslant 2$. Here one should note that $\varphi\notin\mathcal{S}'(\mathbf{R}^n)$. These functions grow faster than r^a for all a and the same is true for the absolute value of the first and second derivatives with respect to r. This can be used to show that (1.3) and (1.4) are satisfied.

3. Notations for the proof

In order to prove Theorem 1.1 we introduce some notations. Let $B_r(x)$ be the open ball in $\mathbf{R}^n$ with center at x and radius r. Numbers denoted by C, c or C' may be different at each occurrence. We let

$$\delta_k=\delta_{k,n}\equiv\gamma(1/(k+1))/\sqrt{n},\qquad k\in\mathbf{N},$$

where γ is the same as in Theorem 1.1. Since γ is strictly increasing it is clear that $(\delta_k)_{k\in\mathbf{N}}$ is strictly decreasing. We also let $(x_j)_{j=1}^{\infty}\subset\mathbf{R}^n$ be chosen such that $x_1,x_2,\dots,x_{m_1}$ denotes all points in $B_1(0)\cap\delta_1\mathbf{Z}^n$, $x_{m_1+1},\dots,x_{m_2}$ denotes all points in $B_2(0)\cap\delta_2\mathbf{Z}^n$ and generally $\{x_{m_k+1},\dots,x_{m_{k+1}}\}=B_{k+1}(0)\cap\delta_{k+1}\mathbf{Z}^n$, for $k\geqslant 1$. Furthermore we choose a strictly decreasing sequence $(t_j)_1^{\infty}$ such that $1>t_1>t_2>\cdots>0$ and

$$\frac{1}{k+2}<t_j<\frac{1}{k+1},\qquad k\in\mathbf{N},$$

for $m_k+1\leqslant j\leqslant m_{k+1}$.

In the proof of Theorem 1.1 we consider the function f_φ, which is defined by the formula

$$\widehat{f_\varphi}(\xi)=|\xi|^{-n}(\log|\xi|)^{-3/4}\sum_{j=1}^{\infty}\chi_j(\xi)e^{-i(x_j\cdot\xi+t_j\varphi(\xi))},\tag{3.1}$$

where χ_j is the characteristic function of $\Omega_j=\{\xi\in\mathbf{R}^n;R_j<|\xi|<R_j'\}$.

Here $(R_j)_1^{\infty}$ and $(R_j')_1^{\infty}$ are sequences in $\mathbf{R}$ which fulfill the following conditions:

(1) $R_1\geqslant 2+R$, $R_1'\geqslant R_1+1$, with R given by Theorem 1.1;
(2) $R_j'=R_j^N$ when $j\geqslant 2$, where N is a large positive number and independent of j, which is specified later on;
(3) $R_j<R_j'<R_{j+1}$, when $j\geqslant 1$;
(4)

$$|\varphi'(r,\omega)|>1\qquad\text{when}\qquad r\geqslant R;\tag{3.2}$$

(5) for $j \geqslant 2$

$$R_j^{\min(\beta,1)} > \max_{l<j} \frac{2^j}{t_l - t_j}, \tag{3.3}$$

where $\beta > 0$ is the same constant as in (1.4) and

$$\inf_{R_j \leqslant r \leqslant R'_j} \big(\inf_{|\omega|=1} |\varphi'(r,\omega)| \big) > \max_{l<j} \frac{2|x_l - x_j|}{t_l - t_j}; \tag{3.4}$$

REMARK 3.1. The sequences $(R_j)_1^\infty$ and $(R'_j)_1^\infty$ can be chosen since φ satisfies condition (1.3).

Furthermore, in order to get convenient approximations of the operator S^φ, we let

$$S_m^\varphi f(x,t) = \frac{1}{(2\pi)^n} \int_{|\xi|<R'_m} e^{ix\cdot\xi} e^{it\varphi(\xi)} \widehat{f}(\xi)\, d\xi. \tag{3.5}$$

Then

$$S_m^\varphi f_\varphi(x,t) = \sum_{j=1}^m A_j^\varphi(x,t), \tag{3.6}$$

where

$$A_j^\varphi(x,t) = \frac{1}{(2\pi)^n} \int_{\Omega_j} e^{i(x-x_j)\cdot\xi} e^{i(t-t_j)\varphi(\xi)} |\xi|^{-n} (\log|\xi|)^{-3/4}\, d\xi. \tag{3.7}$$

By using polar coordinates we get

$$A_j^\varphi(x_k,t_k) = \frac{1}{(2\pi)^n} \int_{|\omega|=1} \left\{ \int_{R_j}^{R'_j} \frac{1}{r(\log r)^{3/4}} e^{iF_\varphi(r,\omega)}\, dr \right\} d\sigma(\omega), \tag{3.8}$$

where $F_\varphi(r,\omega) = r(x_k - x_j)\cdot\omega + (t_k - t_j)\varphi(r,\omega)$, and $d\sigma(\omega)$ is the Euclidean surface measure on the $(n-1)$-dimensional unit sphere. By differentiation we get

$$F'_\varphi(r,\omega) = (x_k - x_j)\cdot\omega + (t_k - t_j)\varphi'(r,\omega), \tag{3.9}$$

$$F''_\varphi(r,\omega) = (t_k - t_j)\varphi''(r,\omega). \tag{3.10}$$

Here recall that $F'_\varphi(r\omega) = F'_\varphi(r,\omega)$ and $F''_\varphi(r,\omega)$ denote the first and second orders of derivatives of $F_\varphi(r,\omega)$ with respect to the r-variable.

By integration by parts in the inner integral of (3.8) we get

$$\int_{R_j}^{R'_j} \frac{1}{r(\log r)^{3/4}} e^{iF_\varphi(r,\omega)}\, dr = A_\varphi - B_\varphi, \tag{3.11}$$

where

$$A_\varphi = \Big[\frac{e^{iF_\varphi(r,\omega)}}{r(\log r)^{3/4} iF'_\varphi(r,\omega)} \Big]_{R_j}^{R'_j}, \tag{3.12}$$

$$B_\varphi = \int_{R_j}^{R'_j} \frac{d}{dr} \Big(\frac{1}{r(\log r)^{3/4} iF'_\varphi(r,\omega)} \Big) e^{iF_\varphi(r,\omega)}\, dr \tag{3.13}$$

4. Proofs

In this section we prove Theorem 1.1. We need some preparing lemmas for the proof. In the following lemma we prove that for fixed $x \in B_k(0)$ there exists sequences $(x_{n_j})_1^\infty$ and $(t_{n_j})_1^\infty$ such that

$$x_{n_j} \in \{x_{m_k+1}, \ldots, x_{m_{k+1}}\}, \qquad \text{and} \qquad t_{n_j} \in \{t_{m_k+1}, \ldots, t_{m_{k+1}}\}$$

and $|x_{n_j} - x| < \gamma(t_{n_j})$.

Lemma 4.1. *Let $x \in \mathbf{R}^n$ be fixed. Then for each $k \geqslant |x|$ there exists $x_{n_j} \in \{x_{m_k+1}, \ldots, x_{m_{k+1}}\}$ and $t_{n_j} \in \{t_{m_k+1}, \ldots, t_{m_{k+1}}\}$ such that $|x_{n_j} - x| < \gamma(t_{n_j})$. In particular $(x_{n_j}, t_{n_j}) \to (x, 0)$ as j turns to infinity.*

Proof. For each $k \geqslant |x|$, x belongs to a cube with vertices in $T_k = B_{k+1}(0) \cap \delta_{k+1}\mathbf{Z}^n$ and side lengths $\gamma(1/(k+2))/\sqrt{n}$. Take a vertex x' in the cube and its diagonal $\gamma(1/(k+2))$ as center and radius of a ball respectively. This ball $B_{\gamma(1/(k+2))}(x')$ contains the whole cube and hence also x. Therefore there exists x_{n_j} for every $k \geqslant |x|$ such that $x \in B_{\gamma(1/(k+2))}(x_{n_j}) \subset B_{\gamma(t_{n_j})}(x_{n_j})$. This proves the first part of the assertion, and the second statement follows from the fact that $\gamma(0) = 0$ and γ is continuous and strictly increasing. $\square$

We want to prove that f_φ in (3.1) belongs to $H^{n/2}(\mathbf{R}^n)$ and fulfills (1.6). The former relation is a consequence of Lemma 4.2 below, which concerns the Sobolev space properties for functions of the form

$$\widehat{g}(\xi) = |\xi|^{-n}(\log|\xi|)^{-\rho/2} \sum_{j=1}^{\infty} \chi_j(\xi) b_j(\xi), \tag{4.1}$$

where χ_j is the characteristic function on disjoint sets Ω_j.

Lemma 4.2. *Assume that $\rho > 1$, Ω_j for $j \in \mathbf{N}$ are disjoint open subsets of $\mathbf{R}^n \setminus B_\rho(0)$, $b_j \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ for $j \in \mathbf{N}$ satisfies $\sup_{j\in\mathbf{N}} \|b_j\|_{L^\infty(\Omega_j)} < \infty$, and let χ_j be the characteristic function for Ω_j. If g is given by (4.1), then $g \in H^{n/2}(\mathbf{R}^n)$.*

Proof. By estimating (1.5) for the function g we get that

$$\begin{aligned}\int_{\mathbf{R}^n} |\widehat{g}(\xi)|^2 (1+|\xi|^2)^{n/2}\, d\xi &\leqslant C \int_{\mathbf{R}^n \setminus B_\rho(0)} |\xi|^{-2n} (\log|\xi|)^{-\rho} (1+|\xi|^2)^{n/2}\, d\xi \\ &\leqslant 2^{n/2} C \int_\rho^\infty \frac{1}{r(\log r)^\rho}\, dr < \infty.\end{aligned}$$

The second inequality holds since $(1+r^2)^{n/2} < (r^2+r^2)^{n/2} = 2^{n/2} r^n$ for $r > 1$. $\square$

In the following lemma we give estimates of the expression A_j^φ.

Lemma 4.3. *Let $A_j^\varphi(x,t)$ be given by (3.7). Then the following is true:*

(1) $\displaystyle\sum_{j=1}^{k-1} |A_j^\varphi(x,t)| \leqslant C(\log R'_{k-1})^{1/4}$, *with C independent of k;*

(2) $A_k^\varphi(x_k, t_k) > c(\log R'_k)^{1/4}$, *with $c > 0$ independent of k.*

PROOF. (1) By triangle inequality and the fact that $|\xi| > 2$, when $\xi \in \Omega_j$, we get

$$\sum_{j=1}^{k-1} |A_j^\varphi(x,t)| \leqslant \frac{1}{(2\pi)^n} \int_{2\leqslant|\xi|\leqslant R'_{k-1}} |\xi|^{-n}(\log|\xi|)^{-3/4}\,d\xi$$
$$= C\int_2^{R'_{k-1}} \frac{1}{r(\log r)^{3/4}}\,dr \leqslant C(\log R'_{k-1})^{1/4},$$

where C is independent of k. In the last equality we have taken the polar coordinates as new variables of integration.

(2) Since $R_j^N = R'_j$ for N sufficiently large, we get

$$A_k^\varphi(x_k,t_k) = C\int_{R_k}^{R'_k} \frac{1}{r(\log r)^{3/4}}\,dr = C\Big((\log R'_k)^{1/4} - \big(\log(R'_k)^{1/N}\big)^{1/4}\Big)$$
$$= C\Big(1 - \frac{1}{N^{1/4}}\Big)(\log R'_k)^{1/4} > c(\log R'_k)^{1/4},$$

for some constant $c > 0$, which is independent of k. □

LEMMA 4.4. *Assume that $S_m^\varphi f_\varphi$ is given by* (3.5). *Then $S_m^\varphi f_\varphi$ is continuous on* $\{(x,t); t>0, x\in\mathbf{R}^n\}$.

PROOF. The continuity for each $S_m^\varphi f_\varphi$ follows from the facts, that for almost every $\xi\in\mathbf{R}^n$, the map $(x,t)\mapsto e^{ix\cdot\xi}e^{it\varphi(\xi)}\widehat{f}_\varphi(\xi)$ is continuous, and that

$$\int_{|\xi|<R'_m} |e^{ix\cdot\xi}e^{it\varphi(\xi)}\widehat{f}_\varphi(\xi)|\,d\xi = \int_{|\xi|<R'_m} |\widehat{f}_\varphi(\xi)|\,d\xi < C.$$ □

When proving Theorem 1.1, we first prove that the modulus of $S_m^\varphi f_\varphi(x_k,t_k)$ turns to infinity as k goes to infinity. For this reason we note that the triangle inequality and (3.6) implies that

$$|S_m^\varphi f_\varphi(x_k,t_k)| \geqslant \big|A_k^\varphi(x_k,t_k)\big| - \Big|\sum_{j=1}^{k-1} A_j^\varphi(x_k,t_k)\Big| - \Big|\sum_{j=k+1}^{m} A_j^\varphi(x_k,t_k)\Big|, \tag{4.2}$$

where $m > k$. We want to estimate the terms in (4.2). From Lemma 4.3 we get estimates for the first two terms. It remains to estimate the last term.

PROOF OF THEOREM 1.1. **Step 1**. For $j > k \geqslant 2$ we shall estimate $|A_j^\varphi(x_k,t_k)|$ in (3.8). We have to find appropriate estimates for A_φ and B_φ in (3.11)–(3.13). By using $t_k - t_j > 0$ and $R_j < r < R'_j$ it follows from (3.4), (3.9), triangle inequality and the Cauchy–Schwarz inequality that

$$\begin{aligned}|F'_\varphi(r,\omega)| &\geqslant (t_k-t_j)|\varphi'(r,\omega)| - |x_k - x_j| \\ &> (t_k-t_j)|\varphi'(r,\omega)| - (t_k-t_j)\frac{|\varphi'(r,\omega)|}{2} = \frac{|\varphi'(r,\omega)|}{2}(t_k-t_j).\end{aligned} \tag{4.3}$$

From (3.2), (3.3) and (4.3) it follows that

$$|A_\varphi| = \Big|\Big[\frac{1}{r(\log r)^{3/4} iF'_\varphi(r,\omega)} e^{iF_\varphi(r,\omega)}\Big]_{R_j}^{R'_j}\Big|$$

$$\leqslant \frac{C}{R_j}\Big(\frac{1}{|F'_\varphi(R_j,\omega)|}+\frac{1}{|F'_\varphi(R'_j,\omega)|}\Big)\leqslant \frac{C}{(t_k-t_j)R_j}\leqslant C2^{-j}.$$

In order to estimate B_φ, using (1.4), (3.10) and (4.3), we have

$$\Big|\frac{d}{dr}\Big(\frac{1}{r(\log r)^{3/4}iF'_\varphi(r,\omega)}\Big)e^{iF_\varphi(r,\omega)}\Big|\leqslant \frac{C}{r^2|F'_\varphi(r,\omega)|}+\frac{C|F''_\varphi(r,\omega)|}{r|F'_\varphi(r,\omega)|^2(\log r)^{3/4}}$$
$$<\frac{C}{r^{1+\min(1,\beta)}(t_k-t_j)}.$$

This together with (3.3) gives us

$$|B_\varphi|=\Big|\int_{R_j}^{R'_j}\frac{d}{dr}\Big(\frac{1}{r(\log r)^{3/4}iF'_\varphi(r,\omega)}\Big)e^{iF_\varphi(r,\omega)}\,dr\Big|$$
$$\leqslant \int_{R_j}^{R'_j}\frac{C}{r^{1+\min(1,\beta)}(t_k-t_j)}\,dr\leqslant \frac{C}{R_j^{\min(1,\beta)}(t_k-t_j)}\leqslant C2^{-j}.$$

From the estimates above and the triangle inequality we get

$$|A_j^\varphi(x_k,t_k)|\leqslant C\big(|A_\varphi|+|B_\varphi|\big)<C2^{-j},\qquad j>k\geqslant 2. \tag{4.4}$$

Here C is independent of j and k.

Using the results from (4.2), (4.4), in combination with Lemma 4.3, and recalling that $R'_j=R_j^N$, gives us

$$|S_m^\varphi f_\varphi(x_k,t_k)|\geqslant c(\log R'_k)^{1/4}-C'(\log R_k)^{1/4}-C\sum_{k+1}^{m}2^{-j} \tag{4.5}$$
$$\geqslant c(\log(R'_k))^{1/4}-\frac{C'}{N^{1/4}}(\log(R'_k))^{1/4}-C\geqslant c(\log R'_k)^{1/4},$$

when $m>k$ and N is chosen sufficiently large. Here $c>0$ is independent of k.

Step 2. Now it remains to show that $S^\varphi f_\varphi$ is continuous when $t>0$, and then it suffices to prove this continuity on a compact subset L of $\{(x,t);\, t>0, x\in \mathbf{R}^n\}$. We want to replace (x_l,t_l) with $(x,t)\in L$ in (3.3) and (3.4). Since we have maximum over all l less than j, we can choose $j_0<\infty$ large enough such that for all $j>l>j_0$ we have that $t_j<t_l<t$. Hence we may replace (x_l,t_l) with $(x,t)\in L$ on the right-hand sides in (3.3) and (3.4) for all $j>j_0$. This in turn implies that (4.4) holds when (x_k,t_k) is replaced by $(x,t)\in L$ and $j>j_0$. We use (4.4) to conclude that

$$|S_m^\varphi f_\varphi(x,t)-S^\varphi f_\varphi(x,t)|$$
$$=\Big|(2\pi)^{-n}\int_{|\xi|<R'_m}e^{ix\cdot\xi}e^{it\varphi(\xi)}\widehat{f}_\varphi(\xi)\,d\xi-(2\pi)^{-n}\int_{\mathbf{R}^n}e^{ix\cdot\xi}e^{it\varphi(\xi)}\widehat{f}_\varphi(\xi)\,d\xi\Big|$$
$$=(2\pi)^{-n}\Big|\int_{|\xi|>R'_m}e^{ix\cdot\xi}e^{it\varphi(\xi)}\widehat{f}_\varphi(\xi)\,d\xi\Big|\leqslant C\sum_{i=m+1}^{\infty}2^{-i}=C2^{-m},$$

when $m>j_0$. Hence $S_m^\varphi f_\varphi$ converge uniformly to $S^\varphi f_\varphi$ on every compact set.

We have now showed that $S^{\varphi}_m f_{\varphi}$ converge uniformly to $S^{\varphi} f_{\varphi}$ on every compact set and from Lemma 4.4 it follows that each $S^{\varphi}_m f_{\varphi}$ is a continuous function. Therefore it follows that $S^{\varphi} f_{\varphi}$ is continuous on $\{(x,t); t>0\}$. In particular there is an $N \in \mathbf{N}$ such that $|S^{\varphi}_m f_{\varphi}(x_k,t_k) - S^{\varphi} f_{\varphi}(x_k,t_k)| < 1$, when $m > N$. Using (4.5) and the triangle inequality we get

$$\begin{aligned} c(\log R'_k)^{1/4} \leqslant |S^{\varphi}_m f_{\varphi}(x_k,t_k)| &\leqslant |S^{\varphi}_m f_{\varphi}(x_k,t_k) - S^{\varphi} f_{\varphi}(x_k,t_k)| + |S^{\varphi} f_{\varphi}(x_k,t_k)| \\ &< 1 + |S^{\varphi} f_{\varphi}(x_k,t_k)|. \end{aligned}$$

This gives us $|S^{\varphi} f_{\varphi}(x_k,t_k)| > c(\log R'_k)^{1/4} - 1 \to +\infty$ as $k \to +\infty$. For any fixed $x \in \mathbf{R}^n$ we can by Lemma 4.1 choose a subsequence (x_{n_j}, t_{n_j}) of (x_k,t_k) that goes to $(x,0)$ as j turns to infinity. This gives the result. $\square$

References

[1] J. Bourgain, *A remark on Schrödinger operators*, Isr. J. Math. **77**(1–2) (1992), 1–16.

[2] C. E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. **40**(1) (1991), 33–69.

[3] P. Sjögren and P. Sjölin, *Convergence properties for the time-dependent Schrödinger equation*, Ann. Acad. Sci. Fenn. Ser. A I, Math. **14**(1) (1989), 13–25.

[4] P. Sjölin, *Regularity of solutions to the Schrödinger equation*, Duke Math. J. **55**(3) (1987), 699–715.

[5] P. Sjölin, *L^p maximal estimates for solutions to the Schrödinger equation*, Math. Scand. **81**(1) (1997), 35–68.

[6] P. Sjölin, *A counter-example concerning maximal estimates for solutions to equations of Schrödinger type*, Indiana Univ. Math. J. **47**(2) (1998), 593–599.

[7] P. Sjölin, *Homogeneous maximal estimates for solutions to the Schrödinger equation*, Bull. Inst. Math. Acad. Sin. **30**(2) (2002), 133–140.

[8] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series, Princeton, New Jersey, 1971.

[9] B. G. Walther, *Sharpness results for L^2-smoothing of oscillatory integrals*, Indiana Univ. Math. J. **50**(1) (2001), 655–669.

[10] B. G. Walther, *Sharp maximal estimates for doubly oscillatory integrals*, Proc. Am. Math. Soc. **130**(12) (2002), 3641–3650.

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