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A LOGIC FOR REASONING ABOUT QUALITATIVE PROBABILITY

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Abstract. We offer extended completeness theorem for probabilistic logic that combines higher-order probabilities (nesting of probability operators) and the qualitative probability operator.

1. Introduction

Qualitative reasoning uses binary relations on events (formulas) instead of exact numerical representations of realizations of events (probabilities, degrees of belief etc). For example, an agent (or expert in some field) will often state something like "A is at least as probable as B", or "A is more probable than B" without any explicit reference to the values of probabilities corresponding to A and B.

There are many relevant papers regarding the subject of qualitative reasoning. For the possibility theory (qualitative possibility and necessity relations), we refer the reader to [**1, 2, 3, 5, 8**]; for qualitative probability and probabilistic logic in general, we refer the reader to [**4, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 19, 20, 21, 22, 23, 24**].

Here we present a probabilistic logic that combines higher order probabilities (nesting of probability operators) and the qualitative probability operator. The main result is the proof of the extended completeness theorem (every consistent set of formulas is satisfiable) for the introduced logic. Our methodology is based on the results presented in [**12, 15, 16**]. Syntactically, instead of countably many probability operators of the form $P_{\geqslant r}A$ (it reads "the probability of A is at least r"), we use rational numbers from the real unit interval as truth constants (similarly as in [**18**]) and the qualitative probability operator $\succeq$. For instance, the above $P_{\geqslant r}A$ we formally express by $A \succeq r$, where r is the name for $r \in [0,1] \cap \mathbb{Q}$. Due to the modal nature of $\succeq$, the standard probabilistic Kripke structures were used for the definition of satisfiability.

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The rest of the paper is organized as follows: syntax and semantics are discussed in Section 2; axioms and inference rules are given in Section 3; extended completeness theorem is proved in Section 4; decidability of the introduced logic is proved in Section 5; concluding remarks are in the last section.

2. Syntax and semantics

Let I denotes the set of all rational numbers from the unit interval. We use $L_{\succeq}$ to denote our logic. The language of the logic consists of:

- a denumerable set $\mathrm{Var} = \{p, q, r \ldots\}$
- classical connectives $\neg$ and $\wedge$
- a binary operator $\succeq$

The set $\mathrm{For}_{L_{\succeq}}$ of formulas is defined as follows:

Definition 2.1. ($\mathrm{For}_{L_{\succeq}}$)

- If $p \in \mathrm{Var}$, then p is formula.
- If α, β are formulas and $r \in I$, then $(\neg\alpha)$, $(\alpha \wedge \beta)$, $(\alpha \succeq r)$ and $(\alpha \succeq \beta)$ are formulas.

The other classical connectives $(\vee, \to, \leftrightarrow)$ can be defined as usual. We use notation $r \preceq \alpha$ for $\alpha \succeq r$. We also denote $(\neg\alpha) \succeq 1 - r$ by $\alpha \preceq r$, $\alpha \preceq r \wedge \neg(r \preceq \alpha)$ by $\alpha \prec r$ and $\neg(\alpha \preceq r)$ by $\alpha \succ r$. Therefore, $r \succeq \alpha$ means $\alpha \preceq r$. Similarly if α and β are formulas, then we use notation $\alpha \preceq \beta$ for $\beta \succeq \alpha$, $\alpha \succ \beta$ denotes $\neg(\alpha \preceq \beta)$ and $\alpha \prec \beta$ means $\beta \succ \alpha$. We also denote $\alpha \succeq \beta \wedge \beta \succeq \alpha$ by $\alpha \asymp \beta$. Finally, we use $\perp$ do denote $\alpha \wedge \neg\alpha$.

Definition 2.2. An $L_{\succeq}$ model is a structure $M = \langle W, \mathrm{Prob}, v\rangle$ where:

- W is a nonempty set of elements called worlds.
- Prob is probability assignment which assigns to every $w \in W$ a probability space $\mathrm{Prob}(w) = \langle W(w), H(w), \mu(w)\rangle$, where:
  - $W(w)$ is a nonempty subset of W
  - $H(w)$ is an algebra of subsets of $W(w)$ and
  - $\mu(w) : H(w) \to [0, 1]$ is a finitely additive probability measure, and
- $v : W \times \mathrm{Var} \to \{\top, \perp\}$ is a valuation which associates with every world $w \in W$ a truth assignment $v(w)$ on the propositional letters.

Definition 2.3. Let $M = \langle W, \mathrm{Prob}, v\rangle$ be an $L_{\succeq}$ model and $w \in W$. The satisfiability relation is inductively defined as follows:

- If $p \in \mathrm{Var}$, then $(w, M) \vDash p$ iff $v(w)(p) = \top$.
- If $\alpha \in \mathrm{For}_{L_{\succeq}}$, then $(w, M) \vDash \neg\alpha$ iff it is not $(w, M) \vDash \alpha$.
- If $\alpha, \beta \in \mathrm{For}_{L_{\succeq}}$, then $(w, M) \vDash \alpha \wedge \beta$ iff $(w, M) \vDash \alpha$ and $(w, M) \vDash \beta$.
- If $\alpha \in \mathrm{For}_{L_{\succeq}}$ and $r \in I$, then $(w, M) \vDash \alpha \succeq r$ iff
  $\mu(w)(\{w' \in W(w) \mid w' \vDash \alpha\}) \geqslant r$.
- If $\alpha, \beta \in \mathrm{For}_{L_{\succeq}}$, then $(w, M) \vDash \alpha \succeq \beta$ iff
  $\mu(w)(\{w' \in W(w) \mid w' \vDash \alpha\}) \geqslant \mu(w)(\{w' \in W(w) \mid w' \vDash \beta\})$

In the sequel, we will omit M from $(w,M) \models \alpha$ and write $w \models \alpha$ if M is clear from the context. In an $L_{\succeq}$ model $M = \langle W, \mathrm{Prob}, v\rangle$ the set $\{w' \in W(w) \mid w' \models \alpha\}$ is denoted by $[\alpha]_{M,w}$ or just by $[\alpha]_w$.

A set of formulas T is $L_{\succeq}$ satisfiable if there is a world w in an $L_{\succeq}$ model M such that for every formula $\alpha \in T$, $w \models \alpha$. A formula α is $L_{\succeq}$ satisfiable if the set $\{\alpha\}$ is $L_{\succeq}$ satisfiable. A formula α is $L_{\succeq}$ valid in an $L_{\succeq}$ model $M = \langle W, \mathrm{Prob}, v\rangle$ (denoted by $\models_M \alpha$) if it is satisfiable in each world of M. A formula α is $L_{\succeq}$ valid (denoted by $\models \alpha$) if it is satisfiable in each world in each model.

3. Axiomatization

The axiom system $AX_{L_{\succeq}}$ involves eleven axiom schemas:

A1: Substitutional instances of tautologies.
A2: $\alpha \succeq 0$
A3: $\alpha \succeq s \to \alpha \succeq r$, $s \geqslant r$
A4: $\alpha \succeq \beta \vee \beta \succeq \alpha$
A5: $\alpha \succeq \beta \wedge \beta \succeq \gamma \to \alpha \succeq \gamma$
A6: $\alpha \succeq \beta \wedge \beta \succ \gamma \to \alpha \succ \gamma$
A7: $(\alpha \succeq r \wedge \beta \succeq s \wedge (\alpha \wedge \beta \asymp 0)) \to (\alpha \vee \beta \succeq r + s)$
A8: $\alpha \preceq r \wedge \beta \preceq s \to \alpha \vee \beta \preceq r + s$
A9: $\alpha \preceq r \wedge \beta \prec s \to \alpha \vee \beta \prec r + s$, $r + s \leqslant 1$
A10: $\alpha \prec r \to \alpha \preceq r$
A11: $\alpha \succeq s \to \alpha \succ r$, $s > r$

and inference rules:

(1) From α and $\alpha \to \beta$ infer β
(2) From α infer $\alpha \asymp 1$
(3) From $\alpha \to (\beta \succeq r - 1/n)$, for every $n \in \mathbf{N}$, $n \geqslant \frac{1}{r}$, infer $\alpha \to (\beta \succeq r)$
(4) From $\gamma \to (\beta \succeq r \to \alpha \succeq r)$, for every $r \in I$, infer $\gamma \to (\alpha \succeq \beta)$

We denote this axiomatic system by $Ax_{L_{\succeq}}$. A formula α is a theorem ($\vdash \alpha$) if there is an at most denumerable sequence (called proof) of formulas $\alpha_0, \alpha_1, \dots, \alpha$ such that every α_i is an axiom or it is derived from the preceding formulas by an inference rule. A formula α is deducible from the set T of formulas (denoted $T \vdash_{Ax_{L_{\succeq}}} \alpha$) if there is an at most denumerable sequence (called proof) of formulas $\alpha_0, \alpha_1, \dots, \alpha$ such that every α_i is an axiom or a formula from the set T, or it is derived from the preceding formulas by an inference rule, with the exception that Rule 2 can be applied only to theorems. $T \nvdash_{Ax_{L_{\succeq}}} \alpha$ means that $T \vdash_{Ax_{L_{\succeq}}} \alpha$ does not hold. A set of formulas T is consistent if there is at least one formula α such that $T \nvdash_{Ax_{L_{\succeq}}} \alpha$. A consistent set T of formulas is said to be maximal consistent if for every formula α either $\alpha \in T$ or $\neg\alpha \in T$. A set T is deductively closed if for every formula α if $T \vdash \alpha$, then $\alpha \in T$.

4. Soundness and completeness

THEOREM 4.1 (Deduction theorem). *If T is a set of formulas and $T \cup \{\alpha\} \vdash \beta$, then $T \vdash \alpha \to \beta$*

Proof. We use transfinite induction of the length of the inference. There are the following cases:

Case 1: β is obtained from γ and $\gamma \to \beta$ by an application of Rule 1. Then, by induction hypothesis $T \vdash \alpha \to \gamma$ and $T \vdash \alpha \to (\gamma \to \beta)$. Since $(\alpha \to (\gamma \to \beta)) \to ((\alpha \to \gamma) \to (\alpha \to \beta))$ is tautology, using Rule 1 two times we obtain $T \vdash \alpha \to \beta$.

Case 2: Let β be formula $(\gamma \asymp 1)$ obtained from γ. In that case γ must be a theorem and therefore β is theorem. Then, from $\vdash \beta \to (\alpha \to \beta)$, using Rule 1 we obtain $T \vdash \alpha \to \beta$.

Case 3: Suppose that $\beta = \gamma \to (\delta \succeq r)$ is obtained from T, α by an application of Rule 3. Then:

$T, \alpha \vdash \gamma \to (\delta \succeq r - 1/n)$ for every $n \geqslant \frac{1}{r}$

$T \vdash \alpha \to (\gamma \to (\delta \succeq r - 1/n))$ for every $n \geqslant \frac{1}{r}$, by the induction hypothesis

$T \vdash (\alpha \wedge \gamma) \to (\delta \succeq r - 1/n)$ for every $n \geqslant \frac{1}{r}$

by classical tautology $(\alpha \to (\beta \to \gamma)) \leftrightarrow ((\alpha \wedge \beta) \to \gamma)$

$T \vdash (\alpha \wedge \gamma) \to (\delta \succeq r)$, by Rule 3.

$T \vdash \alpha \to (\gamma \to \delta \succeq r)$

$T \vdash \alpha \to \beta$.

Case 4: Suppose that $\beta = \gamma \to (\delta \succeq \epsilon)$ is obtained from T, α by an application of Rule 4. Then:

$T, \alpha \vdash \gamma \to (\delta \succeq r \to \epsilon \succeq r)$ for every $r \in I$

$T, \alpha \vdash (\gamma \wedge (\delta \succeq r)) \to \epsilon \succeq r$ for every $r \in I$

$T \vdash \alpha \to ((\gamma \wedge (\delta \succeq r)) \to \epsilon \succeq r)$ for every $r \in I$, by the case 3

$T \vdash \alpha \to (\gamma \to (\delta \succeq r \to \epsilon \succeq r))$ for every $r \in I$

$T \vdash (\alpha \wedge \gamma) \to (\delta \succeq r \to \epsilon \succeq r)$ for every $r \in I$

$T \vdash (\alpha \wedge \gamma) \to (\delta \succeq \epsilon)$ by the Rule 4.

$T \vdash \alpha \to (\gamma \to (\delta \succeq \epsilon))$

$T \vdash \alpha \to \beta$ □

Theorem 4.2. *Every consistent set can be extended to a maximal consistent set.*

Proof. Let T be a consistent theory (set of formulas) and let $\theta_0, \theta_1, \dots$ be an enumeration of all formulas. We define a sequence of theories T_i in the following way:

(1) $T_0 = T$

(2) For every $n \geqslant 0$,

(a) If $T_n \cup \theta_n$ is consistent, then $T_{n+1} = T_n \cup \theta_n$.

(b) Otherwise, if θ_n is formula $\theta \to A \succeq r$, then $T_{n+1} = T_n \cup \{\neg\theta_n, \theta \to A \prec r - 1/n\}$, for some integer n, $n > \frac{1}{r}$ so that T_{n+1} is consistent.

(c) Otherwise, if θ_n is formula $\gamma \to (\epsilon \succeq \delta)$, then $T_{n+1} = T_n \cup \{\neg\theta_n, \gamma \to (\delta \succeq r \wedge \epsilon \prec r)\}$, for some $r \in I$ so that T_{n+1} is consistent.

(d) Otherwise, $T_{n+1} = T_n \cup \{\neg\theta_n\}$.

The sets obtained by the steps 1 and 2a are obviously consistent. The step 2d produces consistent sets, too. For, if $T_n, \theta_n \vdash \perp$, by Deduction Theorem we have $T_n \vdash \neg\theta_n$, and since T_n is consistent so is $T_n \cup \{\neg\theta_n\}$.

Let us first consider step 2b. Suppose that for every integer n, $n > \frac{1}{r}$, $T_{n+1} = T_n \cup \{\neg(\theta \to \alpha \succeq r), \theta \to \alpha \prec r - 1/n\}$ is inconsistent. Then:

$T_n, \neg(\theta \to \alpha \succeq r), \theta \to \alpha \prec r - 1/n \vdash \perp$ for every integer n, $n > \frac{1}{r}$.

$T_n, \neg(\theta \to \alpha \succeq r) \vdash (\theta \to \alpha \prec r - 1/n) \to \perp$ for every integer n, $n > \frac{1}{r}$,
by Deduction theorem.

$T_n, \neg(\theta \to \alpha \succeq r) \vdash \neg(\theta \to \alpha \prec r - 1/n)$ for every integer n, $n > \frac{1}{r}$.

$T_n, \neg(\theta \to \alpha \succeq r) \vdash \theta \wedge \neg(\alpha \prec r - 1/n)$ for every integer n, $n > \frac{1}{r}$,
by classical reasoning.

$T_n, \neg(\theta \to \alpha \succeq r) \vdash \theta \wedge (\alpha \succeq r - 1/n)$ for every integer n, $n > \frac{1}{r}$.

$T_n, \neg(\theta \to \alpha \succeq r) \vdash \theta \to (\alpha \succeq r - 1/n)$ for every integer n, $n > \frac{1}{r}$,
by classical tautology $\alpha \wedge \beta \to (\alpha \to \beta)$

$T_n, \neg(\theta \to \alpha \succeq r) \vdash \theta \to (\alpha \succeq r)$ by Rule 3

$T_n \vdash \neg(\theta \to \alpha \succeq r) \to (\theta \to (\alpha \succeq r))$, by Deduction theorem.

$T_n \vdash (\theta \to \alpha \succeq r)$, by classical reasoning, which contradicts consistency of T_n since $T_n \cup \{\theta \to (\alpha \succeq r)\}$ is not consistent.

Next, consider step 2c. Suppose that for every $r \in I$ set

$$T_{n+1} = T_n \cup \{\neg(\gamma \to (\epsilon \succeq \delta)), \gamma \to (\delta \succeq r \wedge \epsilon \prec r)\}$$

is inconsistent. Then:

$T_n, \neg(\gamma \to (\epsilon \succeq \delta)), \{\gamma \to (\delta \succeq r \wedge \epsilon \prec r)\} \vdash \perp$, for every $r \in I$, $r > 0$.

$T_n, \neg(\gamma \to (\epsilon \succeq \delta)) \vdash (\gamma \to (\delta \succeq r \wedge \epsilon \prec r)) \to \perp$, for every $r \in I$, $r > 0$,
by Deduction theorem.

$T_n, \neg(\gamma \to (\epsilon \succeq \delta)) \vdash \neg(\gamma \to (\delta \succeq r \wedge \epsilon \prec r))$, for every $r \in I$, $r > 0$.

$T_n, \neg(\gamma \to (\epsilon \succeq \delta)) \vdash (\gamma \wedge \neg(\delta \succeq r \wedge \epsilon \prec r))$ for every $r \in I$, $r > 0$.

$T_n, \neg(\gamma \to (\epsilon \succeq \delta)) \vdash (\gamma \wedge (\neg(\delta \succeq r) \vee \neg(\epsilon \prec r)))$ for every $r \in I$, $r > 0$.

$T_n, \neg(\gamma \to (\epsilon \succeq \delta)) \vdash \gamma \wedge (\delta \succeq r \to \epsilon \succeq r)$ for every $r \in I$, $r > 0$,
by classical reasoning.

$T_n, \neg(\gamma \to (\epsilon \succeq \delta)) \vdash \gamma \to (\delta \succeq r \to \epsilon \succeq r)$ for every $r \in I$, $r > 0$,
by classical tautology $(\alpha \wedge \beta) \to (\alpha \to \beta)$

$T_n, \neg(\gamma \to (\epsilon \succeq \delta)) \vdash \gamma \to (\epsilon \succeq \delta)$, by Rule 4

$T_n \vdash \neg(\gamma \to (\epsilon \succeq \delta)) \to (\gamma \to (\epsilon \succeq \delta))$, by Deduction theorem.

$T_n \vdash \gamma \to (\epsilon \succeq \delta)$, by classical reasoning, which contradicts consistency of T_n since $T_n \cup \{\gamma \to (\epsilon \succeq \delta)\}$ is not consistent.

Let $T^* = \bigcup_{n<\omega} T_n$. We have to prove that T^* is maximal consistent.

The steps 2a–2d guarantee that for every formula θ, θ or $\neg\theta$ belongs to T^*, i.e., that T^* is maximal. On the other hand, there is no formula θ, such that θ and $\neg\theta$ belongs to T^*. To prove that, suppose that $\theta = \theta_n$ and $\neg\theta = \theta_m$ for some n and m. If θ, $\neg\theta \in T^*$,, then also $\theta, \neg\theta \in T_{\max(n,m)+1}$, a contradiction with the consistency if $T_{\max(n,m)+1}$.

We continue by showing that T^* is deductively closed, and since it does not contain all formulas, it follows that T^* is consistent.

Next, we can show that if for some n, $T_n \vdash \theta$, it must be $\theta \in T^*$. Suppose that it is not the case. Then, $\neg\theta \in T^*$ so there must be some k such that $T_k \vdash \theta$ and $T_k \vdash \neg\theta$ which contradicts the consistency of T_k.

Let the sequence $\theta_1, \theta_2 \ldots \theta$ form a proof of θ in T^*. If the sequence is finite, there must be a set T_n such that $T_n \vdash \theta$ and $\theta \in T^*$. Thus suppose that the sequence is countably infinite. We can show that for every n, if θ_n is obtained by an application of an inference rule, and all premises belong to T^*, then there must be $\theta_n \in T^*$. If the rule is a finitary one, then there must be a set T_m which contains all premises and $T_m \vdash \theta_n$. Reasoning as above, we conclude that $\theta_n \in T^*$.

So, let us now consider the infinitary rules. Let $\theta_m = \alpha \to (\beta \succeq r)$ be obtained from the set of premises $\{\theta_m^n \mid n > 1/r\}$ by Rule 3, where θ_m^n is the formula $\alpha \to (\beta \succeq r - 1/n)$. Suppose that $\theta_m \notin T^*$. By the induction hypothesis, $\theta_m^n \in T^*$ for every n. By the step 2b of the construction there must be some n and some l such that $\alpha \to (\beta \prec r - 1/n)$ belongs to T_l. It follows that there must be some j such that $\alpha \to (\beta \prec r - 1/n)$ and $\alpha \to (\beta \succeq r - 1/n)$ belongs to T_j. Then $T_j \vdash \alpha \to \perp$ and $T_j \vdash \alpha \to (\beta \succeq r)$. It follows that $\theta_m \in T^*$, a contradiction.

Let $\theta_m = \gamma \to (\alpha \succeq \beta)$ be obtained from the set of premises $\{\theta_m^r \mid r \in I\}$ by Rule 4, where θ_m^r is the formula $\gamma \to (\beta \succeq r \to \alpha \succeq r)$. Suppose that $\theta_m \notin T^*$. By the induction hypothesis, $\theta_m^r \in T^*$ for every r. By the step 2c of the construction there must be some r and some l such that $\gamma \to (\beta \succeq r \wedge \alpha \prec r)$ belongs to T_l. It follows that there must be some j such that $\gamma \to (\beta \succeq r \to \alpha \succeq r\}$ and $\gamma \to (\beta \succeq r \wedge \alpha \prec r)$ belongs to T_j. Then $T_j \vdash \gamma \to \perp$ and $T_j \vdash \gamma \to (\alpha \succeq \beta)$. It follows that $\theta_m \in T^*$, a contradiction. □

THEOREM 4.3. *Let $\alpha, \beta \in \mathrm{For}_{L_\succeq}$, $r, s \in I$ and suppose that T is a maximal consistent set of formulas. Then:*

(1) $T \vdash \alpha \succeq \beta \to (\beta \succeq r \to \alpha \succeq r)$; (2) $\alpha \wedge \beta \in T$ *iff* $\alpha \in T$ *and* $\beta \in T$.

PROOF. (1) $T \vdash \alpha \succeq \beta \wedge \beta \succeq r \to \alpha \succeq r$ by $A5$
$T \vdash \neg(\alpha \succeq \beta \wedge \beta \succeq r) \vee \alpha \succeq r$
$T \vdash \neg(\alpha \succeq \beta) \vee (\neg(\beta \succeq r) \vee \alpha \succeq r)$
$T \vdash \alpha \succeq \beta \to (\beta \succeq r \to \alpha \succeq r)$ by classical reasoning.

(2) Suppose that $\alpha \in T$ and $\beta \in T$. Then:
$T \vdash \alpha$
$T \vdash \beta$
$T \vdash \alpha \wedge \beta$ and since T is deductively closed $\alpha \wedge \beta \in T$.

Let $\alpha \wedge \beta \in T$. Then:
$T \vdash \alpha \wedge \beta$
$T \vdash (\alpha \wedge \beta) \to \alpha$
$T \vdash (\alpha \wedge \beta) \to \beta$ and using Rule 1
$T \vdash \alpha$
$T \vdash \beta$ and since T is deductively closed $\alpha \in T$ and $\beta \in T$. □

Let the tuple $M = \langle W, \mathrm{Prob}, v\rangle$ be defined as follows:

- W is the set of all maximal consistent set of formulas,
- If $\alpha \in \mathrm{For}_{L_\succeq}$, then $[\alpha] = \{w \in W \mid \alpha \in W\}$
- v is a valuation which associated with every world $w \in W$ a truth assignment $v(w) : \mathrm{Var} \to \{\top, \perp\}$ such that for every $p \in \mathrm{Var}$, $v(w)(p) = \top$ iff $p \in w$.

- For every world $w \in W$, $\mathrm{Prob}(w)$ is defined as follows:
  - $W(w) = W$
  - $H(w)$ is a class of all sets of the form $[\alpha] = \{w \in W \mid \alpha \in w\}$ for all $\alpha \in \mathrm{For}_{L_{\succeq}}$
  - $\mu(w)([\alpha]) = \sup\{r \mid \alpha \succeq r \in w\}$

THEOREM 4.4. *Let $M = \langle W, \mathrm{Prob}, v\rangle$ be defined as above. Then the following hold for every $w \in W$.*

(1) *if* $\vdash \alpha \asymp \beta$, *then* $\mu(w)([\alpha]) = \mu(w)([\beta])$.
(2) $\mu(w)([\alpha]) \geqslant 0$, $\alpha \in \mathrm{For}_{L_{\succeq}}$
(3) $\mu(w)([\neg\alpha]) = 1 - \mu(w)([\alpha])$
(4) *if* $\vdash \alpha \wedge \beta \asymp 0$, *then* $\mu(w)([\alpha \vee \beta]) = \mu(w)([\alpha]) + \mu(w)([\beta])$.
(5) *If* $\vdash \alpha \to \beta$, *then* $\mu(w)([\alpha]) \leqslant \mu(w)([\beta])$.
(6) $\vdash \alpha \succeq \beta$ *iff* $\mu(w)([\alpha]) \geqslant \mu(w)([\beta])$.
(7) *If* $\vdash (\alpha \leftrightarrow \beta) \asymp 1$, *then* $\mu(w)([\alpha]) = \mu(w)([\beta])$.
(8) *If* $[\alpha] = [\beta]$, *then* $\vdash \alpha \leftrightarrow \beta$.
(9) *If* $[\alpha] = [\beta]$, *then* $\mu(w)([\alpha]) = \mu(w)([\beta])$.

PROOF. (1) $\vdash \alpha \asymp \beta$ i.e.,
$\vdash \alpha \succeq \beta \wedge \beta \succeq \alpha$
$\vdash \alpha \succeq \beta$ by classical reasoning.
$\vdash \alpha \succeq \beta \to (\beta \succeq r \to \alpha \succeq r)$ for every $r \in I$, by Theorem 4.3, so
$\vdash \beta \succeq r \to \alpha \succeq r$, for every $r \in I$.

Therefore if $\beta \succeq r \in w$, then $\alpha \succeq r \in w$. In the same way we can conclude the opposite, i.e., if $\alpha \succeq r \in w$, then $\beta \succeq r \in w$. Thus $\{r \mid \alpha \succeq r \in w\} = \{r \mid \beta \succeq r \in w\}$ and consequently $\sup\{r \mid \alpha \succeq r \in w\} = \sup\{r \mid \beta \succeq r \in w\}$.

(2) Obvious, according to A2.

(3) Let $r = \sup\{t \mid \alpha \succeq t \in w\}$. If $r = 1$, then $\alpha \succeq 1 \in w$, i.e., $(\neg\alpha) \preceq 0 \in w$. Then, by A2, $(\neg\alpha) \succeq 0 \in w$. If for some $s > 0$, $(\neg\alpha) \succeq s \in w$, then by A11 $(\neg\alpha) \succ 0 \in w$, a contradiction. Therefore $\mu(w)[\neg\alpha] = 0$.

Suppose that $r < 1$. Then, for every $r' \in (r, 1]$, $\neg(\alpha \succeq r') \in w$, i.e., $\alpha \prec r' \in T$. By A10, $\alpha \preceq r' \in w$ so $(\neg\alpha) \succeq 1 - r' \in w$. If there exists $r'' \in [0, r)$ such that $(\neg\alpha) \succeq 1 - r'' \in w$, then $\neg(\alpha \succ r'') \in w$ which is a contradiction. Therefore, $\sup\{t \mid (\neg\alpha) \succeq t \in w\} = 1 - \sup\{t \mid \alpha \succeq t \in w\}$.

(4) By A7, $\mu(w)([\alpha \vee \beta]) \geqslant r + s$ for any r and s such that $\alpha \succeq r \in w$ and $\beta \succeq s \in w$. Therefore $\mu(w)([\alpha \vee \beta]) \geqslant \sup\{r \mid \alpha \succeq r \in w\} + \sup\{s \mid \beta \succeq s \in w\}$ so $\mu(w)([\alpha \vee \beta]) \geqslant \mu(w)([\alpha]) + \mu(w)([\beta])$.

Suppose that $\mu(w)([\alpha]) = r_0$ and $\mu(w)([\beta]) = s_0$. If $r_0 + s_0 = 1$, then equality obviously hold. Let $r_0 + s_0 < 1$. Suppose that $r_0 + s_0 < \mu(w)([\alpha \vee \beta])$. Then, $r_0 + s_0 < t_0 = \sup\{t \mid \alpha \vee \beta \succeq t \in w\}$. If $t' \in (r_0 + s_0, t_0)$ is any rational number, then $\alpha \vee \beta \succeq t' \in w$. Let $r' > r_0$ and $s' > s_0$ be rational numbers such that $r' + s' = t' < 1$, $\alpha \prec r' \in w$ and $\beta \prec s' \in w$. By A10, $\alpha \preceq r' \in w$ and finally, by A9, $\alpha \vee \beta \prec r' + s' \in w$, i.e., $\alpha \vee \beta \prec t' \in w$, which is a contradiction.

(5) Let $\vdash \alpha \to \beta$. Suppose the contrary, i.e., $\mu(w)([\alpha]) > \mu(w)([\beta])$. Then there exists $r \in I$ such that $\alpha \succeq r \in w$ and $\beta \succeq r \notin w$. Therefore $\beta \prec r \in w$. Then:

$\vdash \alpha \to \beta$ by assumption
$\vdash \alpha \to \beta \asymp 1$ by Rule 2.

$\vdash \neg\alpha \vee \beta \succeq 1$ by classical tautology $\vdash \alpha \wedge \beta \rightarrow \alpha$.

Therefore

$\vdash \neg\alpha \vee \beta \succeq 1 \wedge \neg\alpha \preceq 1 - r \wedge \beta \prec r$

$\vdash \neg\alpha \preceq 1 - r \wedge \beta \prec r \rightarrow \neg\alpha \vee \beta \succeq 1$ by classical tautology $\alpha \wedge \beta \rightarrow (\alpha \rightarrow \beta)$.

Applying A9 we have:

$\vdash \neg\alpha \preceq 1 - r \wedge \beta \prec r \rightarrow \neg\alpha \vee \beta \prec 1$, a contradiction.

(6) ($\Leftarrow$) Let $\mu(w)(\alpha) \geqslant \mu(w)(\beta)$. Then $\sup\{r \,|\, \alpha \succeq r \in w\} \geqslant \sup\{r \,|\, \beta \succeq r \in w\}$. Suppose that $\alpha \succeq \beta \notin w$. Then there exists $r_0 \in I$ such that $\beta \succeq r_0 \wedge \alpha \prec r_0 \in w$. Therefore, $\sup\{r \mid \alpha \succeq r \in w\} \geqslant \sup\{r \mid \beta \succeq r \in w\} \geqslant r_0$, so there exists $r \geqslant r_0$ such that $\alpha \succeq r \in w$. However, by A3 $\vdash \alpha \succeq r \rightarrow \alpha \succeq r_0$ and therefore by Rule 1, $\vdash \alpha \succeq r_0$, a contradiction.

($\Rightarrow$). Let $\vdash \alpha \succeq \beta$. Then:

$\vdash \alpha \succeq \beta \rightarrow (\beta \succeq r \rightarrow \alpha \succeq r)$ for every $r \in I$, by Theorem 4.3.

$\vdash (\beta \succeq r \rightarrow \alpha \succeq r)$ for every $r \in I$, by Rule 1.

So, for every $r \in I$, if $\beta \succeq r \in w$, then $\alpha \succeq r \in w$ and consequently $\sup\{r \mid \alpha \succeq r \in w\} \geqslant \sup\{r \mid B \succeq r \in w\}$, i.e., $\mu(w)(\alpha) \geqslant \mu(w)(\beta)$.

(7) Suppose the contrary, for example $\mu(w)([\alpha]) > \mu(w)([\beta])$. Then there exists $r \in I$ such that $\alpha \succeq r \in w$ and $\beta \succeq r \notin w$. Therefore $\beta \prec r \in w$. Then:

$\vdash \alpha \leftrightarrow \beta \asymp 1$, i.e., $\vdash \alpha \leftrightarrow \beta \succeq 1 \wedge 1 \succeq \alpha \leftrightarrow \beta$

$\vdash \alpha \leftrightarrow \beta \succeq 1$, by classical tautology $\alpha \wedge \beta \rightarrow \alpha$

$\vdash (\neg\alpha \vee \beta) \wedge (\neg\beta \vee \alpha) \succeq 1$.

Therefore, by (6), $\mu(w)[((\neg\alpha \vee \beta) \wedge (\neg\beta \vee \alpha))] \geqslant 1$.

Since $\vdash (\neg\alpha \vee \beta) \wedge (\neg\beta \vee \alpha) \rightarrow (\neg\alpha \vee \beta)$, according to (5), we have that $\mu(w)[((\neg\alpha \vee \beta) \wedge (\neg\beta \vee \alpha))] \leqslant \mu(w)[(\neg\alpha \vee \beta)]$. Therefore, $\mu(w)(\neg\alpha \vee \beta) \geqslant 1$. Now, according to (6), $\vdash \neg\alpha \vee \beta \succeq 1$. The rest of the proof is the same as in (5).

(8) Suppose that $[\alpha] \subseteq [\beta]$. Then $\{w \mid \alpha \in w\} \subseteq \{w \mid \beta \in w\}$, i.e., for every w, if $\alpha \in w$, then $\beta \in w$. Therefore, if w is a maximal consistent set, then, if $w \vdash \alpha$, then $w \vdash \beta$, i.e., $w \vdash \alpha \rightarrow \beta$. According to this, there is no maximal consistent set w such that $\neg(\alpha \rightarrow \beta) \in w$. Therefore, $\alpha \wedge \neg\beta$ is inconsistent so $\alpha \wedge \neg\beta \vdash \bot$ i.e., $\vdash \alpha \rightarrow \beta$. If $[\alpha] = [\beta]$, then $[\alpha] \subseteq [\beta]$ and $[\beta] \subseteq [\alpha]$ so $\vdash \alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$ and therefore $\vdash \alpha \leftrightarrow \beta$.

(9) If $[\alpha] = [\beta]$, then by 8, $\vdash \alpha \leftrightarrow \beta$, so applying Rule 2 we have $\vdash \alpha \leftrightarrow \beta \asymp 1$. Therefore, according to (7) we have $\mu(w)([\alpha]) = \mu(w)([\beta])$. □

THEOREM 4.5. *A set of formulas is consistent with respect to $L_{\succeq}$ iff it has an $L_{\succeq}$ model.*

PROOF. ($\Leftarrow$) Since $L_{\succeq}$ is sound, a satisfiable set of formulas is consistent.

($\Rightarrow$). In order to prove this direction we construct a canonical model $M = \langle W, \mathrm{Prob}, v\rangle$ as above and show, by induction on complexity of formulas, that for every world w and every formula α, $w \vDash \alpha$ iff $\alpha \in w$.

- $w \vDash p$ iff $v(w)(p) = \top$ iff $p \in w$ (by definition of canonical model).
- $w \vDash \neg\alpha$ iff it is not $w \vDash \alpha$ iff $\alpha \notin w$ iff $\neg\alpha \in w$.
- $w \vDash \alpha \wedge \beta$ iff $w \vDash \alpha$ and $w \vDash \beta$ iff $\alpha \in w$ and $\beta \in w$ iff $\alpha \wedge \beta \in w$ (by Theorem 3).
- Let $\alpha \succeq r \in w$. Then $\sup\{s \mid \alpha \succeq s \in w\} \geqslant r$, i.e., $\mu(w)[\alpha] \geqslant r$ and therefore $w \vDash \alpha \succeq r$. Let $w \vDash \alpha \succeq r$. Then $\sup\{s \mid \alpha \succeq s \in w\} \geqslant r$. If $\sup\{s \,|\, \alpha \succeq s \in w\} = r$,

then according to Rule 3 and the fact that w is deductively closed, $w \vdash \alpha \succeq r$, i.e., $\alpha \succeq r \in w$. Otherwise, if $\sup\{s \mid \alpha \succeq s \in w\} = \mu(w)([\alpha]) > r$, then according to the properties of supremum and monotonous of function $\mu(w)$, $\alpha \succeq r \in w$.
• Suppose that $\alpha \succeq \beta \in w$. Then $w \vdash \alpha \succeq \beta$ and according to Theorem 4.3 and Rule 1 for every $r \in I$, $w \vdash \beta \succeq r \to \alpha \succeq r$. Then, for every $r \in I$, if $\beta \succeq r \in w$, then $\alpha \succeq r \in w$. Therefore, $\sup\{s \mid \alpha \succeq s \in w\} \geqslant \sup\{s \mid \beta \succeq s \in w\}$, i.e., $\mu(w)([\alpha]) \geqslant \mu(w)([\beta])$ so $w \vDash \alpha \succeq \beta$.

Let $w \vDash \alpha \succeq \beta$. Therefore $\mu(w)([\alpha]) \geqslant \mu(w)([\beta])$, i.e., $\sup\{s \mid \alpha \succeq s \in w\} \geqslant \sup\{s \mid \beta \succeq s \in w\}$. Then, according to the properties of supremum, for every $r \in I$, if $\beta \succeq r \in w$, then $\alpha \succeq r \in w$. Now, for every, $r \in I$, $w \vdash \beta \succeq r \to \alpha \succeq r$. Therefore, according to Rule 4, $w \vdash \alpha \succeq \beta$, i.e., $\alpha \succeq \beta \in w$. □

5. Decidability

In this section we are analyzing decidability of the satisfiability problem for the class $L_{\succeq}$.

THEOREM 5.1. *If a formula α is satisfiable, then it is satisfiable in an $L_{\succeq}$ model with a finite number of worlds. The number of worlds in that model is at most 2^k, where k denotes the number of subformulas of α.*

PROOF. Suppose that α holds in a world of an $L_{\succeq}$ model M$= \langle W, \mathrm{Prob}, v\rangle$. Let $\mathrm{Subf}(\alpha)$ denote the set of all subformulas of α and $k = |\,\mathrm{Subf}(\alpha)|$. Let $\approx$ denote the equivalence relation over W^2, such that $w \approx u$ iff for every $\beta \in \mathrm{Subf}(\alpha)$, $w \vDash \beta$ iff $u \vDash \beta$. The quotient set $W_{/\approx}$ is finite. From every class C_i we choose an element and denote it w_i. We consider the model $\mathrm{M}^* = \langle W^*, \mathrm{Prob}^*, v^*\rangle$, where:

• $W^* = \{w_i\}$.
• Prob^* is defined as follows:
  – $W^* = \{w_j \in W^* : (\exists u \in C_{w_j}) u \in W(w_i)\}$
  – $H^*(w_i)$ is the powerset of $W^*(w_i)$,
  – $\mu^*(w_i)(w_j) = \mu(w_i)(C_{w_j})$, and for any $D \subset H^*(w_i)$, $\mu^*(w_i)(D) = \sum_{w_j \in D} \mu^*(w_i)(w_j)$,
• $v^*(w_i)(p) = v(w_i)(p)$, for every $p \in \mathrm{Var}$.

For every w_i, $\mu^*(w_i)$ is finitely additive probability measure, since
$\mu^*(w_i)(W^*(w_i)) = \sum_{w_j \in W^*(w_i)} \mu^*(w_i)(w_j) = \sum_{C_{w_j} \in W_{/\approx}} \mu^*(w_i)(C_{w_j}) = 1.$

According to the definition of model M^*, it is obvious that it is an $L_{\succeq}$ model.

We can now show that for every $\beta \in \mathrm{Subf}(\alpha)$, β is satisfiable in M iff it is satisfiable in M^*. If $\beta \in \mathrm{Var}$, $(M, w) \vDash \beta$ iff for $w_i \in C_w$, $(M, w_i) \vDash \beta$ iff $(M^*, w_i) \vDash \beta$. The cases related to $\wedge$ and $\neg$ can be proved as usual. We will prove the cases when β is a formula of the form $\gamma \succeq r$ and $\gamma \succeq \delta$.

• $(M, w) \vDash \gamma \succeq r$ iff for $w_i \in C_w$, $(M, w_i) \vDash \gamma \succeq r$ iff

$$r \leqslant \mu(w_i)([\gamma]_{M,w}) = \sum_{C_u:(M,u)\vDash\gamma} \mu(w_i)(C_u) = \sum_{C_u:(M^*,u)\vDash\gamma} \mu^*(w_i)(C_u) = \mu^*(w_i)([\gamma]_{M^*,w})$$

iff $(M^*, w_i) \vDash \gamma \succeq r$.

• $(M, w) \vDash \gamma \succeq \delta$ iff

for $w_i \in C_w$, $(M, w_i) \vDash \gamma \succeq \delta$ iff $\mu(w_i)([\delta]_{M,w}) \leqslant \mu(w_i)([\gamma]_{M,w})$ iff

$\sum_{C_u:(M,u)\vDash\delta}\mu(w_i)(C_u) \leqslant \sum_{C_u:(M,u)\vDash\gamma}\mu(w_i)(C_u)$ iff
$\sum_{C_u:(M^*,u)\vDash\delta}\mu^*(w_i)(C_u) \leqslant \sum_{C_u:(M^*,u)\vDash\gamma}\mu^*(w_i)(C_u)$ iff
$\mu^*(w_i)([\delta]_{M^*,w}) \leqslant \mu^*(w_i)([\gamma]_{M^*,w})$ iff $(M^*,w_i)\vDash\gamma\succeq\delta$.

Finally, it is clear that the number of different classes in $W_{/\approx}$ is at most 2^k, and the same holds for the number of worlds in M^*. □

Theorem 5.2 (Decidability theorem). *The logic* $L_\succeq$ *is decidable.*

Proof. As it is noted above, a formula α is $L_\succeq$-satisfiable iff it is satisfiable in an $L_\succeq$ model with at most 2^k world, where k denotes the numbers of subformulas of α. The next procedure decides the satisfiability problem.

Let $\mathrm{Subf}(\alpha)=\{\beta_1,\dots,\beta_n,\gamma_1,\dots,\gamma_m\}$, and $k=n+m$. In every world w from M exactly one of the formulas of the form $\delta_w=\beta_1\wedge\cdots\wedge\beta_n\wedge\neg\gamma_1\wedge\cdots\wedge\neg\gamma_m$ holds. For every $l\leqslant 2^k$ we will consider l formulas of the above form. The chosen formulas are not necessarily different, but at least one of the formulas must contain the examined formula α. Using probabilistic constraints (i.e., formulas of the form $\beta\succeq r$, $\neg(\beta\succeq r)$, $\beta\succeq\gamma$, $\neg(\beta\succeq\gamma)$) from the formulas we shall examine whether there is an $L_\succeq$ model M with l worlds such that for some world w from the model $w\vDash\alpha$. We do not try to determine probabilities precisely, we just check whether there are probabilities such that probabilistic constraints are satisfied in the corresponding world. To do that, for every world w_i, $i<l$, we consider a system of linear equalities and inequalities of the form (we write $\beta\in\delta_w$ to denote that β occurs positively in the top conjunction of δ_w, i.e., if δ_w can be seen as $\bigwedge_i\delta_i$, then for some i, $\beta=\delta_i$):

$\sum_{j=1}^{l}\mu(w_i)(w_j)=1$
$\mu(w_i)(w_j)\geqslant 0$ for every world w_j
$\sum_{w_j:\beta\in\delta_{w_j}}^{l}\mu(w_i)(w_j)\geqslant r$ for every $\beta\succeq r\in\delta_{w_i}$
$\sum_{w_j:\beta\in\delta_{w_j}}^{l}\mu(w_i)(w_j)< r$ for every $\neg(\beta\succeq r)\in\delta_{w_i}$
$\sum_{w_j:\beta\in\delta_{w_j}}^{l}\mu(w_i)(w_j)\geqslant\sum_{w_j:\gamma\in\delta_{w_j}}^{l}\mu(w_i)(w_j)$ for every $\beta\succeq\gamma\in\delta_{w_i}$
$\sum_{w_j:\beta\in\delta_{w_j}}^{l}\mu(w_i)(w_j)<\sum_{w_j:\gamma\in\delta_{w_j}}^{l}\mu(w_i)(w_j)$ for every $\neg(\beta\succeq\gamma)\in\delta_{w_i}$

The first two rows correspond to the general constraints: the probability of the set of all worlds must be 1, while the probability of every measurable set of worlds must be nonnegative. The last four rows correspond to the probabilistic constraints because $\sum_{w_j:\beta\in\delta_{w_j}}^{l}\mu(w_i)(w_j)=\mu(w_i)([\beta]_{w_i})$.

Such a system is solvable iff there is a probability $\mu(w_i)$ satisfying all probabilistic constraints that appear in δ_{w_i}. Note that there are finitely many such systems that can be solved in a finite number of steps.

If the above test is positively solved, there is an $L_\succeq$ model in which every world $w_i\vDash\delta_{w_i}$. Since α belongs to at least one of the formulas δ_{w_i}, we have that α is satisfiable. If the test fails, and there is another possibility of choosing l and the set of formulas δ_w, we continue with the procedure, otherwise we conclude that α is not satisfiable.

It is easy to see that the procedure terminates in a finite number of steps. Thus the satisfiability problem for the class $L_{\succeq}$ is decidable. Since, $\models \alpha$ iff $\neg\alpha$ is not satisfiable, the $L_{\succeq}$- validity problem is also decidable. $\square$

6. Concluding remarks

We have introduced a probabilistic logic that combines higher order probabilities and the qualitative probability operator. The main result is the proof of the extended completeness theorem for the introduced logic.

Our work is closely related to the methodology presented in [**12, 15, 16**]. The first two of those papers provide formalism that can handle higher-order probabilities and the technique for construction of the canonical model. The last paper gives the formalism that can handle simple probabilities and the qualitative probability, where nesting of operators is not allowed.

The results presented here can be generalized in such a way that they will allow a complete axiomatization of both qualitative probability and higher-order conditional probabilities.

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