PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 87(101) (2010), 9–37 DOI: 10.2298/PIM1001009K

REGULARIZATION OF SOME CLASSES OF ULTRADISTRIBUTION SEMIGROUPS AND SINES

Marko Kostić

Communicated by Stevan Pilipović

Abstract. We systematically analyze regularization of different kinds of ultradistribution semigroups and sines, in general, with nondensely defined generators and contemplate several known results concerning the regularization of Gevrey type ultradistribution semigroups. We prove that, for every closed linear operator A which generates an ultradistribution semigroup (sine), there exists a bounded injective operator C such that A generates a *global differentiable* C-semigroup (C-cosine function) whose derivatives possess some expected properties of operator valued ultradifferentiable functions. With the help of regularized semigroups, we establish the new important characterizations of abstract Beurling spaces associated to nondensely defined generators of ultradistribution semigroups (sines). The study of regularization of ultradistribution sines also enables us to perceive significant ultradifferentiable properties of higher-order abstract Cauchy problems.

1. Introduction and preliminaries

The theory of C-semigroups and cosine functions is an attractive field of investigations of many authors and becomes unavoidable in the analysis of ill-posed abstract Cauchy problems. The essential part of the theory is clearly presented in the monograph [**9**] of deLaubenfels. On the other hand, the abstract Cauchy problems in the framework of the theory of ω-ultradistribution spaces were studied by Beals [**2, 3**], Ciorănescu–Zsidó [**7, 8**] and Kunstmann [**20**]. The foundation of the theory of ultradistribution semigroups with densely defined generators can be attributed to Beals [**3**], Chazarain [**5**], Ciorănescu [**6**], Emami-Rad [**10**], Ushijima [**30**] and it turns out that such a concept plays a crucial role in the analysis

2010 *Mathematics Subject Classification*: Primary 47D06; Secondary 47D09, 47D60, 47D62, 47D99.

Partially supported by grant 144016 of Ministry of Science and Technological Development of Republic Serbia.

of abstract Cauchy problems in the spaces of operator valued Denjoy–Karleman–Komatsu's ultradistributions. The notions of regular ultradistribution semigroups of the Beurling class and abstract Beurling spaces were introduced by Ciorănescu in [**6**] (cf. also [**19**, Example 1.6], [**20**, Examples 6.1, 6.2, 6.3] and Example 3.1 given below). The first comprehensive analysis of ultradistribution semigroups and sines was obtained by Komatsu [**13**]. The definition of ultradistribution semigroup and its generator employed therein has been recently reconsidered in [**16**] following the approaches of Kunstmann [**18**] and Wang [**31**] for distribution semigroups.

The paper is organized as follows. The main objective in Theorem 2.1 and Corollary 2.1 given below is to precisely profile mutual relations between some subclasses of Gevrey type ultradistribution semigroups whose generators possess polynomially bounded resolvent, analytic semigroups of growth order $r > 0$ of Tanaka [**29**] and global C-semigroups. In such a way, we obtain an extension of the well known result of Kunstmann (cf. [**18**, Section 5]) which asserts that every generator of a distribution semigroup of [**18**] is also the generator of a global C-semigroup, and refine several estimates given in [**3**, Lemma 1], [**6**, Remark 2.6, Corollary 4.3], [**9**, Example 22.31] as well as in the proof of [**18**, Theorem 5.5]. Although far from being optimal, we will see in Remark 2.1 how these improvements can be used in a more detailed analysis of the incomplete higher-order Cauchy problems (cf. for instance [**9**, Section XXV] and [**28**]). It is worth pointing out that the essential part of the proof of Theorem 2.1 follows from the important analysis of Beals [**2, 3**], which contains the explicit construction of global C-semigroups, and the construction of complex powers of operators presented by Straub in [**28**]. Furthermore, global C-semigroups constructed in Theorem 2.1 and Corollary 2.1 are C^∞-differentiable in $t \geqslant 0$ and derivatives of such semigroups possess interesting properties of operator valued ultradifferentiable functions of the Beurling type; we parenthetically stress that a similar analysis can be derived in the case of distribution semigroups and refer the interested reader to the paper [**30**] of Ushijima.

The central theme of Section 3 is the regularization of ultradistribution semigroups whose generators possess ultra-polynomially bounded resolvent. In Theorem 3.1, we reveal the important relation between such classes of ultradistribution semigroups and *local* differentiable C-semigroups. The use of local C-semigroups presents the main tool in proving Theorem 3.2, which precisely profiles the solution space of a generator of an ultradistribution semigroup of the Beurling class and extends the assertions of [**6**, Theorem 4.1, Corollary 4.2] to nondensely defined operators. The main result of Section 3, and of the present paper, is Theorem 3.3 stating that for every generator A of an ultradistribution semigroup, there exists a bounded injective operator C such that A generates a global differentiable C-semigroup whose derivatives possess some expected properties of operator valued ultradifferentiable functions. Since we mainly work in the spaces of abstract Beurling ultradistributions, the condition (M.3) is practically imposed throughout the third section. At this place, it is worth noting that it is not clear whether the assertions of Theorem 3.1, Theorem 3.2 and Theorem 3.3 remain true if (M.3) is replaced by a somewhat weaker condition (M.3)′ (cf. [**6**, p. 191]). Finally, in Theorem 3.4 we adapt several results of Beals [**2, 3**] to the present-day definition of regularized semigroups.

Concerning the higher-order abstract Cauchy problems, we recall an old result of Chazarain and Fattorini (cf. for instance [**32**]) which asserts that the problem

$$(ACP_n): \begin{cases} u \in C^n\big([0,\infty):E\big) \cap C\big([0,\infty):[D(A)]\big), \\ u^{(n)}(s) = Au(s),\ s \geqslant 0, \\ u^{(i)}(0) = x_i,\ i \in \{0,1,\dots,n-1\}, \end{cases}$$

is not well posed in the classical sense if A is unbounded and $n \geqslant 3$. Neubrander [**25**] was the first who applied integrated semigroups in the analysis of generalized well-posedness of the problem (ACP_2). In Theorem 4.3, we extend the well known result of Xiao and Liang [**32**, Theorem 6.2, p. 132] which can be viewed as an essential application of regularized semigroups to (ACP_n). As an outcome, we establish remarkable ultradifferentiable properties of entire solutions of (ACP_n). The assumptions of Theorem 4.3 are no longer applicable to generators of ultradistribution sines whose generators possess an ultra-polynomially bounded resolvent. This is the main reason for considering Theorem 4.4 which clarifies an interesting relation between ultradistribution sines and global differentiable C-cosine functions.

We employ the standard terminology; by E and $L(E)$ are denoted a complex Banach space and the Banach algebra of bounded linear operators on E. For a closed linear operator A on E, $D(A)$, $\mathrm{Kern}(A)$, $R(A)$, $\rho(A)$ denote its domain, kernel, range and resolvent set, respectively. Put $D_\infty(A) := \bigcap_{n\in\mathbb{N}_0} D(A^n)$; $[D(A)]$ stands for the Banach space $D(A)$ equipped with the graph norm.

Definition 1.1. Let $\tau \in (0,\infty]$. A strongly continuous family $(T(t))_{t\in[0,\tau)}$, resp. $(C(t))_{t\in[0,\tau)}$, in $L(E)$ is said to be a (local, if $\tau < \infty$) *C-semigroup*, resp. *C-cosine function*, if:

(i.1) $T(t+s)C = T(t)T(s)$, for all $t,\ s \in [0,\tau)$ with $t+s<\tau$, and
(i.2) $T(0) = C$,

resp.,

(ii.1) $C(t+s)C + C(|t-s|)C = 2C(t)C(s)$, for all $t,\ s \in [0,\tau)$ with $t+s<\tau$, and
(ii.2) $C(0) = C$.

The (integral) *generator* of $(T(t))_{t\in[0,\tau)}$, resp. $(C(t))_{t\in[0,\tau)}$, is defined by

$$\left\{(x,y) \in E \times E : T(t)x - Cx = \int_0^t T(s)y\,ds,\ t\in[0,\tau)\right\},\ \text{resp.}$$

$$\left\{(x,y) \in E \times E : C(t)x - Cx = \int_0^t (t-s)C(s)y\,ds,\ t\in[0,\tau)\right\},$$

and it is a closed linear operator.

Let us recall now the basic definitions and notions from the theory of ultradistributions. In the rest of the paper, we will always assume that (M_p) is a sequence of positive real numbers such that $M_0 = 1$ and that the following condition holds:

(M.1) $M_p^2 \leqslant M_{p+1}M_{p-1}$, $p \in \mathbb{N}$. Every employment of the conditions:

(M.2) $M_n \leqslant AH^n \min_{p+q=n} M_pM_q$, $n \in \mathbb{N}$, for some $A > 1$ and $H > 1$,

(M.3)$'$ $\sum\limits_{p=1}^{\infty} \dfrac{M_{p-1}}{M_p} < \infty$, and the condition

(M.3) $\sup\limits_{p\in\mathbb{N}} \sum\limits_{q=p+1}^{\infty} \dfrac{M_{p+1}M_{q-1}}{pM_pM_q} < \infty$, which is slightly stronger than (M.3)$'$, will be explicitly emphasized.

If $s > 1$, then the Gevrey sequences $(p!^s)$, (p^{ps}) and $(\Gamma(1 + ps))$ satisfy the above conditions, where $\Gamma(\cdot)$ denotes the Gamma function. Put $m_p := M_p/M_{p-1}$, $p \in \mathbb{N}$; then (M.1) implies that (m_p) is increasing and (M.3)$'$ simply means that $\sum_{p=1}^{\infty} \frac{1}{m_p} < \infty$. The associated function of (M_p) is defined by $M(\rho) := \sup_{p\in\mathbb{N}} \ln \frac{|\rho|^p}{M_p}$, $\rho \in \mathbb{C} \smallsetminus \{0\}$, $M(0) := 0$. We know that the function $t \mapsto M(t)$, $t \geqslant 0$ is increasing as well as that $\lim_{|\lambda|\to\infty} M(\lambda) = +\infty$ and that the function M vanishes in some open neighborhood of zero. Let us point out that the use of symbols A and M in the continuation of the paper is clear from the context.

Suppose $l > 0$, $\alpha > 0$, $\beta \in \mathbb{R}$ and denote by $\Lambda_{\alpha,\beta,l}$ the ultra-logarithmic region of type l; notice that such regions were introduced by Chazarain in [**5**] (cf. also [**23**, Section 2.3]) as follows: $\Lambda_{\alpha,\beta,l} := \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \geqslant \alpha M(l|\operatorname{Im}(\lambda)|) + \beta\}$. We assume that the boundary of $\Lambda_{\alpha,\beta,l}$, denoted by Γ_l, is upwards oriented. Next, for given $\theta \in (0, \pi]$ and $d \in (0, 1]$, put $\Sigma_\theta := \{\lambda \in \mathbb{C} : \lambda \neq 0,\ |\arg(\lambda)| < \theta\}$, $B_d := \{\lambda \in \mathbb{C} : |\lambda| \leqslant d\}$, $\Omega_{\theta,d} := \Sigma_\theta \cup B_d$, $\lfloor \beta \rfloor := \sup\{k \in \mathbb{Z} : k \leqslant \beta\}$ and $\lceil \beta \rceil := \inf\{k \in \mathbb{Z} : \beta \leqslant k\}$. In the rest of the first section, we assume that (M_p) additionally satisfies (M.2) and (M.3)$'$. Recall ([**11**]–[**13**]), the spaces $\mathcal{D}^{(M_p)}$ and $\mathcal{D}^{\{M_p\}}$ of the Beurling, resp. the Roumieu ultradifferentiable functions, are defined by setting:

$$\mathcal{D}^{(M_p)} := \text{ind} \lim_{K \subseteq\subseteq \mathbb{R}} \mathcal{D}_K^{(M_p)}, \text{ resp., } \mathcal{D}^{\{M_p\}} := \text{ind} \lim_{K \subseteq\subseteq \mathbb{R}} \mathcal{D}_K^{\{M_p\}},$$

where

$$\mathcal{D}_K^{(M_p)} := \text{proj} \lim_{h\to\infty} \mathcal{D}_K^{M_p,h}, \text{ resp., } \mathcal{D}_K^{\{M_p\}} := \text{ind} \lim_{h\to 0} \mathcal{D}_K^{M_p,h},$$

$$\mathcal{D}_K^{M_p,h} =: \bigl\{\phi \in C^\infty(\mathbb{R}) : \operatorname{supp}\phi \subseteq K,\ \|\phi\|_{M_p,h,K} < \infty\bigr\} \text{ and}$$

$$\|\phi\|_{M_p,h,K} := \sup\Bigl\{\frac{h^p|\phi^{(p)}(t)|}{M_p} : t \in K,\ p \in \mathbb{N}_0\Bigr\}.$$

We refer to [**11**]–[**13**] for a more detailed analysis of locally convex space valued ultradifferentiable functions defined on $\mathbb{R}$ and corresponding ultradistributions of the Beurling, resp., Roumieu type. The classes of Beurling, resp., Roumieu ultradistributions which take values in a Banach space E are denoted by $\mathcal{D}'^{(M_p)}(E)$, resp., $\mathcal{D}'^{\{M_p\}}(E)$ or simply $\mathcal{D}'^{(M_p)}$ and $\mathcal{D}'^{\{M_p\}}$ in the case $E = \mathbb{R}$. In what follows, we denote by $*$ either (M_p) or $\{M_p\}$. The similar terminology is used for the spaces of Beurling and Roumieu type ultradifferentiable functions. The space of all ultradifferentiable functions of $*$-class with the support in $[0,\infty)$ is denoted by $\mathcal{D}_0^*$ ($\mathcal{D}_0^{'*}(E)$ in the case of E-valued ultradistributions); further on, $\mathcal{E}_0^{'*}$ denotes the space of ultradistributions whose supports are compact subsets of $[0,\infty)$. The convolution of operator valued ultradifferentiable functions and operator valued ultradistributions is understood in the sense of [**12**]. Recall [**11**], an entire function of the form $P(\lambda)=\sum_{p=0}^{\infty}a_p\lambda^p$, $\lambda\in\mathbb{C}$, is of class (M_p), resp., of class $\{M_p\}$, if there exist $l>0$ and $C>0$, resp., for every $l>0$ there exists a constant $C>0$, such that $|a_p|\leqslant Cl^p/M_p$, $p\in\mathbb{N}$. The corresponding ultradifferential operator $P(D)=\sum_{p=0}^{\infty}a_pD^p$ is of class (M_p), resp., of class $\{M_p\}$.

If $f,\ g:\mathbb{R}\to\mathbb{C}$ are measurable, put $f*_0 g(t):=\int_0^t f(t-s)\,g(s)\,ds$. Then $*_0:\mathcal{D}^*\times\mathcal{D}^*\to\mathcal{D}^*$ is continuous and this justifies the following definition introduced in [**16**].

Definition 1.2. Let $G\in\mathcal{D}_0^{'*}(L(E))$. If G satisfies

$$G(\phi *_0 \psi)=G(\phi)G(\psi),\quad \phi,\psi\in\mathcal{D}^*, \tag{U.1}$$

then G is called a *pre*-(UDSG) of $*$-class. If G additionally satisfies

$$\mathcal{N}(G):=\bigcap_{\phi\in\mathcal{D}_0^*}\operatorname{Kern}(G(\phi))=\{0\}, \tag{U.2}$$

then we say that G is an *ultradistribution semigroup* of $*$-class, (UDSG) for short. Further on, G is called *dense* if (U.3) holds, where

$$\mathcal{R}(G):=\bigcup_{\phi\in\mathcal{D}_0^*}R(G(\phi)) \text{ is dense in } E. \tag{U.3}$$

The *generator* A of G is defined by

$$\big\{(x,y)\in E\times E : G(-\varphi,)x=G(\varphi)y \text{ for all } \varphi\in\mathcal{D}_0^*\big\}.$$

Notice that Definition 1.2 consider ultradistribution semigroups of [**5, 6**], [**10**], [**13**], [**23**] and [**30**] in a great generality.

Let G be a (UDSG) of $*$-class and $T\in\mathcal{E}_0^{'*}$. Then $T*\phi\in\mathcal{D}_0^*$, $\phi\in\mathcal{D}_0^*$ and the following definition of the operator $G(T)$ makes sense:

$$G(T):=\big\{(x,y)\in E\times E : G(T*\phi)x=G(\phi)y \text{ for all } \phi\in\mathcal{D}_0^*\big\}.$$

Note that $A=G(-\delta')$ and that $G(T)$ is a closed linear operator [**16**].

The fiollowing definition of a regular (M_p)-ultradistribution semigroup and its generator was introduced by Ciorănescu in [**6**].

Definition 1.3. Let A be a closed linear operator and let G belong to the space $\mathcal{D}_0^{'(M_p)}\big(L(E,[D(A)])\big)$. Then we say that G is a *regular* (M_p)*-ultradistribution semigroup* generated by A if:

(i) $G*P=\delta'\otimes\mathrm{Id}_{[D(A)]}$ and $P*G=\delta'\otimes\mathrm{Id}_E$, where $P=\delta'\otimes I-\delta\otimes A\in\mathcal{D}_0^{'(M_p)}\big(L([D(A)],E)\big)$,

(ii) the same as (U.2),

(iii) the linear hull of $\mathcal{R}(G)$, denoted by $\langle\mathcal{R}(G)\rangle$, is dense in E.

It could be of importance to state the following useful facts concerning ultradistribution semigroups. Arguing as in the proof of [**6**, Proposition 2.6], one can verify that the polynomial boundedness of $\|R(\cdot : A)\|$ existing on a suitable ultra-logarithmic region implies that A generates a (UDSG) of (M_p)-class. The previous assertion does not remain true in the case of ultra-polynomial boundedness; more precisely, there exists a closed linear operator A and an element $\mathcal{D}_0^{'(M_p)}\big(L(E,[D(A)])\big)$ so that the condition (i) quoted in the formulation of Definition 3 holds and that (U.2) does not hold for G (cf. [**7**, p. 156] and [**16**]). The condition (U.2) plays a crucial role in our investigation. Suppose now $G \in \mathcal{D}_0^{'(M_p)}\big(L(E,[D(A)])\big)$ and A is a closed, densely defined operator. Then G is a regular (M_p)-ultradistribution semigroup generated by A iff G is a dense (UDSG) of (M_p)-class generated by A [**16**].

The class of ultradistribution sines can be introduced following the approaches of Miana [**24**] and the author [**14**] for (almost-)distribution cosine functions, or by means of convolution type equations as it has been done by Komatsu [**13**]. The concepts presented in [**13**], [**24**] and [**14**] are not so easily comparable in the ultradistribution case, and in order to simplify our exposition, we shall say that a closed linear operator A generates an ultradistribution sine in E iff the operator $\mathcal{A} \equiv \left(\begin{smallmatrix} 0 & I \\ A & 0 \end{smallmatrix}\right)$ generates an ultradistribution semigroup in $E \times E$.

2. Regularization of Gevrey type ultradistribution semigroups

In this section, we use the construction of complex powers of operators given by Straub in [**28**] and refer to [**28, 29**] for the notion of (analytic) semigroups of growth order $r > 0$. Our aim is to find the precise relations between Gevrey type ultradistribution semigroups, analytic semigroups of growth order $r > 0$ and global C-semigroups. In order to do that, suppose that (N_p) and (R_p) are two sequences of positive numbers which satisfy (M.1). Following Chou (cf. for example [**11**, Definition 3.9, p. 53]), we write $N_p \prec R_p$ if and only if, for every $\delta \in (0,\infty)$, $\sup_{p\in\mathbb{N}_0} N_p\delta^p/R_p < \infty$. The assertions (i), (iv) and (v) of the next theorem can be attributed to Straub [**28**]. Herein we notice that the denseness of A is not used in the proofs of Propositions 2.2, 2.5, 2.6 and 2.8 as well as Lemmas 2.7 and 2.10 of [**28**] and that the assertion (v) extends [**3**, Lemma 1] and some estimates used in the proof of [**18**, Lemma 5.4] (cf. also [**1**, Lemma II-1, Theorem II-3]). The main problem in regularization of ultradistribution semigroups whose generators do not have polynomially bounded resolvent appears exactly at this place. Actually, if $\|R(\cdot : A)\|$ is not polynomially bounded on an appropriate ultra-logarithmic region, then it is not clear whether there exists $n \in \mathbb{N}_0$ such that, for every $x \in D(A^{n+2})$, the operator $T_b(t)$, defined in the formulation of the next theorem, fulfills $\lim_{t\to 0+} T_b(t)x = x$. Then it is not clear how to show that the operator $T_b(t)$ is injective; see also [**3**, Lemma 3], [**18**, Lemma 5.4] and the proof of [**28**, Proposition 2.8].

THEOREM 2.1. *Suppose, in addition, that there exists a number* $b \in (0,1)$ *such that*

$$p^{p/b} \prec M_p \tag{2.1}$$

and that (M_p) *satisfies* (M.1) *and* (M.2). *If* A *is a closed linear operator such that there exist* $\alpha > 0$, $l > 0$, $M > 0$, $\beta \in \mathbb{R}$ *and* $n \in \mathbb{N}$ *satisfying*

$$\Lambda_{\alpha,\beta,l} \subseteq \rho(A) \text{ and } \|R(\lambda : A)\| \leqslant M(1+|\lambda|)^n,\ \lambda \in \Lambda_{\alpha,\beta,l},$$

then, for every $\gamma \in \big(0, \arctan(\cos(\frac{b\pi}{2}))\big)$, *there are an* $\omega \in \mathbb{R}$ *and an analytic operator family* $(T_b(t))_{t\in\Sigma_\gamma}$ *of growth order* $\frac{n+1}{b}$ *such that the following holds.*

(i) *For every* $t \in \Sigma_\gamma$, *the operator* $T_b(t)$ *is injective.*
(ii) *For every* $t \in \Sigma_\gamma$, A *generates a global* $T_b(t)$*-semigroup* $(S_{b,t}(s))_{s\geqslant 0}$.
(iii) *Let* $K \subseteq [0,\infty)$ *be a compact set,* $t \in \Sigma_\gamma$ *and* $x \in E$. *Then the mapping* $s \mapsto S_{b,t}(s)x$ *is infinitely differentiable in* $s \geqslant 0$ *and, for every* $h > 0$,
$$\sup_{p\in\mathbb{N}_0,\, s\in K} \frac{h^p}{M_p}\Big\|\frac{d^p}{ds^p}S_{b,t}(s)x\Big\| < \infty.$$
(iv) *There is an* $L > 0$ *with* $\|T_b(t)\| \leqslant L\big(\tan(\gamma)\operatorname{Re}(t) - |\operatorname{Im}(t)|\big)^{-(n+1)/b}$, $t \in \Sigma_\gamma$.
(v) *If* $x \in D(A^{n+2})$, *then there exists* $\lim_{t\to 0+}\big(T_b(t)x - x\big)/t$ *and, in particular,* $\lim_{t\to 0+} T_b(t)x = x$.

Furthermore, if A *is densely defined, then* $(T_b(t))_{t\in\Sigma_\gamma}$ *is an analytic semigroup of growth order* $\frac{n+1}{b}$ *whose c.i.g. is* $-(\omega - A)^b$.

PROOF. Let $\varphi : \mathbb{R} \to [0,\infty)$ be an infinitely differentiable function satisfying $\operatorname{supp}\varphi \subseteq [0,1]$ and $\int_{\mathbb{R}} \varphi(t)\,dt = 1$. Put $M_1(t) := \int_0^t M(t-s)\,\varphi(s)\,ds$, $t \in \mathbb{R}$ and notice that $M_1 \in C^\infty(\mathbb{R})$ and that [**4**, Lemma 2.1.3, (2.11)] implies that there exists a constant $K > 0$ such that

$$\frac{M(t-s)}{M(t)} \geqslant \frac{M(t-1)}{M(t)} \geqslant \frac{M(t-1)}{\frac{3}{2}\frac{t}{t-1}M(t-1)+K} = \frac{1}{\frac{3}{2}\frac{t}{t-1} + \frac{K}{M(t-1)}} \to \frac{2}{3}, \quad t \to +\infty.$$

Thereby, there exist $m > 0$ and $M > 0$ such that:

$$M(t) \leqslant mM_1(t) + M \leqslant mM(t) + M, \quad t \geqslant 0. \tag{2.2}$$

Suppose $(0,1) \ni b$ satisfies $p^{p/b} \prec M_p$ and designate by $N(\cdot)$ the associated function of the sequence $(p^{p/b})$. Then $N(|\lambda|) \sim \frac{1}{be}|\lambda|^b$, $|\lambda| \to \infty$ and an application of [**11**, Lemma 3.10] gives that, for every $\mu > 0$, there exist positive real constants c_μ and C_μ such that $\lim_{\mu\to 0} c_\mu = 0$ and that

$$M_1(l\lambda) \leqslant M(l\lambda) \leqslant N(\mu l\lambda) + C_\mu \leqslant c_\mu|\lambda|^b + C_\mu, \quad \lambda \geqslant 0. \tag{2.3}$$

Denote, for $\sigma > 0$ and $\varsigma \in \mathbb{R}$, $\Lambda^1_{\sigma,\varsigma,l} := \big\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \geqslant \sigma M_1(l|\operatorname{Im}(\lambda)|) + \varsigma\big\}$. By (2.2)–(2.3), we have $\Lambda^1_{\alpha m,\beta+\alpha M,l} \subseteq \Lambda_{\alpha,\beta,l} \subseteq \rho(A)$. Let $\gamma \in (0, \arctan(\cos(\frac{b\pi}{2})))$ and let $a \in (0, \frac{\pi}{2})$ satisfy $b \in (0, \frac{\pi}{2(\pi-a)})$ and $\gamma \in \big(0, \arctan(\cos(b(\pi-a)))\big)$. Recall, $\Omega_{a,d} = B_d \cup \Sigma_a$. Thanks to (2.3), one obtains the existence of numbers $d \in (0,1]$ and $\omega \in \mathbb{R}$ such that $\Omega_{a,d} \subseteq \Lambda^1_{\alpha m,\beta+\alpha M-\omega,l} \subseteq \rho(A-\omega)$. Let $\Gamma_{a,d}$ and Γ denote the upwards oriented boundaries of $\Omega_{a,d}$ and $\Lambda^1_{\alpha m,\,\beta+\alpha M-\omega,\,l}$, respectively. Define $T_b(t),\ t\in\Sigma_\gamma$ by

$$T_b(t)x := \frac{1}{2\pi i}\int\limits_{\Gamma_{a,d}} e^{-t(-\lambda)^b}R(\lambda : A-\omega)x\,d\lambda, \quad x\in E.$$

By the arguments given in [**28**, Section 2], we have that $(T_b(t))_{t\in\Sigma_\gamma}$ is an analytic operator family which satisfies the claimed properties (i), (iv) and (v). Furthermore, if A is densely defined, we have that $(T_b(t))_{t\in\Sigma_\gamma}$ is an analytic semigroup of growth order $\frac{n+1}{b}$ whose c.i.g. is $-(\omega-A)^b$ [**28**]. Define now, for every $t=t_1+it_2\in\Sigma_\gamma$, $s\geqslant 0$ and $x\in E$,

$$S^1_{b,t}(s)x := \frac{1}{2\pi i}\int\limits_{\Gamma} e^{-t(-\lambda)^b}e^{\lambda s}R(\lambda : A-\omega)x\,d\lambda.$$

To prove that $S^1_{b,t}(s)\in L(E)$, notice that, for every $\lambda\notin\Omega_{a,d}$, we have $b\arg(-\lambda)\in(b(-\pi+a),b(\pi-a))$, $\cos(b\arg(-\lambda))\in(\cos(b(\pi-a)),1]$, $\tan(\gamma)<\cos(b(\pi-a))$ and

$$\begin{aligned}|e^{-t(-\lambda)^b}| &= e^{-t_1|\lambda|^b\cos(b\arg(-\lambda))+t_2|\lambda|^b\sin(b\arg(-\lambda))}\\ &\leqslant e^{-(t_1\cos(b\arg(-\lambda))-|t_2|)|\lambda|^b}\leqslant e^{-(t_1\cos(b(\pi-a))-|t_2|)|\lambda|^b}\leqslant e^{-(t_1\tan(\gamma)-|t_2|)|\lambda|^b}.\end{aligned}$$

This inequality and (2.3) imply that, for all sufficiently small $\mu>0$:

$$\begin{aligned}&\left|e^{-t(-\lambda)^b}e^{\lambda s}\|R(\lambda : A-\omega)\|\right|\\ &\quad\leqslant Me^{s(\alpha mM_1(l|\operatorname{Im}(\lambda)|)+\beta+\alpha M-\omega)}e^{-(t_1\tan(\gamma)-|t_2|)|\lambda|^b}(1+|\lambda|+|\omega|)^n\\ &\leqslant M_\mu e^{s(\beta+\alpha M-\omega)}e^{s\alpha mc_\mu|\lambda|^b}e^{-(t_1\tan(\gamma)-|t_2|)|\lambda|^b}(1+|\lambda|+|\omega|)^n,\ \lambda\in\Gamma,\ |\lambda|\geqslant r.\end{aligned}\tag{2.4}$$

The use of (2.4) with sufficiently small μ implies that $S^1_{b,t}(s)\in L(E)$, as required. Further on, the Cauchy formula and the previous argumentation enable one to see that

$$\int\limits_{\Gamma} e^{\lambda s}e^{-t(-\lambda)^b}\,d\lambda = 0, \quad s\geqslant 0,\ t\in\Sigma_\gamma \tag{2.5}$$

and that $T_b(t)=S^1_{b,t}(0)$, $t\in\Sigma_\gamma$. It is also clear that $S^1_{b,t}(s)T_b(t)=T_b(t)S^1_{b,t}(s)$ and that $S^1_{b,t}(s)(A-\omega)\subseteq(A-\omega)S^1_{b,t}(s)$, $s\geqslant 0$, $t\in\Sigma_\gamma$. Using the Fubini theorem, the resolvent equation and (2.5), one obtains

$$\begin{aligned}(A-\omega)\int\limits_0^s S^1_{b,t}(r)x\,dr &= \frac{1}{2\pi i}\int\limits_0^s\int\limits_\Gamma e^{\lambda r}e^{-t(-\lambda)^b}(A-\omega)R(\lambda : A-\omega)x\,d\lambda\,dr\\ &= \frac{1}{2\pi i}\int\limits_0^s\int\limits_\Gamma e^{\lambda r}e^{-t(-\lambda)^b}(\lambda R(\lambda : A-\omega)x-x)\,d\lambda\,dr\\ = \frac{1}{2\pi i}\int\limits_0^s\int\limits_\Gamma e^{\lambda r}e^{-t(-\lambda)^b}\lambda R(\lambda : A-\omega)x\,d\lambda\,dr &- \frac{1}{2\pi i}\int\limits_0^s\int\limits_\Gamma e^{\lambda r}e^{-t(-\lambda)^b}x\,d\lambda\,dr\end{aligned}$$

$$= \frac{1}{2\pi i}\int_0^s \int_\Gamma e^{\lambda r} e^{-t(-\lambda)^b} \lambda R(\lambda : A-\omega)x\,d\lambda\,dr$$

$$= \frac{1}{2\pi i}\int_\Gamma \left[\int_0^s e^{\lambda r} e^{-t(-\lambda)^b} \lambda R(\lambda : A-\omega)x\,dr\right] d\lambda$$

$$= \frac{1}{2\pi i}\int_\Gamma (e^{\lambda s}-1) e^{-t(-\lambda)^b} R(\lambda : A-\omega)x\,d\lambda$$

$$= \frac{1}{2\pi i}\int_\Gamma e^{\lambda s} e^{-t(-\lambda)^b} R(\lambda : A-\omega)x\,d\lambda - \frac{1}{2\pi i}\int_\Gamma e^{-t(-\lambda)^b} R(\lambda : A-\omega)x\,d\lambda$$

$$= S^1_{b,t}(s)x - T_b(t)x, \quad s \geqslant 0,\ t \in \Sigma_\gamma,\ x \in E.$$

This implies that $(S^1_{b,t}(s))_{s\geqslant 0}$ is a global $T_b(t)$-semigroup generated by $A-\omega$. In order to prove differentiability of $(S^1_{b,t}(s))_{s\geqslant 0}$, note that the arguments used in the proof of boundedness of the operator $S^1_{b,t}(s)$ also show that the integral $\frac{1}{2\pi i}\int_\Gamma \lambda^p e^{\lambda s} e^{-t(-\lambda)^b} R(\lambda : A-\omega)\,d\lambda$ converges for all $p \in \mathbb{N}$. Then the elementary inequality $|e^{\lambda h}-1| \leqslant h|\lambda|e^{\operatorname{Re}(\lambda)h}$, $\lambda \in \mathbb{C}$, $h > 0$ and the dominated convergence theorem yield

$$\frac{d}{ds}S^1_{b,t}(s) = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda s} e^{-t(-\lambda)^b} R(\lambda : A-\omega)\,d\lambda, \quad s \geqslant 0.$$

Inductively,

$$\frac{d^p}{ds^p}S^1_{b,t}(s) = \frac{1}{2\pi i}\int_\Gamma \lambda^p e^{\lambda s} e^{-t(-\lambda)^b} R(\lambda : A-\omega)\,d\lambda,\ p \in \mathbb{N}_0, \quad s \geqslant 0. \tag{2.6}$$

Taking into account (2.3) and (2.6), we easily infer that, for every compact set $K \subseteq [0,\infty)$, $t \in \Sigma_\gamma$ and $\mu > 0$:

$$\sup_{p\in\mathbb{N}_0,\, s\in K} \frac{h^p}{M_p}\Big\|\frac{d^p}{ds^p}S^1_{b,t}(s)\Big\| \leqslant \frac{M}{2\pi}\int_\Gamma e^{M(h\lambda)} e^{\operatorname{Re}(\lambda)s} e^{-(t_1\cos(b(\pi-a))-|t_2|)|\lambda|^b} \times e^{-(t_1\tan(\gamma)-|t_2|)|\lambda|^b}(1+|\lambda|+|\omega|)^n|d\lambda|$$

$$\leqslant \frac{M}{2\pi}\int_\Gamma e^{M(h\lambda)} e^{s[\alpha M_1(l|\operatorname{Im}(\lambda)|)+\beta+\alpha M-\omega]} e^{-(t_1\cos(b(\pi-a))-|t_2|)|\lambda|^b} \times e^{-(t_1\tan(\gamma)-|t_2|)|\lambda|^b}(1+|\lambda|+|\omega|)^n|d\lambda|$$

$$\leqslant \frac{M}{2\pi}e^{\sup K(\beta+\alpha M-\omega)}\int_\Gamma e^{c_\mu\frac{h^b}{l^b}|\lambda|^b+C_\mu} e^{\alpha m\sup K[c_\mu\frac{h^b}{l^b}|\lambda|^b+C_\mu]} \times e^{-(t_1\cos(b(\pi-a))-|t_2|)|\lambda|^b}(1+|\lambda|+|\omega|)^n|d\lambda|$$

$$\leqslant \frac{M}{2\pi}e^{\sup K[\beta+\alpha M-\omega+\alpha m C_\mu]+C_\mu}\int_\Gamma e^{c_\mu\frac{h^b}{l^b}|\lambda|^b(1+\sup K)} \times e^{-(t_1\cos(b(\pi-a))-|t_2|)|\lambda|^b}(1+|\lambda|+|\omega|)^n|d\lambda|.$$

Choosing μ sufficiently small, we obtain that

$$\sup_{p\in\mathbb{N}_0,\, s\in K}\frac{h^p}{M_p}\Big\|\frac{d^p}{ds^p}S^1_{b,t}(s)\Big\|<\infty.$$

Put now $S_{b,t}(s):=e^{\omega s}S^1_{b,t}(s)$, $s\geqslant 0$, $t\in\Sigma_\gamma$ and notice that $(S_{b,t}(s))_{s\geqslant 0}$ is a global $T_b(t)$-semigroup generated by A. Since (M_p) satisfies (M.1) and $M_0=1$, it can be easily seen that $M_{p+q}\geqslant M_pM_q$, $p,q\in\mathbb{N}_0$ (cf. for instance [**4**, Lemma 2.1.1]). Hence, we have that, for every $h_1\in[h(2+2|\omega|),\infty)$ and $x\in E$:

$$\begin{aligned}
&\sup_{p\in\mathbb{N}_0,\, s\in K}\frac{h^p}{M_p}\Big\|\frac{d^p}{ds^p}S_{b,t}(s)x\Big\|\\
&\qquad\leqslant e^{|\omega|\sup K}\sup_{p\in\mathbb{N}_0,\, s\in K}\frac{h^p2^p(1+|\omega|)^p}{M_p}\sum_{i=0}^{p}\Big\|\frac{d^{p-i}}{ds^{p-i}}S^1_{b,t}(s)x\Big\|M_p\\
&\qquad\leqslant e^{|\omega|\sup K}\sup_{p\in\mathbb{N}_0,\, s\in K}h^p(2+2|\omega|)^p\sum_{i=0}^{p}\frac{CM_{p-i}}{h_1^{p-i}M_p}\\
&\qquad\leqslant e^{|\omega|\sup K}\sup_{p\in\mathbb{N}_0,\, s\in K}\Big(\frac{h(2+2|\omega|)}{h_1}\Big)^p\sum_{i=0}^{p}\frac{Ch_1^i}{M_i}\\
&\qquad\leqslant Ce^{|\omega|\sup K}\sum_{i=0}^{\infty}\frac{h_1^i}{M_i}\leqslant Ce^{|\omega|\sup K}\sum_{i=0}^{\infty}\frac{h_1^i}{(2h_1)^i}\sup_{p\in\mathbb{N}_0}\frac{(2h_1)^p}{M_p}\\
&\qquad\leqslant 2Ce^{|\omega|\sup K}e^{M(2h_1)}<\infty,
\end{aligned}$$

where

$$C=\sup_{p\in\mathbb{N}_0,\, s\in K}\frac{h_1^p}{M_p}\Big\|\frac{d^p}{ds^p}S^1_{b,t}(s)x\Big\|.$$

Therefore, the property (iii) also holds and this completes the proof. □

Before proceeding further, let us notice that every Gevrey sequence satisfies (2.1) with $b\in(\frac{1}{s},1)$.

Corollary 2.1. *Suppose that A is a closed operator and that there exist $c\in(0,1)$, $\sigma>0$, $M>0$, $n\in\mathbb{N}$ and $\varsigma\in\mathbb{R}$ such that*

$$\Pi_{c,\sigma,\varsigma}:=\big\{\lambda\in\mathbb{C}:\operatorname{Re}(\lambda)\geqslant\sigma|\operatorname{Im}(\lambda)|^c+\varsigma\big\}\subseteq\rho(A),\tag{2.7}$$

$$\|R(\lambda:A)\|\leqslant M(1+|\lambda|)^n,\quad \lambda\in\Pi_{c,\sigma,\varsigma}.\tag{2.8}$$

Then, for every $b\in(c,1)$ and $\gamma\in\big(0,\arctan(\cos(\frac{b\pi}{2}))\big)$, there is an analytic operator family $(T_b(t))_{t\in\Sigma_\gamma}$ in $L(E)$ satisfying the properties (ii), (iv) *and* (v) *stated in the formulation of Theorem* 2.1. *Furthermore, the property* (iii) *holds for every compact set $K\subseteq[0,\infty)$ and $M_p=p^{p/c}$, and in the case when $D(A)$ is dense in E, we have that $(T_b(t))_{t\in\Sigma_\gamma}$ is an analytic semigroup of growth order $\frac{n+1}{b}$ and that there exists $\omega\in\mathbb{R}$ such that the c.i.g. of $(T_b(t))_{t\in\Sigma_\gamma}$ is $-(\omega-A)^c$.*

Proof. Clearly, $p^{p/b}\prec M_p$ and $M(|\lambda|)\sim\frac{1}{ce}|\lambda|^c$, $|\lambda|\to\infty$. This implies that there exist $\alpha>0$, $l>0$ and $\beta\in\mathbb{R}$ with $\Lambda_{\alpha,\beta,l}\subseteq\Pi_{c,\sigma,\varsigma}$. An application of Theorem 2.1 ends the proof. □

REMARK 2.1. Suppose A generates a distribution semigroup of [**18**]. Then one can employ [**18**, Corollary 3.12] in order to conclude that, for every $c > 0$, there exist $\sigma > 0$, $M > 0$, $n \in \mathbb{N}$ and $\varsigma \in \mathbb{R}$ such that (2.7) and (2.8) hold. Hence, for every $b \in (0,1)$ and $\gamma \in \big(0, \arctan(\cos(b\frac{\pi}{2}))\big)$, A generates a global $T_b(t)$-semigroup, where we define $T_b(t)$ as before; let us remind that Kunstmann [**18**] proved that this statement holds for every $b \in (0,1)$ and $\gamma \in \big(0, \frac{\pi(1-b)}{4}\big)$ (cf. also [**3**, p. 302]). Our estimate is better if $b \in (0, \frac{2}{\pi}]$. This follows from the following simple observation:

$$\Big|\arctan\Big(\cos\Big(b\frac{\pi}{2}\Big)\Big) - \frac{\pi}{4}\Big| \leqslant \Big|1 - \cos\Big(b\frac{\pi}{2}\Big)\Big| = 2\sin^2\Big(b\frac{\pi}{4}\Big) < \frac{b^2\pi^2}{8} \leqslant b\frac{\pi}{4}.$$

In conclusion, we obtain that there exists $\omega \in \mathbb{R}$ such that the solution of the incomplete Cauchy problem $u^{(k)}(t) = (-1)^{k+1}(A-\omega)u(t)$, $t > 0$, given by $T_{1/k}(\cdot)$, $k \in \mathbb{N} \smallsetminus \{1\}$, can be analytically extended to the larger sector $\Sigma_{\arctan(\cos(b\frac{\pi}{2}))}$.

3. Regularization of ultradistribution semigroups whose generators possess ultra-polynomially bounded resolvent

In this section, we assume that (M_p) satisfies (M.1), (M.2) and (M.3). We define the abstract Beurling space of (M_p) class associated to a closed linear operator A as in [**6**]. Put $E^{(M_p)}(A) := \operatorname{proj}\lim_{h\to+\infty} E_h^{\{M_p\}}(A)$, where

$$E_h^{\{M_p\}}(A) =: \Big\{x \in D_\infty(A) : \|x\|_h^{\{M_p\}} = \sup_{p\in\mathbb{N}_0} \frac{h^p\|A^p x\|}{M_p} < \infty\Big\}.$$

Then $\big(E_h^{\{M_p\}}(A),\ \|\cdot\|_h^{\{M_p\}}\big)$ is a Banach space, $E_{h'}^{\{M_p\}}(A) \subseteq E_h^{\{M_p\}}(A)$ if $0 < h < h' < \infty$ and $E^{(M_p)}(A)$ is a dense subspace of E whenever A is the generator of a regular (M_p)-ultradistribution semigroup ([**6**]). In general, we do not know whether the space $E^{(M_p)}(A)$ is nontrivial (cf. [**3**, p. 301] and [**6**, p. 185]). Notice that the simple inequality

$$\sup_{p\in\mathbb{N}_0} \frac{h^p\|(A-z)^p x\|}{M_p} \leqslant 2e^{M(h(4+4|z|))}\|x\|_{h(2+2|z|)}^{\{M_p\}}, \quad h > 0,\ z \in \mathbb{C},$$

implies $E^{(M_p)}(A) = E^{(M_p)}(A-z)$, $z \in \mathbb{C}$ and that, thanks to (M.2), we have that the part of A in $E^{(M_p)}(A)$ is a continuous mapping from $E^{(M_p)}(A)$ into $E^{(M_p)}(A)$. The next entire function of exponential type zero [**11**] plays a crucial role in our investigation: $\omega(z) = \prod_{i=1}^{\infty}(1 + \frac{iz}{m_p})$, $z \in \mathbb{C}$. We know the following (cf. for instance [**6**, pp. 169, 171, 182 and Lemma 3.2, p. 179]):

- (P.1) there exist $l_0 \geqslant 1$ and $c_0 > 0$ such that $|\omega^n(z)| \leqslant c_0^n A^{n-1} e^{M(l_0 H^{n-1}|z|)}$, $z \in \mathbb{C}$, $n \in \mathbb{N}$,
- (P.2) there exist $L > 0$ and $\sigma \in (0,1]$ such that $|\omega(iz)| \geqslant L|\omega(|z|)|^\sigma$, $z \in \overline{(\Lambda_{\alpha,\beta,l})^c}$,
- (P.3) due to [**11**, Proposition 4.6], the operator $\omega(lD) = \prod_{p=1}^{\infty}\big(1 + \frac{ilD}{m_p}\big)$, $l \in \mathbb{C}$, is an ultradifferential operator of class (M_p). Denote $\omega^n(z) = \sum_{p=0}^{\infty} a_{n,p} z^p$ and notice that $|a_{n,p}| \leqslant \operatorname{Const} \frac{(l_0 H^{n-1})^p}{M_p}$, $p \in \mathbb{N}_0$, $n \in \mathbb{N}$, which implies

that, for every $n \in \mathbb{N}$ and $l \in \mathbb{C}$, the operator $\omega^n(lD)$ is an ultradifferential operator of class (M_p) as long as (M_p) satisfies (M.3),

(P.4) for every $\alpha \geqslant 1$ and $z \in \mathbb{C}$: $|\omega(|z|)|^{\alpha} \geqslant \frac{1}{c_0}|\omega(\alpha l_0^{-1}|z|)|$, and

(P.5) $e^{(k+1)M(|z|)} \leqslant A^k e^{M(H^k|z|)}$, $z \in \mathbb{C}$.

Herein H denotes the constant appearing in the formulation of the condition (M.2). Suppose that A is the generator of a (UDSG) G of (M_p)-class. Then there exist constants $l \geqslant 1$, $\alpha > 0$ and $\beta > 0$ (cf. [**5**], [**6**, Theorem 1.5 and p. 181], [**13**], [**16**] and [**23**]) which satisfy:

$$\Lambda_{\alpha,\beta,l} \subseteq \rho(A) \text{ and } \|R(\lambda : A)\| \leqslant \text{Const}\,\frac{e^{M(Hl|\lambda|)}}{|\lambda|^k}, \quad \lambda \in \Lambda_{\alpha,\beta,l},\ k \in \mathbb{N}. \tag{3.1}$$

Let $n \in \mathbb{N}$ and $n > Hl_0 l\sigma^{-1}$. Following the proof of [**6**, Proposition 3.1], we define a bounded linear operator D_n by setting $D_n := \frac{1}{2\pi i}\int_{\Gamma_l}\frac{R(\lambda:A)}{\omega^n(i\lambda)}d\lambda$, where Γ_l denotes the upwards oriented boundary of $\Lambda_{\alpha,\beta,l}$. Then $D_{nk} = D_n^k$, $k \in \mathbb{N}$ and proceeding similarly as in the proofs of [**6**, Proposition 3.1] and [**6**, Theorem 3.8], one can prove: $\mathcal{R}(G) \subseteq R(D_n)$, $E^{(M_p)}(A) = \bigcap_{k\in\mathbb{N}} R(D_{nk})$, and since we have assumed that G satisfies (U.2) (cf. Definition 1.2), D_n is injective. Unfortunately, it is not clear whether $R(D_n) \subseteq E^{(M_p)}(A)$. Now we clarify the following important relationship between ultradistribution semigroups and local differentiable C-semigroups.

Theorem 3.1. *Suppose that A is the generator of a (UDSG) G of (M_p)-class. Then, for every $\tau \in (0,\infty)$, there exists an injective operator $C_\tau \in L(E)$ such that A generates a local C_τ-semigroup $(S(t))_{t\in[0,\tau)}$. Furthermore, $(S(t))_{t\in[0,\tau)}$ is infinitely differentiable in $[0,\tau)$ and there exists an $h \in (0,\infty)$, independent of $\tau \in (0,\infty)$, such that the next inequality holds:*

$$\sup_{t\in[0,\tau),\,p\in\mathbb{N}_0}\frac{h^p}{M_p}\Big\|\frac{d^p}{dt^p}S(t)\Big\| < \infty. \tag{3.2}$$

Proof. The arguments given in the final part of the proof of Theorem 2.1 imply that we can translate A by a convenient multiple of the identity and assume that constants $l \geqslant 1$, $\alpha > 0$ and $\beta > 0$ satisfy (3.1). Clearly,

$$|\omega(s)| = \prod_{k=1}^{\infty}\Big|1 + \frac{is}{m_k}\Big| \geqslant \sup_{p\in\mathbb{N}}\prod_{k=1}^{p}\frac{s}{m_k} = \sup_{p\in\mathbb{N}}\frac{s^p}{M_p} \geqslant e^{M(s)}, \quad s > 0. \tag{3.3}$$

Put $n_0 = \lfloor Hl_0 l\sigma^{-1}\rfloor + 1$, $k = \max(\lceil\tau\alpha\rceil, 2)$ and fix afterwards an element $x \in E$, an integer $n \in \mathbb{N}$ with $n \geqslant H^k + 2$ and a number $t \in [0,\tau)$. Then we have

$$(n-1)n_0\sigma \geqslant (n-1)Hll_0 > n-1 > 1, \tag{3.4}$$

$$(n-1)n_0\sigma l_0^{-1} \geqslant (H^k+1)Hl_0 l\sigma^{-1}\sigma l_0^{-1} > H^k l. \tag{3.5}$$

We define the bounded linear operator $S(t)$ (cf. also [**6**, pp. 188–189]) by

$$S(t) := \frac{1}{2\pi i}\int_{\Gamma_l} e^{\lambda t}\frac{R(\lambda : A)}{\omega^{nn_0}(i\lambda)}\,d\lambda.$$

In fact, $S(0) = D_{nn_0} := C_\tau \in L(E)$ is injective since G satisfies (U.2) (see the previous discussion). Notice that $n_0\sigma > 1$ and that (3.3)–(3.4), (P.2) and (P.4)–(P.5) together imply that, for every $p \in \mathbb{N}_0$:

$$\begin{aligned}
\Big\|\lambda^p \frac{e^{\lambda t}R(\lambda : A)}{\omega^{nn_0}(i\lambda)}\Big\| &\leqslant \mathrm{Const}\,|\lambda|^p \frac{e^{t(\alpha M(l|\lambda|)+\beta)}e^{M(Hl|\lambda|)}}{|\omega^{(n-1)n_0}(i\lambda)||\omega^{n_0}(i\lambda)|}\\
&\leqslant \mathrm{Const}\,|\lambda|^p e^{t\beta}\frac{A^{\lceil t\alpha\rceil-1}e^{M(H^{\lceil t\alpha\rceil-1}l|\lambda|)}e^{M(Hl|\lambda|)}}{|\omega^{(n-1)n_0}(i\lambda)||\omega^{n_0}(i\lambda)|}\\
&\leqslant \frac{\mathrm{Const}\,|\lambda|^p e^{t\beta}A^{\lceil t\alpha\rceil-1}}{|\omega(|\lambda|)|^{n_0\sigma}}\frac{e^{2M(H^{k-1}l|\lambda|)}}{L^{nn_0}|\omega(|\lambda|)|^{(n-1)n_0\sigma}}\\
&\leqslant \frac{\mathrm{Const}\,|\lambda|^p e^{t\beta}L^{-nn_0}A^{\lceil t\alpha\rceil}}{|\omega(|\lambda|)||\omega(|\lambda|)|^{n_0\sigma-1}}\frac{c_0|\omega(H^k l|\lambda|)|}{|\omega((n-1)n_0\sigma l_0^{-1}|\lambda|)|}\\
&\leqslant \frac{\mathrm{Const}\,|\lambda|^p e^{t\beta}L^{-nn_0}A^{\lceil t\alpha\rceil}}{|\omega(|\lambda|)|e^{M(|\lambda|(n_0\sigma-1))}}\frac{c_0|\omega(H^k l|\lambda|)|}{|\omega((n-1)n_0\sigma l_0^{-1}|\lambda|)|}\\
&\leqslant \mathrm{Const}\,|\lambda|^p \frac{1}{e^{M(|\lambda|(n_0\sigma-1))}|\lambda|^2},
\end{aligned} \tag{3.6}$$

where Const. is independent of $p \in \mathbb{N}_0$. The Fubini theorem implies $S(s)C_\tau = C_\tau S(s)$, $s \in [0,\tau)$, and furthermore, it is checked at once that $S(s)A \subseteq AS(s)$, $s \in [0,\tau)$. Since $\rho(A) \neq \emptyset$, we have $C_\tau^{-1}AC_\tau = A$. In order to see that $(S(t))_{t\in[0,\tau)}$ is a local C_τ-semigroup generated by A (cf. [**9**], [**17**] and [**22**]) it is enough to prove that $A\int_0^t S(s)x\,ds = S(t)x - C_\tau x$, $t \in [0,\tau)$. To see this, we will first prove the next equality:

$$\int\limits_{\Gamma_l} \frac{e^{\lambda t}}{\omega^{nn_0}(i\lambda)}\,d\lambda = 0. \tag{3.7}$$

For a sufficiently large $R > 0$, put $\Gamma_R = \{z \in \mathbb{C} : |z| = R,\ z \notin \Lambda_{\alpha,\beta,l}\}$. As above, (3.4) and (P.4) imply

$$|\omega(|z|)|^{(n-1)n_0\sigma} \geqslant \frac{1}{c_0}\big|\omega((n-1)n_0\sigma l_0^{-1}|z|)\big|, \quad z \in \mathbb{C}. \tag{3.8}$$

Taking into account (P.2) and (3.3), we have the following:

$$\begin{aligned}
|\omega^{nn_0}(i\lambda)| &= |\omega^{(n-1)n_0}(i\lambda)||\omega^{n_0}(i\lambda)|\\
&\geqslant L^{nn_0}|\omega^{(n-1)n_0\sigma}(R)|e^{M(R)n_0\sigma} \geqslant \mathrm{Const}\,|\omega^{(n-1)n_0\sigma}(R)|R^2,\ \lambda \in \Gamma_R.
\end{aligned}$$

An employment of (P.5) implies

$$\begin{aligned}
\Big|\frac{e^{\lambda t}}{\omega^{nn_0}(i\lambda)}\Big| &\leqslant \frac{e^{t(\alpha M(l|\operatorname{Im}(\lambda)|)+\beta)}}{|\omega^{nn_0}(i\lambda)|} \leqslant \frac{\mathrm{Const}}{R^2}e^{t\beta}\frac{e^{t\alpha M(lR)}}{|\omega^{(n-1)n_0\sigma}(R)|}\\
&\leqslant \frac{\mathrm{Const}}{R^2}e^{t\beta}\frac{A^{\lceil t\alpha\rceil-1}e^{M(H^{\lceil t\alpha\rceil-1}lR)}}{|\omega^{(n-1)n_0\sigma}(R)|}.
\end{aligned}$$

Owing to (3.8), we can continue the calculation as follows:

$$\leqslant \frac{\mathrm{Const}}{R^2} c_0 e^{t\beta} A^{\lceil t\alpha \rceil - 1} \frac{|\omega(H^{\lceil t\alpha \rceil - 1} l R)|}{|\omega((n-1) n_0 \sigma l_0^{-1} R)|}.$$

The last inequality and (3.5) imply

$$\int_{\Gamma_R} \frac{e^{\lambda t}}{\omega^{n n_0}(i\lambda)}\, d\lambda \to 0, \quad R \to +\infty.$$

Then the Cauchy theorem yields (3.7). Applying the Fubini theorem, the resolvent equation and (3.7), one obtains:

$$\begin{aligned}
A \int_0^t S(s) x\, ds &= \frac{1}{2\pi i} \int_0^t \int_{\Gamma_l} e^{\lambda s} \frac{A R(\lambda : A) x}{\omega^{n n_0}(i\lambda)}\, d\lambda\, ds = \frac{1}{2\pi i} \int_0^t \int_{\Gamma_l} e^{\lambda s} \frac{\lambda R(\lambda : A) x - x}{\omega^{n n_0}(i\lambda)}\, d\lambda\, ds \\
&= \frac{1}{2\pi i} \int_0^t \int_{\Gamma_l} e^{\lambda s} \frac{\lambda R(\lambda : A) x}{\omega^{n n_0}(i\lambda)}\, d\lambda\, ds - \frac{1}{2\pi i} \int_0^t \int_{\Gamma_l} e^{\lambda s} \frac{x}{\omega^{n n_0}(i\lambda)}\, d\lambda\, ds \\
&= \frac{1}{2\pi i} \int_0^t \int_{\Gamma_l} e^{\lambda s} \frac{\lambda R(\lambda : A) x}{\omega^{n n_0}(i\lambda)}\, d\lambda\, ds = \frac{1}{2\pi i} \int_{\Gamma_l} \left[\int_0^t e^{\lambda s} \frac{\lambda R(\lambda : A) x}{\omega^{n n_0}(i\lambda)} ds \right] d\lambda \\
&= \frac{1}{2\pi i} \int_{\Gamma_l} (e^{\lambda t} - 1) \frac{R(\lambda : A) x}{\omega^{n n_0}(i\lambda)}\, d\lambda \\
&= \frac{1}{2\pi i} \int_{\Gamma_l} e^{\lambda t} \frac{R(\lambda : A) x}{\omega^{n n_0}(i\lambda)}\, d\lambda - \frac{1}{2\pi i} \int_{\Gamma_l} \frac{R(\lambda : A) x}{\omega^{n n_0}(i\lambda)}\, d\lambda = S(t) x - C_\tau x.
\end{aligned}$$

As before, we have that, for every $p \in \mathbb{N}$, the integral

$$\frac{1}{2\pi i} \int_{\Gamma_l} \lambda^p e^{\lambda t} \frac{R(\lambda : A)}{\omega^{n n_0}(i\lambda)}\, d\lambda, \quad t \in [0, \tau)$$

is convergent and that

$$\frac{d}{dt} S(t) = \frac{1}{2\pi i} \int_{\Gamma_l} \lambda e^{\lambda t} \frac{R(\lambda : A)}{\omega^{n n_0}(i\lambda)}\, d\lambda.$$

Inductively,

$$\frac{d^p}{dt^p} S(t) = \frac{1}{2\pi i} \int_{\Gamma_l} \lambda^p e^{\lambda t} \frac{R(\lambda : A)}{\omega^{n n_0}(i\lambda)}\, d\lambda, \quad p \in \mathbb{N}_0,\ t \in [0, \tau). \tag{3.9}$$

It remains to be shown (3.2). Choose arbitrarily a number $h \in (0, n_0\sigma - 1)$. An application of (3.6) and (3.9) gives

$$\sup_{t \in [0,\tau),\, p \in \mathbb{N}_0} \frac{h^p}{M_p} \left\| \frac{d^p}{dt^p} S(t) \right\| \leqslant \frac{1}{2\pi} \sup_{t \in [0,\tau),\, p \in \mathbb{N}_0} \frac{h^p}{M_p} \int_{\Gamma_l} \frac{|\lambda|^p |e^{\lambda t}|\, \|R(\lambda : A)\|}{|\omega^{n n_0}(i\lambda)|} |d\lambda|$$

$$\leqslant \text{Const} \int_{\Gamma_l} \frac{e^{M(h|\lambda|)}}{e^{M(|\lambda|(n_0\sigma-1))}|\lambda|^2}|d\lambda| \leqslant \text{Const} \int_{\Gamma_l} \frac{|d\lambda|}{|\lambda|^2} < \infty.$$

The proof is now completed. □

EXAMPLE 3.1. ([**19**], [**20**], [**15**]) Define

$$E_{M_p} =: \left\{ f \in C^\infty[0,1] : \|f\|_{M_p} := \sup_{p \geqslant 0} \frac{\|f^{(p)}\|_\infty}{M_p} < \infty \right\},$$
$$A_{M_p} := -d/ds, \quad D(A_{M_p}) =: \{f \in E_{M_p} : f' \in E_{M_p},\ f(0) = 0\}.$$

Proceeding as in [**19**, Example 1.6], one can verify that A_{M_p} is not stationary dense and that A_{M_p} cannot be the generator of a distribution semigroup. Furthermore, $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \geqslant 0\} \subseteq \rho(A_{M_p})$ and $\|R(\lambda : A_{M_p})\| \leqslant Ce^{M(\tilde{r}|\lambda|)}$, $\operatorname{Re}(\lambda) \geqslant 0$, for some $C > 0$ and $\tilde{r} > 0$ [**11, 19**]. Put now

$$(G(\varphi)f)(x) := \int_0^x \varphi(x-t)f(t)\,dt, \quad \varphi \in \mathcal{D}^{(M_p)},\ f \in E_{M_p},\ x \in [0,1].$$

Clearly, $G(\varphi)f \in C^\infty[0,1]$ and

$$\frac{d^p}{dx^p}(G(\varphi)f)(x) = \int_0^x \varphi^{(p)}(x-t)f(t)\,dt + \sum_{k=0}^{p-1} \varphi^{(p-1-k)}(0)f^{(k)}(x),$$

for every $\varphi \in \mathcal{D}^{(M_p)}$, $f \in E_{M_p}$, $x \in [0,1]$ and $p \in \mathbb{N}_0$. Since $M_{p+q} \geqslant M_pM_q$, $p, q \in \mathbb{N}_0$, the preceding equality implies that, for every $p \in \mathbb{N}_0$, $x \in [0,1]$, $\varphi \in \mathcal{D}^{(M_p)}$ and $f \in E_{M_p}$:

$$\left|\frac{d^p}{dx^p}(G(\varphi)f)(x)\frac{1}{M_p}\right| \leqslant \|\varphi\|_{M_p,1,[0,1]}\|f\| + \sum_{k=0}^{p-1}\left|\frac{\varphi^{(p-1-k)}(0)}{M_{p-k}}\right|\|f\|$$
$$\leqslant \|\varphi\|_{M_p,1,[0,1]}\left(1 + \sum_{k=0}^{p-1}\frac{1}{m_{p-k}}\right)\|f\| \leqslant \|\varphi\|_{M_p,1,[0,1]}\left(1 + \sum_{p=0}^{\infty}\frac{1}{m_p}\right)\|f\|.$$

Hence, $\|G(\varphi)\| \leqslant \|\varphi\|_{M_p,1,[0,1]}(1 + \sum_{p=0}^\infty \frac{1}{m_p})$ and $G \in \mathcal{D}_0^{'(M_p)}(L(E))$. The conditions (U.1) and (U.2) can be proved trivially, and consequently, G is a (UDSG) of (M_p)-class whose generator is obviously the operator A_{M_p}. By Theorem 3.1, we have that there exists an injective operator $C \in L(E_{M_p})$ such that A_{M_p} generates a differentiable local C-semigroup $(S(t))_{t\in[0,2)}$. Put, for every fixed $f \in E_{M_p}$, $x \in [0,1]$ and $t \in [0,1]$, $u(t,x) := (S(t)f)(x)$. According to the differentiability of $(S(t))_{t\in[0,2)}$ and the proof of Theorem 3.1, one immediately obtains that u is a solution of the problem

$$(P): \begin{cases} u \in C^1([0,1]\times[0,1]) \\ u_x + u_t = 0 \\ u(0,x) = (Cf)(x),\ u(t,0) = 0. \end{cases}$$

Hence, for every $(t,x) \in [0,1] \times [0,1]$,

$$(S(t)f)(x) = \begin{cases} 0, & 0 \leqslant x \leqslant t \\ [Cf](x-t), & 1 \geqslant x > t. \end{cases}$$

In particular, $S(t) = 0$, $t \in [1,2)$. Define now $\tilde{S}(t)$, $t \geqslant 0$ by $\tilde{S}(t) := S(t)$, $t \in [0,1]$ and $\tilde{S}(t) := 0$, $t > 1$. Then $(\tilde{S}(t))_{t \geqslant 0}$ is a global differentiable C-semigroup generated by A_{M_p}. The previous analysis and Theorem 3.2 given below imply that there exists an injective operator $C_1 \in L(E)$ such that A_{M_p} generates a global differentiable C_1-semigroup $(\tilde{S}_1(t))_{t \geqslant 0}$ such that $\tilde{S}_1(t) = 0$, $t \geqslant 1$ and that

$$\sup_{t \geqslant 0,\, p \in \mathbb{N}_0} \frac{h^p}{M_p} \Big\| \frac{d^p}{dt^p} \tilde{S}_1(t) \Big\| < \infty$$

for every fixed number $h > 0$.

The proof of the following lemma essentially follows from the corresponding one of [**6**, Theorem 3.8].

Lemma 3.1. *Suppose G is a (UDSG) of (M_p)-class generated by A, $l \geqslant 1$, $\alpha > 0$, $\beta > 0$, $n \in \mathbb{N}$, $n > Hl_0 l\sigma^{-1}$ (cf.* (P.1)–(P.5)) *and* (3.1) *holds. Then $E^{(M_p)}(A) = \bigcap_{k \in \mathbb{N}} D_{nk}(D^\infty(A))$ and*

(3.10)
$$D_{nk}^{-1}(E^{\{M_p\}}_{2l_0 H^{nk+1}}) \subseteq \Big\{ x \in D^\infty(A) : \sup_{p \in \mathbb{N}_0} \frac{\varsigma^p \|A^p x\|}{M_p} < \infty \text{ for all } \varsigma \in (0, 2l_0 H^{nk}) \Big\}.$$

Proof. Fix an integer $k \in \mathbb{N}$ and a number $\varsigma \in (0, 2l_0 H^{nk})$. Further, put $h = 2l_0 H^{nk+1}$ and suppose that $y \in E_h^{\{M_p\}}$ and that $\omega^{nk}(iz) = \sum_{p=0}^\infty a_{k,p} z^p$, $z \in \mathbb{C}$. By (P.3), we have that $|a_{k,p}| \leqslant \text{Const}\, \frac{(l_0 H^{nk})^p}{M_k}$, $p \in \mathbb{N}$ and that the series $\sum_{p=0}^\infty a_{k,p} A^p y := x$ is convergent since

$$|a_{k,p}|\, \|A^p y\| \leqslant \text{Const}\, \frac{h^p \|A^p y\|}{M_p} \Big(\frac{l_0 H^{nk}}{h} \Big)^p \leqslant \text{Const}\, \|y\|_h^{\{M_p\}} \Big(\frac{1}{2H} \Big)^p.$$

Proceeding as in the proof of [**6**, Theorem 3.8, p. 187], one gets that $y = D_{nk} x$ and the proof is completed if one shows that $x \in D^\infty(A)$ and that (3.10) holds with ς. First of all, let us observe that the series $\sum_{p=0}^\infty a_{k,p} A^{m+p} y$ is also convergent for all $m \in \mathbb{N}$. Indeed, (M.2) yields

$$\begin{aligned} (3.11) \quad & |a_{k,p}|\, \|A^{m+p} y\| \\ & \leqslant \text{Const}\, \frac{h^{p+m} \|A^{p+m} y\|}{M_{p+m}} \Big(\frac{l_0 H^{nk}}{h} \Big)^p \frac{M_{p+m}}{M_p h^m} \leqslant \text{Const}\, \|y\|_h^{\{M_p\}} \Big(\frac{1}{2H} \Big)^p \frac{M_{p+m}}{M_p h^m} \\ & \leqslant \text{Const}\, \|y\|_h^{\{M_p\}} \Big(\frac{1}{2H} \Big)^p \frac{A H^{p+m} M_m}{h^m} \leqslant \text{Const}\, \|y\|_h^{\{M_p\}} \Big(\frac{1}{2} \Big)^p \Big(\frac{1}{2l_0 H^{nk}} \Big)^m M_m. \end{aligned}$$

By (3.11), we have that

$$\sum_{p=0}^\infty |a_{k,p}|\, \|A^{p+m} y\| \leqslant \text{Const}\, \|y\|_h^{\{M_p\}} \Big(\frac{1}{2l_0 H^{nk}} \Big)^m M_m, \quad x \in D^\infty(A)$$

and $A^m x = \sum_{p=0}^{\infty} a_{k,p} A^{m+p} y$. Then the proof of Lemma 3.1 completes an employment of the estimate (3.11):

$$\sup_{m\in\mathbb{N}_0} \frac{\varsigma^m \|A^m x\|}{M_m} \leqslant \mathrm{Const}\, \|y\|_h^{\{M_p\}} \sup_{m\in\mathbb{N}_0} \Big(\frac{\varsigma}{2l_0 H^{nk}}\Big)^m \leqslant \mathrm{Const}\, \|y\|_h^{\{M_p\}}. \qquad \square$$

Now we are in a position to clarify the following analogue of [**6**, Theorem 4.1, Corollary 4.2] for nondense ultradistribution semigroups of (M_p)-class.

THEOREM 3.2. *Suppose that A generates a $(UDSG)$ of (M_p)-class. Then the abstract Cauchy problem*

$$(ACP): \begin{cases} u \in C^\infty([0,\infty) : E) \cap C([0,\infty) : [D(A)]), \\ u'(t) = Au(t),\ t \geqslant 0, \\ u(0) = x, \end{cases}$$

has a unique solution for all $x \in E^{(M_p)}(A)$. Furthermore, for every compact set $K \subseteq [0,\infty)$ and $h > 0$, the solution u of (ACP) satisfies

$$\sup_{t\in K,\, p\in\mathbb{N}_0} \frac{h^p}{M_p} \Big\| \frac{d^p}{dt^p} u(t) \Big\| < \infty.$$

PROOF. We basically follow the terminology given in the proofs of Theorem 3.1 and Lemma 3.1 (cf. also (P.1)–(P.5)). The uniqueness of solution of (ACP) is a consequence of the Ljubich uniqueness theorem (cf. for instance [**23**, p. 29]). To prove the existence of solutions of (ACP), let us observe that the proof of Theorem 3.1 implies that there exist a number $n_0 \in \mathbb{N}$ and a strictly increasing sequence (k_l) in $\mathbb{N}$ such that $n_0 > H l_0 l \sigma^{-1}$ and that, for every $l \in \mathbb{N}$, the operator A is the generator of a differentiable $D_{n_0 k_l}$-semigroup $(S_l(t))_{t\in[0,l)}$. This implies that the abstract Cauchy problem

$$\begin{cases} u_l \in C^1([0,l) : E) \cap C([0,l) : [D(A)]), \\ u_l'(t) = Au_l(t),\ t \geqslant 0, \\ u_l(0) = x, \end{cases}$$

has a unique solution for every $x \in D_{n_0 k_l}(D(A))$ given by $u_l(t) = D_{n_0 k_l}^{-1} S_l(t)x$, $t \in [0,l)$. If $x \in E^{(M_p)}(A)$, then Lemma 3.1 implies that $u_l(t) = S_l(t) D_{n_0 k_l}^{-1} x$, $t \in [0,l)$, and due to Theorem 3.1, we get $u_l \in C^\infty([0,l) : E)$. Therefore, we automatically obtain the existence of a solution of (ACP) for all $x \in E^{(M_p)}(A)$. Let $K \subseteq [0,\infty)$ be a compact set, $K \subseteq [0,l)$ for some $l \in \mathbb{N}$, $h > 0$, $l' \in \mathbb{N}$, $l' > l$ and $2l_0 H^{n_0 k_{l'}} > h$. Then Lemma 3.1 and the proof of Theorem 3.1 imply that, for every $t \in K$:

$$\begin{aligned} \frac{h^p}{M_p} \Big\| \frac{d^p}{dt^p} u(t) \Big\| &= \frac{h^p}{M_p} \Big\| \frac{d^p}{dt^p} u_{l'}(t) \Big\| = \frac{h^p}{M_p} \|A^p u_{l'}(t)\| = \frac{h^p}{M_p} \|A^p D_{n_0 k_{l'}}^{-1} S_{l'}(t)x\| \\ &= \frac{h^p}{M_p} \|S_{l'}(t) A^p D_{n_0 k_{l'}}^{-1} x\| \leqslant \sup_{s\in[0,l]} \|S_{l'}(s)\| \frac{h^p}{M_p} \|A^p D_{n_0 k_{l'}}^{-1} x\| < \infty \end{aligned}$$

for all $x \in E^{(M_p)}(A)$, which completes the proof of the theorem. $\square$

Lemma 3.2. *There exists a sequence (N_p) of positive real numbers satisfying $N_0 = 1$, $(M.1)$, $(M.2)$, $(M.3)$ and $N_p \prec M_p$.*

Proof. Define a sequence (r_p) of positive real numbers recursively by:

$$r_1 := 1 \text{ and } r_{p+1} := r_p\Big[\frac{m_p}{m_{p+1}} + \min\Big(1 - \frac{m_p}{m_{p+1}}, \frac{1}{p}\frac{m_p}{m_{p+1}}\Big)\Big],\ p \in \mathbb{N}.$$

Then:

$$1 \geqslant \frac{r_{p+1}}{r_p} \geqslant \frac{m_p}{m_{p+1}} \text{ and } r_{p+1} \leqslant r_p\Big(1 + \frac{1}{p}\Big)\frac{m_p}{m_{p+1}},\ p \in \mathbb{N}. \tag{3.12}$$

Using (3.12), one obtains inductively:

$$r_p \leqslant p\frac{m_1}{m_p} \text{ and } \prod_{i=1}^{p} r_i \leqslant p!\frac{m_1^p}{M_p},\ p \in \mathbb{N}.$$

Since $p! \prec M_p$ (cf. [**11**, p. 74] and [**4**, Lemma 2.1.2]), one gets that, for every $\sigma > 0$:

$$\sup_{p\in\mathbb{N}_0} \sigma^p \prod_{i=1}^{p} r_i < \infty. \tag{3.13}$$

Put now $N_0 := 1$ and $N_p := M_p \prod_{i=1}^{p} r_i$, $p \in \mathbb{N}$. Keeping in mind (3.12), one can simply prove that (N_p) satisfies (M.1), (M.2) (with the same constants A and H) and (M.3). By (3.13), $N_p \prec M_p$ and this completes the proof. $\square$

Now we are able to state the following important result.

Theorem 3.3. *Suppose that A generates a (UDSG) G of (M_p)-class. Then there exists an injective operator $C \in L(E)$ such that A generates a global differentiable C-semigroup $(S(t))_{t\geqslant 0}$. Furthermore, for every compact set $K \subseteq [0,\infty)$ and $h > 0$, we have:*

$$\sup_{t\in K,\, p\in\mathbb{N}_0} \frac{h^p}{M_p}\Big\|\frac{d^p}{dt^p}S(t)\Big\| < \infty.$$

Proof. By Lemma 3.2, we have the existence of a sequence (N_p) of positive real numbers satisfying $N_0 = 1$, (M.1), (M.2), (M.3) and $N_p \prec M_p$. As in the proof of Theorem 3.1, we may assume that numbers $l \geqslant 1$, $\alpha > 0$ and $\beta > 0$ satisfy (3.1). Denote by $N(\cdot)$ the associated function of (N_p) and notice that the previously given arguments combined with [**11**, Lemma 3.10] indicate that there exist $\alpha_1 > 0$ and $\beta_1 > 0$ such that $\Lambda^1_{\alpha_1,\beta_1,l} \subseteq \Lambda_{\alpha,\beta,l} \subseteq \rho(A)$. Furthermore, one has that, for every $\mu > 0$, there exists $C_\mu > 0$ such that $M(\lambda) \leqslant N(\mu\lambda) + C_\mu$, $\lambda \geqslant 0$, and thanks to [**26**, Lemma 1.7, p. 140] (cf. also [**4**, Lemma 2.1.3]), we know that, for every $L \geqslant 1$, there exist a constant $B > 0$ and a constant $E_L > 0$ such that

$$LN(\lambda) \leqslant N(B^{L-1}\lambda) + E_L, \quad \lambda \geqslant 0. \tag{3.14}$$

Let Γ_1 and Γ_2 denote the upwards oriented boundaries of $\Lambda_{\alpha,\beta,l}$ and $\Lambda^1_{\alpha_1,\beta_1,l}$, respectively. Suppose that $\varrho \in \mathcal{D}^{(N_p)}_{[0,1]}$ satisfies $\varrho(t) \geqslant 0$, $t \in \mathbb{R}$, $\int_{\mathbb{R}} \varrho(t)\,dt = 1$ and put $\varrho_n(t) := n\varrho(nt)$, $t \in \mathbb{R}$, $n \in \mathbb{N}$. Then it can be simply verified that, for every $\varphi \in \mathcal{D}^{(M_p)}$, we have $\varrho_n * \varphi \in \mathcal{D}^{(N_p)} \subseteq \mathcal{D}^{(M_p)}$, $n \in \mathbb{N}$ and that $\lim_{n\to\infty} \varrho_n * \varphi = \varphi$

in $\mathcal{D}^{(M_p)}$. Define $G_1(\varphi) := G(\varphi)$, $\varphi \in \mathcal{D}^{(N_p)}$. Then $G_1 \in \mathcal{D}_0^{'(N_p)}(L(E))$ and satisfies $(U.1)$. To prove (U.2), suppose $G_1(\varphi)x = 0$ for all $\varphi \in \mathcal{D}_0^{(N_p)}$. Then $G(\psi)x = \lim_{n\to\infty} G(\varrho_n * \psi)x = \lim_{n\to\infty} G_1(\varrho_n * \psi)x = 0$ for all $\psi \in \mathcal{D}_0^{(M_p)}$. So, $x = 0$, G_1 is a (UDSG) of (N_p)-class and it can be simply checked that the generator of G_1 is A. Set $\omega_{N_p}(z) := \prod_{i=1}^{\infty}\big(1 + \frac{izN_{p-1}}{N_p}\big)$, $z \in \mathbb{C}$ and notice that $|\omega_{N_p}(s)| \geqslant e^{N(s)}$, $s \geqslant 0$ and that, owing to (P.2), there exist $L_1 > 0$ and $\sigma_1 \in (0,1]$ such that $|\omega_{N_p}(iz)| \geqslant L_1|\omega_{N_p}(|z|)|^{\sigma_1}$, $z \in \overline{(\Lambda^1_{\alpha_1,\beta_1,l})^c}$. Since G_1 is a (UDSG) generated by A, we have that there exists a sufficiently large $n \in \mathbb{N}$ such that $n \geqslant \lceil \frac{1}{\sigma_1} \rceil$ and that the bounded linear operator

$$C := \frac{1}{2\pi i}\int_{\Gamma_2} \frac{R(\lambda : A)}{\omega^n_{N_p}(i\lambda)}\, d\lambda$$

is injective. An elementary application of the Cauchy formula implies that

$$C = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{R(\lambda : A)}{\omega^n_{N_p}(i\lambda)}\, d\lambda.$$

Set now

$$S(t)x := \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{e^{\lambda t}R(\lambda : A)x}{\omega^n_{N_p}(i\lambda)}\, d\lambda, \quad t \geqslant 0,\ x \in E.$$

Taking into account the simple equality

$$\int_{\Gamma_1} \frac{\lambda^p}{\omega^n_{N_p}(i\lambda)}\, d\lambda = 0,$$

one can repeat literally the proof of Theorem 3.1 in order to deduce that $(S(t))_{t\geqslant 0}$ is a global differentiable C-semigroup generated by A and that, for every $p \in \mathbb{N}_0$,

$$\frac{d^p}{dt^p}S(t) = \frac{1}{2\pi i}\int\limits_{\Gamma_1} \lambda^p e^{\lambda t}\frac{R(\lambda : A)}{\omega^n_{N_p}(i\lambda)}\, d\lambda, \quad p \in \mathbb{N}_0,\ t \geqslant 0.$$

Suppose now that $K \subseteq [0,\infty)$ is a compact set and that $h > 0$. By (3.14), we get that, for every $\mu > 0$, there exists $M_\mu > 0$ such that:

$$\begin{aligned}
\sup_{t\in K,\, p\in\mathbb{N}_0} \frac{h^p}{M_p}\Big\|\frac{d^p}{dt^p}S(t)\Big\| &\leqslant \text{Const}\, e^{\beta \sup K}\int\limits_{\Gamma_1} \frac{e^{M(h|\lambda|)+\alpha \sup K M(l|\lambda|)}e^{M(Hl|\lambda|)}}{|\omega^n_{N_p}(i\lambda)||\lambda|^2}|d\lambda| \\
&\leqslant \text{Const}\, e^{\beta \sup K}\int\limits_{\Gamma_1} \frac{e^{M(h\lambda)+\alpha \sup K M(l|\lambda|)+M(Hl|\lambda|)}}{|\lambda|^2 L_1^n|\omega_{N_p}(|\lambda|)|^{n\sigma_1}}|d\lambda| \\
&\leqslant \text{Const}\int\limits_{\Gamma_1} \frac{e^{M(h\lambda)+\alpha \sup K M(l|\lambda|)+M(Hl|\lambda|)}}{|\lambda|^2 e^{n\sigma_1 N(|\lambda|)}}|d\lambda| \\
&\leqslant M_\mu\int\limits_{\Gamma_1} \frac{e^{N(h|\lambda|\mu)+\alpha \sup K N(l|\lambda|\mu)+N(Hl|\lambda|\mu)}}{|\lambda|^2 e^{n\sigma_1 N(|\lambda|)}}|d\lambda| \qquad (3.15)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant M_\mu \int\limits_{\Gamma_1} \frac{e^{N(h|\lambda|\mu)+N(B^{\alpha\sup K}l|\lambda|\mu)+E_{K,\alpha}+N(Hl|\lambda|\mu)}}{|\lambda|^2 e^{n\sigma_1 N(|\lambda|)}}|d\lambda|\\
&\leqslant M_\mu e^{E_{\alpha\sup K}} \int\limits_{\Gamma_1} \frac{e^{3N(|\lambda|\mu[h+lB^{\alpha\sup K}+Hl])}}{|\lambda|^2 e^{n\sigma_1 N(|\lambda|)}}|d\lambda|\\
&\leqslant M_\mu e^{E_{\alpha\sup K}+E_3} \int\limits_{\Gamma_1} \frac{e^{N(B^2|\lambda|\mu[h+lB^{\alpha\sup K}+Hl])}}{|\lambda|^2 e^{n\sigma_1 N(|\lambda|)}}|d\lambda|\\
&\leqslant M_\mu e^{E_{\alpha\sup K}+E_3} \int\limits_{\Gamma_1} \frac{e^{N(B^2|\lambda|\mu[h+lB^{\alpha\sup K}+Hl])}}{|\lambda|^2 e^{N(|\lambda|)}}|d\lambda|.
\end{aligned}
$$

Suppose now that $\mu \in \big(0, 1/B^2(h+lB^{\alpha\sup K}+Hl)\big)$. Then we obtain from (3.15):

$$
\sup_{t\in K,\,p\in\mathbb{N}_0} \frac{h^p}{M_p}\Big\|\frac{d^p}{dt^p}S(t)\Big\| \leqslant M_\mu e^{E_{\alpha\sup K}+E_3}\int\limits_{\Gamma_1}\frac{|d\lambda|}{|\lambda|^2} < \infty.
$$

The proof of the theorem is completed. $\square$

Finally, we consider regularization of ω-ultradistribution semigroups in the sense of [**3**]. We use the terminology given in [**3**, p. 308] and suppose that $\omega : [0,\infty) \to [0,\infty)$ is a continuous, concave, increasing function satisfying

$$
\lim_{t\to\infty}\omega(t)=\infty,\quad \lim_{t\to\infty}\frac{\omega(t)}{t}=0 \text{ and } \int_1^\infty \frac{\omega(t)}{t^2}\,dt<\infty.
$$

Let $x_0 \in (0,\infty)$ be fixed. Put $\Omega(\omega) := \big\{\lambda\in\mathbb{C} : \operatorname{Re}(\lambda) \geqslant \max(x_0, \omega(|\operatorname{Im}(\lambda)|))\big\}$.

Theorem 3.4. *Suppose A is a closed, linear operator which satisfies $\Omega(\omega)\subseteq\rho(A)$ and $\|R(\lambda : A)\| \leqslant M(1+|\lambda|)^n$, $\lambda\in\Omega(\omega)$, for some $M>0$ and $n\in\mathbb{N}$. Then there is a family of bounded injective operators $(C(k,\varepsilon))_{\varepsilon>0}$ such that, for every $\varepsilon>0$, A generates a global $C(k,\varepsilon)$-semigroup.*

Proof. Let Γ be the curve employed in the proof of [**2**, Theorem 1]; note that Γ is oriented from the lower to the upper half plane and that zero lies to the right of Γ. If W is the region to the left of Γ, then, without loss of generality, one may translate A and assume that $R(\cdot : A)$ exists for all $z\notin W$. We refer to [**2**, pp. 287–290] for more details. If $\varepsilon>0$ is given, define the function $h_\varepsilon(\cdot)$ in the same way as in the proof of [**2**, Theorem 1] (cf. [**2**, p. 290]) and an operator $C(k,t,\varepsilon)\in L(E)$ by

$$
C(k,t,\varepsilon)x := \frac{1}{2\pi i}\int\limits_{\Gamma} h_\varepsilon(\lambda)e^{\lambda t}R(\lambda : A)x\,d\lambda,\quad \varepsilon>0,\ t\geqslant 0,\ x\in E.
$$

Put $C(k,\varepsilon)=C(k,0,\varepsilon)$, $\varepsilon>0$. It is obvious that the operator $C(k,\varepsilon)$ is injective; see also [**3**, Lemma 3 and p. 308]. Put $G^*(k,\varepsilon) := R(C(k,\varepsilon))$ and $G^*(k) := \bigcup_{\varepsilon>0}G^*(k,\varepsilon)$. Although we do not have the intrinsic characterization of $G^*(k)$ as a subspace of E, we know that the abstract Cauchy problem (ACP) is well-posed if $x\in G^*(k)$ [3]. An application of [**9**, Theorem 4.15] gives that, for every

$\varepsilon > 0$, A generates a global $C(k,\varepsilon)$-semigroup $(C(k,t,\varepsilon))_{t\geqslant 0}$. Furthermore, the constructed $C(k,\varepsilon)$-semigroup $(C(k,t,\varepsilon))_{t\geqslant 0}$ is infinitely differentiable for $t > 0$ (cf. [**2**, p. 290]). □

4. Ultradifferentiable properties of entire solutions of higher-order abstract Cauchy problems and regularization of ultradistribution sines

First of all, we recall the assertion of [**32**, Theorem 6.2, p. 132] with $\alpha = N \in \mathbb{N}$:

Theorem 4.1. *Suppose $n \in \mathbb{N}$, $n \geqslant 2$, $\theta \in (0, \frac{\pi}{2})$, $M > 0$, $z_0 \in \mathbb{C}$ and $z_0 \neq 0$. If a closed linear operator A satisfies:*

$$e^{i\arg(z_0)}(|z_0| + \Sigma_\theta) \subseteq \rho(A), \tag{4.1}$$

$$\|R(\lambda : A)\| \leqslant M(1+|\lambda|)^N,\ \ \lambda \in e^{i\arg(z_0)}(|z_0| + \Sigma_\theta), \tag{4.2}$$

(i) *For every $\epsilon > 0$, there exists a unique solution u of the abstract Cauchy problem (ACP_n) with initial data $x_0, \dots, x_{n-1} \in R(C_\epsilon)$ and*

$$\|u(t)\| \leqslant M(t) \sum_{i=0}^{n-1} \|C_\epsilon^{-1} x_i\|, \quad t \geqslant 0, \tag{4.3}$$

for some non-negative and locally bounded function $M(t)$, $t \geqslant 0$.

(ii) $\bigcup_{\epsilon>0} C_\epsilon(D(A^{N+2}))$ *is dense in $D(A^{N+2})$.*

Our essential contribution is related to the estimate (4.3) quoted in the formulation of Theorem 4.1. Actually, by inspecting the proof of [**32**, Theorem 6.2, p. 132], we are in a position to conclude that, for every $\epsilon > 0$, the solution u of (ACP_n) can be analytically extended to an entire function taking values in E. Furthermore, the derivatives of such a solution possess some interesting properties of operator valued ultradifferentiable functions and this is the substantial part of the following assertion which reads as follows:

Theorem 4.2. *Let (M_p) satisfy* (M.1) *and* (M.3)$'$. *Suppose $n \in \mathbb{N}$, $n \geqslant 2$, $\theta \in (0, \frac{\pi}{2})$, $M > 0$, $z_0 \in \mathbb{C}$, $z_0 \neq 0$, $N \in \mathbb{N}$ and A is a closed linear operator satisfying* (4.1) *and* (4.2). *Then, for every $b \in \big(\frac{1}{n}, \frac{\pi}{2(\pi-\theta)}\big)$ and $\gamma \in \big(0, \arctan(\cos(b(\pi-\theta)))\big)$, there exists an analytic operator family $(T_b(t))_{t\in\Sigma_\gamma}$ so that:*

(i) *For every $t \in \Sigma_\gamma$, $T_b(t)$ is injective and there exists $C > 0$ with $\|T_b(t)\| \leqslant C(\tan(\gamma)\operatorname{Re}(t) - |\operatorname{Im}(t)|)^{-(N+1)/b}$, $t \in \Sigma_\gamma$.*

(ii) *If $x \in D(A^{N+2})$, then there exists $\lim_{t\to 0+}(T_b(t)x - x)/t$, and particularly, we have $\lim_{t\to 0+} T_b(t)x = x$.*

(iii) *For every $t \in \Sigma_\gamma$, there exists a unique solution $u(\cdot;t)$ of the abstract Cauchy problem (ACP_n) with initial data $x_0, \dots, x_{n-1} \in R(T_b(t))$ and $u(\cdot;t)$ can be analytically extended to the whole complex plane. Furthermore, for every compact set $K \subseteq \mathbb{C}$, $h > 0$ and $t \in \Sigma_\gamma$:*

$$\sum_{l=0}^{n-1} \sup_{z\in K,\, p\in\mathbb{N}} \frac{h^p}{M_{np-1+l}} \Big\| A^p \frac{d^l}{dz^l} u(z;t) \Big\| \leqslant \text{Const} \sum_{i=0}^{n-1} \|T_b(t)^{-1} x_i\|. \tag{4.4}$$

Proof. Let $t \in \Sigma_\gamma$ and $d \in (0,1]$ be fixed. Choose a set of initial data $\{x_0, \dots, x_{n-1}\} \subseteq R(T_b(t))$, an $\omega \in (|z_0| + d, \infty)$ and an $a \in (0, \theta)$ such that $b \in (\frac{1}{n}, \frac{\pi}{2(\pi - a)})$ and $\gamma \in \big(0, \arctan(\cos(b(\pi - a)))\big)$. Here we would like to notice that $\frac{1}{n} < \frac{\pi}{2(\pi-\theta)}$ since $n \geqslant 2$. Put $A_0 := e^{-i \arg(z_0)} A$. Then we know that $|z_0| + \Sigma_\theta \subseteq \rho(A_0)$ and that $R(\lambda : A_0) = e^{i \arg(z_0)} R(e^{i \arg(z_0)} \lambda : A)$, $\lambda \in |z_0| + \Sigma_\theta$. Thereby, $\|R(\lambda : A_0)\| \leqslant M(1 + |\lambda|)^N$, $\lambda \in |z_0| + \Sigma_\theta$. Designate by $\Gamma_{a,d}$ the upwards oriented boundary of $\Omega_{a,d}$ and define the bounded operator $T_b(t)$ by:

$$T_b(t)x := \frac{1}{2\pi i} \int\limits_{\Gamma_{a,d}} e^{-t(-\lambda)^b} R(\lambda : A_0 - \omega)x \, d\lambda, \quad x \in E.$$

As before, $T_b(t)$ is injective and $(T_b(z))_{z \in \Sigma_\gamma}$ is an analytic operator family which satisfies the items (i) and (ii) stated in the formulation of the theorem. In order to prove that there exists a solution $u(\cdot; t)$ of the abstract Cauchy problem (ACP_n) with initial data $x_0, \dots, x_{n-1}$ and that $u(\cdot; t)$ can be analytically extended to $\mathbb{C}$, we will slightly modify the arguments given in the proof of [**32**, Theorem 6.2, p. 132]. Put, for every $z \in \mathbb{C}$ and $k \in \{0, 1, \dots, n-1\}$:

$$S_k(z;t)x := \frac{1}{2\pi i} \int\limits_{\Gamma_{a,d}} e^{-t(-\lambda)^b} \sum_{j=0}^{\infty} \frac{z^{nj+k}(\lambda+\omega)^j}{(nj+k)!} R(\lambda : A_0 - \omega)x \, d\lambda, \quad x \in E,$$

where $0^0 := 1$ by common consent. Notice that:

$$\left| \sum_{j=0}^{\infty} \frac{z^{nj+k}(\lambda+\omega)^j}{(nj+k)!} \right| \leqslant |z|^k e^{|z||\lambda+\omega|^{1/n}} \leqslant |z|^k e^{|z|(|\lambda|^{1/n} + |\omega|^{1/n})}. \tag{4.5}$$

Put $\eta := \operatorname{Re}(t)\tan(\gamma) - |\operatorname{Im}(t)|$. Then $\eta > 0$ and, for every $\lambda \in \Gamma_{a,d}$:

$$\begin{aligned} |e^{-t(-\lambda)^b}| &= e^{-\operatorname{Re}(t)|\lambda|^b \cos(b \arg(-\lambda)) + \operatorname{Im}(t)|\lambda|^b \sin(b \arg(-\lambda))} \\ &\leqslant e^{-(\operatorname{Re}(t)\cos(b\arg(-\lambda)) - |\operatorname{Im}(t)|)|\lambda|^b} \leqslant e^{-(\operatorname{Re}(t)\cos(b(\pi - a)) - |\operatorname{Im}(t)|)|\lambda|^b} \\ &\leqslant e^{-(\operatorname{Re}(t)\tan(\gamma) - |\operatorname{Im}(t)|)|\lambda|^b} = e^{-\eta|\lambda|^b}. \end{aligned}$$

These arguments and (4.5) imply $S_k(z;t) \in L(E)$. An elementary application of the Cauchy formula (see also the proof of Theorem 2.1) yields:

$$\frac{1}{2\pi i} \int\limits_{\Gamma_{a,d}} e^{-t(-\lambda)^b} \sum_{j=0}^{\infty} \frac{z^{nj+k}(\lambda+\omega)^j}{(nj+k)!} \, d\lambda = 0,$$

and the argumentation given in the proof of [**32**, Theorem 6.2] (cf. [**32**, (6.5)–(6.12), pp. 132–133]) implies that, for every $z \in \mathbb{C}$ and $k, l \in \{0, 1, \dots, n-1\}$:

$$\frac{d^n}{dz^n} S_k(z;t) = A_0 S_k(z;t) \text{ and } S_k^{(l)}(0;t) = \delta_{kl} T_b(t), \tag{4.6}$$

where δ_{kl} is the Kronecker delta. Define

$$u(z;t) := \sum_{k=0}^{n-1} S_k\big(e^{i\frac{\arg(z_0)}{n}} z; t\big) S_0(0;t)^{-1} e^{-i\frac{k\arg(z_0)}{n}} x_k, \quad z \in \mathbb{C}.$$

Clearly, the mapping $z \mapsto u(z;t)$, $z \in \mathbb{C}$ is analytic and the use of (4.6) enables one to show that the restriction $u(z;t)_{|[0,\infty)}$ solves (ACP_n) with initial data $x_0, \ldots, x_{n-1}$. The uniqueness follows as in [**32**] and it remains to be proved (4.4). In order to simplify the notation, let us reach the agreement $\frac{z^{nj+k-l}}{(nj+k-l)!} := 0$, if $z \in \mathbb{C}$, $k, l \in \{0, 1, \ldots, n-1\}$, $j \in \mathbb{N}_0$ and $nj + k - l < 0$. Put $M_1 := \sum_{i=0}^{n-1} \|T_b(t)^{-1} x_i\|$ and assume $K \subseteq \mathbb{C}$ is a compact set, $p \in \mathbb{N}$ and $|z| \leqslant L$, $z \in K$ for an appropriate $L \geqslant 1$. Due to the resolvent equation and Cauchy formula, we get:

$$\Big\|A^p \frac{d^l}{dz^l} u(z;t)\Big\| \leqslant M_1 \Big\| \sum_{k=0}^{n-1} \frac{e^{i(p+1)\arg(z_0)}}{2\pi i}$$
$$\times \int\limits_{\Gamma_{a,d}} e^{-t(-\lambda)^b} \sum_{j=0}^{\infty} \frac{z^{nj+k-l}(e^{i\frac{\arg(z_0)}{n}})^{nj+k}(\lambda+\omega)^{p+j}}{(nj+k-l)!} R(e^{i\arg(z_0)}(\lambda+\omega) : A)\, d\lambda \Big\|.$$

Since $|\lambda+\omega|^{p+j}(1+|\lambda+\omega|)^N \leqslant (1+\omega)^{p+j+N}(1+|\lambda|)^{p+j+N}$, $j \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$, one can continue the calculus:

$$\Big\|A^p \frac{d^l}{dz^l} u(z;t)\Big\|$$
$$\leqslant MM_1 \sum_{k=0}^{n-1} \frac{1}{2\pi} \Big| \int\limits_{\Gamma_{a,d}} e^{-\eta|\lambda|^b} \sum_{j=0}^{\infty} \frac{|z|^{nj+k-l}}{(nj+k-l)!} (1+\omega)^{p+j+N}(1+|\lambda|)^{p+j+N} d\lambda \Big|$$
$$\leqslant MM_1(1+\omega)^{N+p} \sum_{k=0}^{n-1}\sum_{j=0}^{\infty} \frac{|z|^{nj+k-l}(1+\omega)^j}{(nj+k-l)!} \Big| \frac{1}{2\pi} \int\limits_{\Gamma_{a,d}} e^{-\eta|\lambda|^b}(1+|\lambda|)^{p+j+N} d\lambda \Big|,$$

and by the proofs of [**28**, Proposition 2.2] and Theorem 2.1,

$$\Big| \frac{1}{2\pi} \int\limits_{\Gamma_{a,d}} e^{-\eta|\lambda|^b}(1+|\lambda|)^{p+j+N} d\lambda \Big|$$
$$\leqslant 2^{p+j+N}\Big[(1-d)e^{-\eta d^b} + \frac{1}{b}\Gamma\Big(\frac{p+j+N+1}{b}\Big)\eta^{-\frac{p+j+N+1}{b}}\Big] + 2^{p+j+N} e^{-\eta d^b}$$
$$\leqslant 2^{p+j+N}\Big[2 + \frac{1}{b}\Gamma\Big(\frac{p+j+N+1}{b}\Big)\eta^{-\frac{p+j+N+1}{b}}\Big], \quad j \in \mathbb{N}_0.$$

Hence,

$$\Big\|A^p \frac{d^l}{dz^l} u(z;t)\Big\| \leqslant MM_1 2^{p+N+1}(1+\omega)^{N+p} \sum_{k=0}^{n-1}\sum_{j=0}^{\infty} \frac{L^{nj+k-l}(2+2\omega)^j}{(nj+k-l)!}$$
$$+ \frac{MM_1(2+2\omega)^{N+p}}{\eta^{\frac{N+p+1}{b}} b} \sum_{k=0}^{n-1}\sum_{j=0}^{\infty} \frac{L^{nj+k-l}(2+2\omega)^j}{(nj+k-l)!} \frac{\Gamma(\frac{p+j+N+1}{b})}{\eta^{\frac{j}{b}}}.$$

Denote $A=\sum_{k=0}^{n-1}\sum_{j=0}^{\infty}\frac{L^{nj+k-l}(2+2\omega)^j}{(nj+k-l)!}$. Then

$$\begin{aligned}A&\leqslant\sum_{k=0}^{n-1}\Big(\frac{L^{k-l}}{(k-l)!}+\sum_{j=1}^{\infty}\frac{L^{nj+k-l}(2+2\omega)^j}{(nj+k-l)!}\Big)\\&\leqslant e^L+\sum_{k=0}^{n-1}e^L\sum_{j=1}^{\infty}\frac{(2+2\omega)^jL^j}{j!}\leqslant e^L+ne^{(3+2\omega)L}\leqslant(n+1)e^{(3+2\omega)L}.\end{aligned}$$

Fix an $h>0$. Then

$$\begin{aligned}&\sup_{p\in\mathbb{N}}MM_12^{p+N+1}(1+\omega)^{N+p}\frac{Ah^p}{M_p}\\&\qquad\leqslant MM_12^{N+1}(1+\omega)^N(n+1)e^{(3+2\omega)L}e^{M(2h(1+\omega))}<\infty,\end{aligned}$$

which simply gives

$$\sup_{p\in\mathbb{N}}MM_12^{p+N+1}(1+\omega)^{N+p}\frac{Ah^p}{M_{np-1+l}}<\infty.$$

Denote $\kappa=(2+2\omega)\eta^{-1/b}$; the proof is completed if one shows that:

$$S:=\sup_{p\in\mathbb{N}}\Big(\frac{\kappa^p}{M_{np-1+l}}\sum_{k=0}^{n-1}\sum_{j=0}^{\infty}\frac{L^{nj+k-l}}{(nj+k-l)!}\kappa^j\Gamma\Big(\frac{p+j+N+1}{b}\Big)\Big)<\infty. \tag{4.7}$$

Note, the choice of b implies $\frac{p+j+N+1}{b}>(p+j+N+1)\frac{2(\pi-a)}{\pi}\geqslant 2$ and since $\Gamma(\cdot)$ is increasing in (ξ,∞), where $\xi\sim 1.4616...$, we have $\Gamma(\frac{p+j+N+1}{b})\leqslant(\lceil\frac{p+j+N+1}{b}\rceil-1)!$. Further on, $M_0=1$ and (M.1) imply $M_{p+q}\geqslant M_pM_q$, $p,q\in\mathbb{N}_0$ and, as a consequence of (M.1) and (M.3)$'$, we obtain $p!\prec M_p$, i.e., for every $h_1>0$: $\sup_{p\in\mathbb{N}}h_1^pp!/M_p<\infty$. Therefore,

$$\begin{aligned}S&\leqslant\sup_{p\in\mathbb{N}}\frac{\kappa^p(np+nN)!}{M_{np-1}}\sum_{k=0}^{n-1}\sum_{j=0}^{\infty}\frac{L^{nj+k-l}\kappa^j}{M_l(nj+k-l)!(np+nN)!}\Big(\Big\lceil\frac{p+j+N+1}{b}\Big\rceil-1\Big)!\\&\leqslant 2^{nN}\sup_{p\in\mathbb{N}}\frac{2^{np}\kappa^p(np-1)!(nN+1)!}{M_{np-1}}\\&\qquad\times\sum_{k=0}^{n-1}\sum_{j=0}^{\infty}\frac{L^{nj+k-l}\kappa^j}{M_l(nj+k-l)!(np+nN)!}\Big(\Big\lceil\frac{p+j+N+1}{b}\Big\rceil-1\Big)!\\&\leqslant 2^{nN}(nN+1)!\sup_{p\in\mathbb{N}}\frac{(2^n\kappa)^p(np-1)!}{M_{np-1}}\\&\qquad\times\sum_{k=0}^{n-1}\sum_{j=0}^{\infty}\frac{L^{nj+k-l}\kappa^j2^{nj+np+nN+k-l}}{M_l(nj+np+nN+k-l)!}\Big(\Big\lceil\frac{p+j+N+1}{b}\Big\rceil-1\Big)!\\&\leqslant 2^{(2N+1)n}(nN+1)!\sup_{p\in\mathbb{N}}\frac{(4^n\kappa)^p(np-1)!}{M_{np-1}}\end{aligned}$$

$$\times \sum_{k=0}^{n-1}\sum_{j=0}^{\infty}\frac{L^{nj+k-l}(2^n\kappa)^j}{M_l(nj+np+nN+k-l)!}\Big(\Big\lceil\frac{p+j+N+1}{b}\Big\rceil-1\Big)!$$

$$\leqslant 2^{(2N+1)n}(nN+1)!\sup_{p\in\mathbb{N}}\frac{(4^n\kappa)^p(np-1)!}{M_{np-1}}\times$$

$$\times L^{n-1}\sum_{k=0}^{n-1}\sum_{j=0}^{\infty}\frac{L^{nj}(2^n\kappa)^j\lceil\frac{1}{b}\rceil!2^{nj+np+nN+k-l+\lceil\frac{1}{b}\rceil}}{M_l\big(nj+np+nN+k-l+\lceil\frac{1}{b}\rceil\big)!}\Big(\Big\lceil\frac{p+j+N+1}{b}\Big\rceil-1\Big)!$$

$$\leqslant \Big\lceil\frac{1}{b}\Big\rceil!L^{n-1}2^{(3N+1)n+n-1+\lceil\frac{1}{b}\rceil}(nN+1)!\sup_{p\in\mathbb{N}}\frac{(8^n\kappa)^p(np-1)!}{M_{np-1}}$$

$$\times \sum_{k=0}^{n-1}\sum_{j=0}^{\infty}\frac{L^{nj}(4^n\kappa)^j}{M_l\big(nj+np+nN+k-l+V\lceil\frac{1}{b}\rceil\big)!}\Big(\Big\lceil\frac{p+j+N+1}{b}\Big\rceil-1\Big)!$$

$$\leqslant \text{Const}\sum_{k=0}^{n-1}C_{p,k},$$

where Const. is independent of k, j and p and

$$C_{p,k}:=\sum_{j=0}^{\infty}\frac{L^{nj}(4^n\kappa)^j}{M_l\big(nj+np+nN+k-l+\lceil\frac{1}{b}\rceil\big)!}\Big(\Big\lceil\frac{p+j+N+1}{b}\Big\rceil-1\Big)!.$$

Let $k\in\{0,1,\dots,n-1\}$ be fixed. It is clear that (4.7) holds if we prove that $\sup_{p\in\mathbb{N}}C_{p,k}<\infty$. This follows from the next computation:

$$C_{p,k}\leqslant\sum_{j=0}^{\infty}\frac{(L^n4^n\kappa)^j}{M_l\big(nj+np+nN-l+\lceil\frac{1}{b}\rceil-\lceil\frac{p+j+N+1}{b}\rceil+1\big)!}$$

$$\leqslant\sum_{j=0}^{\infty}\frac{(L^n4^n\kappa)^j}{M_l\big(nj+np+nN-l+\lceil\frac{1}{b}\rceil-\lceil\frac{p}{b}\rceil-\lceil\frac{j}{b}\rceil-\lceil\frac{N}{b}\rceil-\lceil\frac{1}{b}\rceil+1\big)!}$$

$$\leqslant\sum_{j=0}^{\infty}\frac{(L^n4^n\kappa)^jl!}{M_l\big(nj-\lceil\frac{j}{b}\rceil-l+1\big)!\,l!}\leqslant 2\sup_{m\in\mathbb{N}_0}\frac{m!}{M_m}\sum_{j=0}^{\infty}\frac{(8^nL^n\kappa)^j}{\big(nj-\lceil\frac{j}{b}\rceil+1\big)!}<\infty,$$

since $p!\prec M_p$ and, for every $s\in\mathbb{R}$: $\sum_{j=0}^{\infty}\frac{s^j}{(nj-\lceil j/b\rceil+1)!}<\infty$. □

Concerning regularization of ultradistribution sines whose generators possess ultra-polynomially bounded resolvent, we have the following analogue of Theorem 3.3.

Theorem 4.3. *Suppose* (M_p) *satisfies* (M.1), (M.2) *and* (M.3). *If* A *generates an ultradistribution sine of* (M_p)*-class, then there exists an injective operator* $C\in L(E)$ *such that* A *generates a global* C*-cosine function* $(C(t))_{t\geqslant 0}$. *Furthermore, the mapping* $t\mapsto C(t)$, $t\geqslant 0$ *is infinitely differentiable and, for every compact set* $K\subseteq[0,\infty)$ *and* $h>0$*:*

$$\sup_{p\in\mathbb{N}_0,\,t\in K}\frac{h^p}{M_p}\Big(\Big\|\frac{d^{p+1}}{dt^{p+1}}C(t)\Big\|+\Big\|\frac{d^p}{dt^p}C(t)\Big\|\Big)<\infty. \tag{4.8}$$

PROOF. We will use the same terminology as in the preceding section. The operator $\mathcal{A}$ generates a (UDSG) of (M_p)-class and one can argue as in the proofs of Theorem 3.1 and [**14**, Lemma 1.10] to deduce that there exist constants $l \geqslant 1$, $\alpha > 0$ and $\beta > 0$ such that $\Lambda_{\alpha,\beta,l} \subseteq \rho(\mathcal{A})$ and that:

$$\|R(\lambda^2 : A)\| \leqslant \|R(\lambda : \mathcal{A})\| \leqslant \mathrm{Const}\, e^{M(Hl|\lambda|)}|\lambda|^{-k}, \quad \lambda \in \Lambda_{\alpha,\beta,l},\ k \in \mathbb{N}.$$

By Lemma 3.2, we infer that there exists a sequence (N_p) of positive real numbers satisfying $N_0 = 1$, (M.1), (M.2), (M.3) and $N_p \prec M_p$. Furthermore, there exists $n \in \mathbb{N}$ such that the operator $\mathcal{D}_n \in L(E \times E)$, defined by

$$\mathcal{D}_n(x\ y)^T := \frac{1}{2\pi i}\int_{\Gamma_l} \frac{R(\lambda : \mathcal{A})}{\omega^n_{N_p}(i\lambda)}(x\,y)^T d\lambda, \quad x, y \in E$$

is injective (cf. Section 3) and that the following expression defines a bounded linear operator for every $t \geqslant 0$:

$$C(t)x := \frac{1}{2\pi i}\int_{\Gamma_l} \frac{\lambda\cosh(\lambda t)R(\lambda^2 : A)x}{\omega^n_{N_p}(i\lambda)}\, d\lambda, \quad x \in E. \tag{4.9}$$

By standard argumentation, $C(t)A \subseteq AC(t)$, $t \geqslant 0$, $(C(t))_{t\geqslant 0}$ is strongly continuous and:

$$\frac{1}{2\pi i}\int_{\Gamma_l} \frac{e^{\pm\lambda s}\lambda^p}{\omega^n_{N_p}(i\lambda)}\, d\lambda = 0, \quad p \in \mathbb{N}_0,\ s \geqslant 0. \tag{4.10}$$

Let us prove that $A\int_0^t (t-s)C(s)x\,ds = C(t)x - Cx$, $x \in E$, $t \geqslant 0$, where $C = C(0)$. Fix, for the time being, a number $t \geqslant 0$ and note that, for every $\lambda \in \Gamma_l$, we have $\lambda^3\int_0^t (t-s)\cosh(\lambda s)\,ds = \lambda\cosh(\lambda t) - \lambda$. Then the Fubini theorem, the simple equality $AR(\lambda^2 : A)x = \lambda^2 R(\lambda^2 : A)x - x$, $\lambda \in \Gamma_l$, $x \in E$ and (4.10) imply:

$$\begin{aligned}
A\int_0^t (t-s)C(s)x\,ds &= \int_0^t (t-s)\frac{1}{2\pi i}\int_{\Gamma_l}\left[\lambda\cosh(\lambda s)\frac{\lambda^2 R(\lambda^2 : A)x - x}{\omega^n_{N_p}(i\lambda)}d\lambda\right]ds \\
&= \int_0^t (t-s)\frac{1}{2\pi i}\int_{\Gamma_l}\left[\lambda^3\cosh(\lambda s)\frac{R(\lambda^2 : A)x}{\omega^n_{N_p}(i\lambda)}d\lambda\right]ds \\
&= \frac{1}{2\pi i}\int_{\Gamma_l}\left[\lambda^3\int_0^t (t-s)\cosh(\lambda s)\,ds\right]\frac{R(\lambda^2 : A)x}{\omega^n_{N_p}(i\lambda)}\,d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma_l}(\lambda\cosh(\lambda t) - \lambda)\frac{R(\lambda^2 : A)x}{\omega^n_{N_p}(i\lambda)}\,d\lambda = C(t)x - Cx,
\end{aligned}$$

for every $x \in E$. Arguing as in the proof of Theorem 3.1, one can differentiate (4.9) under the integral sign and, in such a way, one gets that, for every $t \geqslant 0$ and $x \in E$:

$$\frac{d^p}{dt^p}C(t)x = \frac{1}{2\pi i}\int\limits_{\Gamma_l} \frac{\lambda^{p+1}\cosh(\lambda t)R(\lambda^2 : A)x}{\omega^n_{N_p}(i\lambda)}\,d\lambda, \quad 2 \mid p,\ p \in \mathbb{N} \text{ and} \tag{4.11}$$

$$\frac{d^p}{dt^p}C(t)x = \frac{1}{2\pi i}\int\limits_{\Gamma_l} \frac{\lambda^{p+1}\sinh(\lambda t)R(\lambda^2 : A)x}{\omega^n_{N_p}(i\lambda)}\,d\lambda, \quad 2 \mid p-1,\ p \in \mathbb{N}. \tag{4.12}$$

The proof of (4.8) follows by means of (4.11)–(4.12) and the estimations given in the proof of Theorem 3.1. It remains to be shown that the operator C is injective. Suppose $Cx = 0$, for some $x \in E$. Put $C(-t) := C(t)$, $t > 0$ and notice that the previous argumentation simply implies that, for every $y \in E$ and $t, s \in \mathbb{R}$:

$$\int\limits_0^s (s-r)C(r)(C(t)y - Cy)\,dr = (C(s) - C)\int\limits_0^t (t-r)C(r)y\,dr.$$

Now an application of [**27**, Theorem 1.2] yields:

$$C(t+s)Cy + C(|t-s|)Cy = 2C(t)C(s)y, \quad y \in E,\ t \geqslant 0,\ s \geqslant 0.$$

Thereby, $C(t)x = 0$, $t \geqslant 0$ and the use of (4.12), with $n = 1$ and $t = 0$, gives:

$$\frac{1}{2\pi i}\int\limits_{\Gamma_l} \frac{\lambda^2 R(\lambda^2 : A)x}{\omega^n_{N_p}(i\lambda)}\,d\lambda = 0. \tag{4.13}$$

According to the proof of [**14**, Lemma 1.10]:

$$R(\lambda : \mathcal{A})\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda R(\lambda^2 : A)x \\ AR(\lambda^2 : A)x \end{pmatrix}, \quad \lambda \in \Lambda_{\alpha,\beta,l},$$

and, as a consequence of (4.10), (4.13) and the resolvent equation, we easily infer that

$$\frac{1}{2\pi i}\int\limits_{\Gamma_l} \frac{AR(\lambda^2 : A)x}{\omega^n_{N_p}(i\lambda)}\,d\lambda = \frac{(-1)}{2\pi i}\int\limits_{\Gamma_l} \frac{d\lambda}{\omega^n_{N_p}(i\lambda)} + \frac{1}{2\pi i}\int\limits_{\Gamma_l} \frac{\lambda^2 R(\lambda^2 : A)x}{\omega^n_{N_p}(i\lambda)}\,d\lambda = 0.$$

Therefore, $\mathcal{D}_n(x\,0)^T = 0$ and $x = 0$, as required. This completes the proof of the theorem. $\square$

It can be straightforwardly justified that the following profiling of the abstract Beurling space associated to the operator $\mathcal{A}$ holds whenever the corresponding sequence (M_p) satisfies (M.1), (M.2) and (M.3):

$$(E^2)^{(M_p)}(\mathcal{A}) = E^{(M_p)}(A) \times E^{(M_p)}(A).$$

Keeping in mind Theorem 3.2, the preceding equality immediately implies the following theorem which ends the paper.

THEOREM 4.4. *Suppose that* (M_p) *satisfies* (M.1), (M.2) *and* (M.3) *and that* G *is an ultradistribution sine of* (M_p)*-class generated by* A*. Then, for every* $x \in E^{(M_p)}(A)$ *and* $y \in E^{(M_p)}(A)$*, the abstract Cauchy problem*

$$(ACP_2): \begin{cases} u \in C^\infty([0,\infty) : E) \cap C([0,\infty) : [D(A)]), \\ u''(t) = Au(t), \quad t \geqslant 0, \\ u(0) = x, \; u'(0) = y, \end{cases}$$

has a unique solution. Furthermore, for every compact set $K \subseteq [0,\infty)$ *and* $h > 0$*, the solution* u *of* (ACP_2) *satisfies*

$$\sup_{t \in K,\, p \in \mathbb{N}_0} \frac{h^p}{M_p} \Big(\Big\| \frac{d^p}{dt^p} u(t) \Big\| + \Big\| \frac{d^p}{dt^{p+1}} u(t) \Big\| \Big) < \infty.$$

References

[1] M. Balabane, *Puissances fractionnaires d'un opérateur générateur d'un semi-groupe distribution régulier*, Ann. Inst. Fourier. Grenoble **26** (1976), 157–203.
[2] R. Beals, *On the abstract Cauchy problem*, J. Funct. Anal. **10** (1972), 281–299.
[3] R. Beals, *Semigroups and abstract Gevrey spaces*, J. Funct. Anal. **10** (1972), 300–308.
[4] R. Carmicheal, A. Kamiński, S. Pilipović, *Notes on Boundary Values in Ultradistribution Spaces*, Lecture Notes Series **49**, Seul National University, 1999.
[5] J. Chazarain, *Problèmes de Cauchy abstraits et applications à quelques problèmes mixtes*, J. Funct. Anal. **7** (1971), 386–446.
[6] I. Ciorănescu, *Abstract Beurling spaces of class* (M_p) *and ultadistribution semi-groups*, Bull. Sci. Math. **102** (1978), 167–192.
[7] I. Ciorănescu, L. Zsidó, *ω-ultradistributions and their applications to operator theory*, in: *Spectral Theory*, Banach Center Publications **8**, Warszawa, 1982, 77–220.
[8] I. Ciorănescu, L. Zsidó, *ω-ultradistributions in the abstract Cauchy problem*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.), **25** (1979), 79–94.
[9] R. deLaubenfels, *Existence Families, Functional Calculi and Evolution Equations*, Lect. Notes Math. **1570**, Springer, Berlin, 1994.
[10] H. A. Emamirad, *Les semi-groupes distributions de Beurling*, C. R. Acad. Sci. Sér. A **276** (1976), 117–119.
[11] H. Komatsu, *Ultradistributions, I. Structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **20** (1973), 25–105.
[12] H. Komatsu, *Ultradistributions, III. Vector valued ultradistributions and the theory of kernels*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29** (1982), 653–718.
[13] H. Komatsu, *Operational calculus and semi-groups of operators*, in: *Functional Analysis and Related Topics* (Kioto), Springer, Berlin, 1991, pp. 213–234.
[14] M. Kostić, *Distribution cosine functions*, Taiwan. J. Math. **3** (2006), 739–775.
[15] M. Kostić, S. Pilipović, *Global convoluted semigroups*, Math. Nachr. **280** (2007), 1727–1743.
[16] M. Kostić, S. Pilipović, *Generalized semigroups*, Sibirsk. Mat. Zh., to appear.
[17] M. Kostić, S. Pilipović, *Convoluted C-cosine functions and semigroups. Relations with ultradistribution and hyperfunction sines*, J. Math. Anal. Appl. **338** (2008), 1224–1242.
[18] P. C. Kunstmann, *Distribution semigroups and abstract Cauchy problems*, Trans. Amer. Math. Soc. **351** (1999), 837–856.
[19] P. C. Kunstmann, *Stationary dense operators and generation of non-dense distribution semigroups*, J. Operator Theory **37** (1997), 111–120.
[20] P. C. Kunstmann, *Banach space valued ultradistributions and applications to abstract Cauchy problems*, preprint.
[21] E. Larsson, *Generalized distribution semi-groups of bounded linear operators*, Ann. Scuola. Norm. Sup. Pisa **21** (1967), 137–159.

[22] M. Li, F. Huang, Q. Zheng, *Local integrated C-semigroups*, Studia Math. **145** (2001), 265–280.

[23] I. V. Melnikova, A. I. Filinkov, *Abstract Cauchy Problems: Three Approaches*, Chapman and Hall/CRC, Washington, 2001.

[24] P. J. Miana, *Almost-distribution cosine functions and integrated cosine functions*, Studia Math. **166** (2005), 171–180.

[25] F. Neubrander, *Integrated semigroups and their application to complete second order Cauchy problems*, Semigroup Forum **38** (1989), 233-251.

[26] H. J. Petzsche, *Generalized functions and the boundary values of holomorphic functions*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984), 391–431.

[27] S.-Y. Shaw, *Cosine operator functions and Cauchy problems*, Conf. Sem. Mat. Univ. Bari. Dipart. Interuniv. Mat. **287**, ARACNE, Roma, 2002, 1–75.

[28] B. Straub, *Fractional powers of operators with polynomially bounded resolvents and semigroups generated by them*, Hiroshima Math. J. **24** (1994), 529–548.

[29] N. Tanaka, *Holomorphic C-semigroups and holomorphic semigroups*, Semigroup Forum **38** (1989), 253–261.

[30] T. Ushijima, *On the generation and smothness of semi-groups of linear operators*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **19** (1972), 65–126.

[31] S. Wang, *Quasi-distribution semigroups and integrated semigroups*, J. Funct. Anal. **146** (1997), 352–381.

[32] T.-J. Xiao, J. Liang, *The Cauchy Problem for Higher-Order Abstract Differential Equations*, Springer-Verlag, Berlin, 1998.

Faculty of Technical Sciences
University of Novi Sad
21000 Novi Sad
Serbia
marco.s@verat.net

(Received 28 01 2009)