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OSCILLATOR WITH STRONG QUADRATIC DAMPING FORCE

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Abstract. Oscillations of a system with strong quadratic damping are considered. For the exact analytical form of the energy-displacement function the explicit form of the maximal amplitudes of vibration are obtained by introducing the Lambert W function. Comparing the neighbor maximal amplitudes and the corresponding energies the conclusions about the energy dissipation is given. The approximate solution for a strong nonlinear differential equation which describes the motion of the oscillator with quadratic damping is calculated applying the elliptic-harmonic-balance method. The accuracy of the solution is affirmed by comparing the maximal displacements obtained using the approximate method with the exact one obtained by energy method.

1. Introduction

In the standard textbooks usually the systems with linear damping are considered. Due to their simplicity and the existence of an exact analytical solution, the problem is discussed in details. Unfortunately, in reality the systems and damping are usually not linear. Very often two kinds of oscillators with damping are considered: either, the systems with non-linear elasticity and linear damping, or the systems with linear elasticity and non-linear damping. The first group of the problems is widely discussed in the papers of Mickens [**1**], Waluya and van Horssen [**2**], Andrianov and van Horssen [**3**] and Pilipchuk [**4**]. The second group of the problems is usually restricted to the small damping. Nayfeh and Mook [**5**] applied the analytical method of multiple scales for solving the lightly damped systems. Timoshenko [**6**] gave an example of the application of the method of successive approximations for solving a differential equation with a small square damping term. The same problem is considered by Bogolyubov and Mitropolski [**7**], who modified the averaging method for solving the equation. Magnus [**8**] and Andronov et al. [**9**] provided a qualitative analysis of the oscillator with strong quadratic damping.

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The trajectories for various initial energy values are plotted in the phase plane. In [**10**] also some analytical solutions for the differential equation of the oscillator with strong quadratic damping are presented. In this paper the extension to the previously obtained results is given.

The mathematical model of the system is

$$\ddot{x} + x + 2\delta\dot{x}\,|\dot{x}| = 0,$$

i.e.,

$$\ddot{x} + x \pm 2\delta\dot{x}^2 = 0, \tag{1}$$

with the initial conditions

$$x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0, \tag{2}$$

where δ is a damping coefficient which need not to be small. The sign “minus" corresponds to the velocity direction which is opposite to the direction of x axis, and “plus" holds if the direction of the velocity and x axis are the same. In [**11**] the same notation is introduced in the equation which describes the vibrations of an oscillator with quadratic geometric nonlinearity. In Section 2 the energy-displacement function is determined. Based on that function, exact analytical values for the maximal displacements of the system are calculated. By analyzing the results, the values of the parameters for an energy dissipation are obtained. In Section 3 the energy-displacement function for the system with linear damping is determined. The energy-displacement curve is compared with that obtained for the quadratic damping. In Section 4, by using the elliptic-harmonic-balance method, an approximate analytical solution for the differential equation (1) and the initial conditions (2) is obtained. The solution is assumed in the form of an even order Jacobi elliptic function cn. The approximate analytical values of the maximal displacements are compared with the exact one, calculated in the previous section. The theoretical consideration are illustrated with numerical examples.

2. Energy-displacement function for quadratic damping

Multiplying the differential equation of motion (1) by $\dot{x}$, it follows

$$\frac{d}{dt}\Big(\frac{\dot{x}^2}{2} + \frac{x^2}{2}\Big) = \mp 2\delta\dot{x}^3. \tag{3}$$

Introducing the energy function $E = \dfrac{\dot{x}^2}{2} + \dfrac{x^2}{2}$, (3) can be rewritten as

$$\frac{dE}{dx} = (\mp 4\delta)\Big(E - \frac{x^2}{2}\Big). \tag{4}$$

Introducing the new variable

$$\frac{y^2}{2} = E - \frac{x^2}{2}, \tag{5}$$

(4) is transformed into the linear Bernoulli first-order equation [**12**]

$$\frac{dy}{dx} + (\pm 4\delta)y + x = 0. \tag{6}$$

The solution of (6) and transformation (5) give the exact energy-displacement solution

$$E = \frac{x^2}{2} + \frac{x}{(\pm 4\delta)} + \frac{1}{(\pm 4\delta)^2} + \Big(\eta - \frac{1}{(\pm 4\delta)^2}\Big)\exp((\pm 4\delta)x), \tag{7}$$

where η is a constant of integration. Two energy functions exist: one with a positive sign in front of δ and the other with a negative sign. The transient from one to the other energy function (7) is for $E_{\min}$, i.e., when $\dot{x}$ is zero and the velocity changes its direction. It is satisfied for

$$x + \frac{1}{(\pm 4\delta)} + (\pm 4\delta)\Big(\eta - \frac{1}{(\pm 4\delta)^2}\Big)\exp((\pm 4\delta)x) = 0, \tag{8}$$

when the amplitude of vibration is maximal. The upper sign in (8) corresponds to the energy function during the motion from the left side to the right side and the lower sign for the motion in the opposite direction.

The motion in one direction is in the displacement interval (x_{i-1}, x_i), where $i = 1, 2, \ldots$. Based on the initial conditions: initial position x_{i-1} and the initial energy $E_{i-1} = x_{i-1}^2/2$, the integration constant η in (7) can be calculated

$$\eta_i = \frac{1}{16\delta^2} + \Big(E_{i-1} - \frac{x_{i-1}^2}{2} - \frac{x_{i-1}}{(\pm 4\delta)} - \frac{1}{16\delta^2}\Big)\exp(-(\pm 4\delta)x_{i-1}). \tag{9}$$

The energy dissipation in the interval is

$$E = \frac{x^2}{2} + \frac{x}{(\pm 4\delta)} + \frac{1}{(\pm 4\delta)^2} + \Big(\eta_i - \frac{1}{(\pm 4\delta)^2}\Big)\exp((\pm 4\delta)x).$$

The motion in one direction stops for zero velocity when

$$x_i - \frac{1}{(\pm 4\delta)} = (\pm 4\delta)\Big(\eta_i - \frac{1}{16\delta^2}\Big)\exp(-(\pm 4\delta)x_i). \tag{10}$$

Introducing (9) into (10) and transforming it, the following relation is obtained

$$\Big(x_i - \frac{1}{(\pm 4\delta)}\Big)\exp(\pm 4\delta)x_i = -\Big(x_{i-1} - \frac{1}{(\pm 4\delta)}\Big)\exp(-(\pm 4\delta)x_{i-1}). \tag{11}$$

The relation enables one to compute all of the local maximal amplitudes of vibration for the given initial conditions.

The main disadvantage of relation (10) i.e., (11) is that x_i has an implicit form, which is not suitable for discussion. For a better analysis, let us transform relation (10) into an explicit form. Introducing a new variable r

$$r = \frac{x_i}{(\pm 4\delta)(\eta_i - 1/16\delta^2)} - \frac{1}{16\delta^2(\eta_i - 1/16\delta^2)},$$

relation (10) is transformed to

$$r = \exp\Big(-16\delta^2\Big(\eta_i - \frac{1}{16\delta^2}\Big)r - 1\Big),$$

i.e.,

$$\ln r + 1 = -16\delta^2\Big(\eta_i - \frac{1}{16\delta^2}\Big)r. \tag{12}$$

The left side of equation (12) represents the Lambert W function [**13**]

$$w = -(\ln r + 1) = 16\delta^2\Big(\eta_i - \frac{1}{16\delta^2}\Big)r. \tag{13}$$

The expression for r in (13) is

$$r = \frac{1}{16\delta^2(\eta_i - 1/16\delta^2)} w\Big(\frac{16\delta^2(\eta_i - 1/16\delta^2)}{e}\Big),$$

and for the coordinate x_i

$$x_i = \frac{1}{(\pm 4\delta)}\Big(1 + w\Big(\frac{16\delta^2\eta_i - 1}{e}\Big)\Big). \tag{14}$$

Using (9) and (14), the maximal displacement x_i as a function of the neighbor maximal amplitude x_{i-1} is obtained

$$x_i = \frac{1}{(\pm 4\delta)}\bigg(1 + w\Big(-\frac{1}{16\delta^2 e}\Big(\frac{x_{i-1}}{(\pm 4\delta)} + \frac{1}{16\delta^2}\Big)\exp\big(-(\pm 4\delta)x_{i-1}\big)\Big)\bigg). \tag{15}$$

Using the Lambert W series expansion

$$\begin{aligned} &w\Big(-\frac{1}{16\delta^2 e}\Big(\frac{x_{i-1}}{(\pm 4\delta)} + \frac{1}{16\delta^2}\Big)\exp\big(-(\pm 4\delta)x_{i-1}\big)\Big) \\ &\quad = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}n^{n-2}}{(n-1)!}\bigg(\frac{x_{i-1}}{(\pm 4\delta)^3} + \frac{1}{(16\delta^2)^2}\bigg)^n \exp\big(-n(1 + (\pm 4\delta)x_{i-1})\big), \end{aligned} \tag{16}$$

and substituting (16) into (15), the value of x_i as a function of the initial conditions, i.e., the previous maximal amplitude, and damping coefficient δ, is obtained.

3. Energy-displacement function for linear damping

For a system with linear damping

$$\ddot{x} + x + 2\delta\dot{x} = 0,$$

the energy variation is described with the differential equation

$$\frac{dE}{dx} = -2\delta\sqrt{2}\Big(E - \frac{x^2}{2}\Big)^{1/2}.$$

An exact analytical solution for $\delta < 1$ (see [12]) is:

$$\ln\Big|2E \pm 2\delta x\sqrt{2E - x^2}\Big| - \frac{2\delta}{\sqrt{1-\delta^2}}\tan^{-1}\frac{\pm\sqrt{2E - x^2} + \delta x}{x\sqrt{1-\delta^2}} = \eta,$$

where η is a constant of integration.

Due to the change of the motion direction, the energy variation is divided into two intervals ($\dot{x} > 0$ and $\dot{x} < 0$), whose boundaries are defined by velocity. Besides, the change of the sign of x requires dividing each of these intervals into two parts: $x > 0$ and $x < 0$.

For the first part of the interval, the energy variation is

$$\ln\Big|2E + 2\delta x\sqrt{2E - x^2}\Big| - \frac{2\delta}{\sqrt{1-\delta^2}}\tan^{-1}\frac{\sqrt{2E - x^2} + \delta x}{x\sqrt{1-\delta^2}} = \eta_i, \tag{17}$$

where $i = 1, 2, \dots,$ and

$$\eta_i = \ln|2E_{i-1}| - \frac{2\delta}{\sqrt{1-\delta^2}} \tan^{-1}\frac{\delta}{\sqrt{1-\delta^2}}.$$

η_i is calculated using the initial conditions $x_{i-1} \neq 0$, $\dot{x}_{i-1} = 0$ and $E_{i-1} = x_{i-1}^2/2$. The end of this part is for $x = 0$, when, due to (17), the energy E_{i-1}^0 is

$$\ln\left|2E_{i-1}^0\right| + \frac{\delta\pi}{\sqrt{1-\delta^2}} = \eta_i.$$

For $x = 0$ and energy E_{i-1}^0, the constant of integration for the second part of the interval is calculated

$$\eta_i^0 = \ln\left|2E_{i-1}^0\right| - \frac{\pi\delta}{\sqrt{1-\delta^2}}.$$

The energy dissipation is in accordance with the relation

$$\ln\left|2E + 2\delta x\sqrt{2E - x^2}\right| - \frac{2\delta}{\sqrt{1-\delta^2}} \tan^{-1}\frac{\sqrt{2E-x^2} + \delta x}{x\sqrt{1-\delta^2}} = \eta_i^0. \tag{18}$$

The maximal displacement (for $\dot{x}_i = 0$) and the corresponding energy at the end of the interval are

$$x_i = \pm\left[\exp\left(\frac{2\delta}{\sqrt{1-\delta^2}} \tan^{-1}\frac{\delta}{\sqrt{1-\delta^2}} + \eta_i^0\right)\right]^{1/2},$$

$$E_i = \frac{1}{2}x_i^2 = \frac{1}{2}\exp\left(\frac{2\delta}{\sqrt{1-\delta^2}} \tan^{-1}\frac{\delta}{\sqrt{1-\delta^2}} + \eta_i^0\right).$$

The same procedure of the energy dissipation has to be done for the next interval of motion.

To compare the results obtained for the quadratic and linear damping numerical calculations are carried out.

3.1. Examples. 1. Let us consider a system with quadratic damping, with the value of the damping parameter $\delta = 0.5$. The initial conditions of motion are $x(0) = 0.5$ and $\dot{x}(0) = 0$, i.e., $E_0 = 0.125$. Based on the suggested procedure, the integration constant for the first interval is calculated. The energy-displacement function for the first interval is

$$E_{\leftarrow} = \frac{x^2}{2} + \frac{x}{2} + \frac{1}{4} + \left(0.06606 - \frac{1}{4}\right)\exp(2x).$$

At the end of the first interval: $x_1 = -0.29681$, $E_1 = 4.4048 \times 10^{-2}$. In the next interval, when the system moves from the left side to the right one, the energy decrease is

$$E_{\rightarrow} = \frac{x^2}{2} - \frac{x}{2} + \frac{1}{4} + \left(0.0029951 - \frac{1}{4}\right)\exp(-2x).$$

The end of the interval is for $x_2 = 0.21197$, $E_2 = 2.2466 \times 10^{-2}$. The motion direction changes and

$$E_{\leftarrow} = E = \frac{x^2}{2} + \frac{x}{2} + \frac{1}{4} + \left(0.017021 - \frac{1}{4}\right)\exp(2x).$$

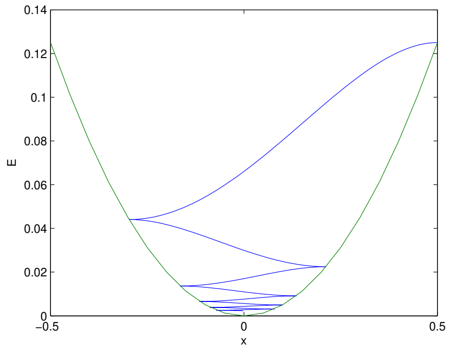

FIGURE 1. E-x diagram with E_i-x_i curve for the system with quadratic damping, damping coefficient $\delta = 0.5$, and initial conditions $x_0 = 0.5$, $E_0 = 0.125$.

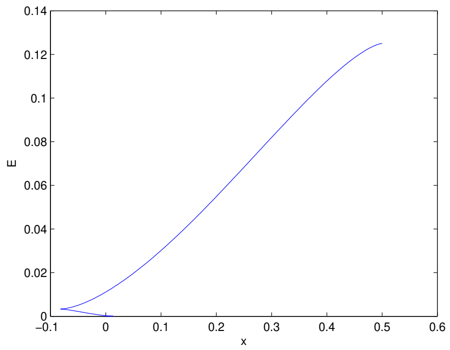

FIGURE 2. E-x diagram for the system with linear damping, damping coefficient $\delta = 0.5$, and initial conditions $x_0 = 0.5$, $E_0 = 0.125$.

The motion in that direction stops for $x_3 = -0.16504$, $E_3 = 1.3619 \times 10^{-2}$. This procedure can be further continued. In Fig. 1 the energy-displacement diagram is plotted. In the same figure the $E_i - x_i$ curve of the maximal displacements is plotted.

2. Let us consider a mechanical system with linear damping, where the damping parameter is $\delta = 0.5$, and the initial conditions are $x_0 = 0.5$ and $\dot{x}_0 = 0$ with the energy level $E_0 = 0.125$. The energy displacement function in the first part of the first interval is

$$\ln\left|2E + x\sqrt{2E - x^2}\right| - 1.1547\tan^{-1}\frac{\sqrt{2E - x^2} + 0.5x}{x\sqrt{0.75}} = -1.9909.$$

For $x = 0$ the energy is $E_1^0 = 1.1133 \times 10^{-2}$. These values are the initial conditions for the second part of the first interval. The energy dissipates, according to the relation (18),

$$\ln\left|2E + x\sqrt{2E - x^2}\right| - 1.1547\tan^{-1}\frac{\sqrt{2E - x^2} + 0.5x}{x\sqrt{0.75}} = -5.6185,$$

and decreases to $E_1 = 3.3224 \times 10^{-3}$ for $x_1 = -8.1516 \times 10^{-2}$.

The second interval starts with the following energy variation

$$\ln\left|2E + x\sqrt{2E - x^2}\right| + 1.1547\tan^{-1}\frac{\sqrt{2E - x^2} + 0.5x}{x\sqrt{0.75}} = -4.4093.$$

For $x = 0$, the energy value is $E_2^0 = 9.9154 \times 10^{-4}$. The second part of the second interval is

$$\ln\left|2E + x\sqrt{2E - x^2}\right| + 1.1547\tan^{-1}\frac{\sqrt{2E - x^2} + 0.5x}{x\sqrt{0.75}} = -8.0369.$$

The maximal displacement is $x_2 = 1.3290 \times 10^{-2}$ when the energy is $E_2 = 8.8312 \times 10^{-5}$. The procedure is repeating, and the energy-displacement curve is plotted in Fig. 2.

3.2. Comparison of the results. Comparing the curves plotted in the figures 1 and 2 for the systems with quadratic and linear damping where the initial conditions ($x_0 = 0.5$, $\dot{x}_0 = 0$), i.e., the energy supply ($E_0 = 0.125$) and also the damping coefficient ($\delta = 0.5$) are equal, it can be concluded that the energy dissipation is faster for the linear damping than for the quadratic damping.

The differential equation with quadratic damping (1) is solved numerically for $\delta = 1$ and various initial conditions i.e., various values of the initial energy: a) $x_0 = 0.2$, $E_0 = 0.02$; b) $x_0 = 0.5$, $E_0 = 0.125$ and c) $x_0 = 1$, $E_0 = 0.5$. The energy-displacement curves are plotted in figures 3–5, respectively. Comparing the curves in these figures it can be concluded that the energy dissipation is faster for a smaller initial energy.

In figures 1 and 3 the energy-displacement curves for various values of the damping coefficient ($\delta = 0.5$ and $\delta = 1$) are plotted. The initial conditions for both systems with strong quadratic damping are equal. Comparing the curves we see that the energy dissipation is faster for higher values of the damping coefficient, as it is known for a system with linear damping.

4. Approximate analytical solution for a system with quadratic damping

To solve the differential equation with quadratic damping, the elliptic-harmonic-balance method [**11**] is applied. A solution is assumed in the form of Jacobi elliptic function cn [**14**]

$$x = A + B\,\mathrm{cn}^2(\omega t + \alpha, k^2) \equiv A + B\,\mathrm{cn}^2, \tag{19}$$

where ω is the frequency, k is the modulus of the function and A, B and α are constants.

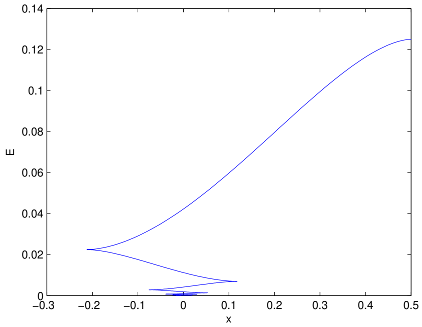

FIGURE 3. E-x diagram for the system with quadratic damping, damping coefficient $\delta = 1$, and initial conditions $x_0 = 0.5$, $E_0 = 0.125$.

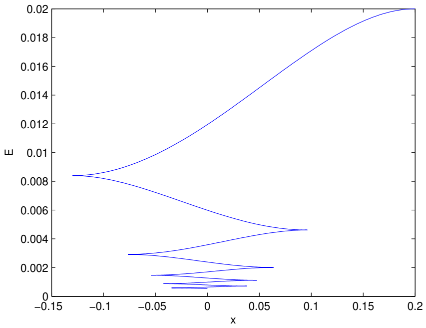

FIGURE 4. E-x diagram for the system with quadratic damping, damping coefficient $\delta = 1$, and initial conditions $x_0 = 0.2$, $E_0 = 0.02$.

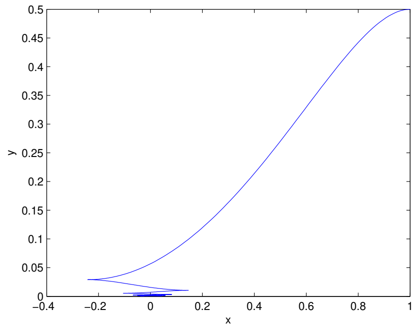

FIGURE 5. E-x diagram for the system with quadratic damping, damping coefficient $\delta = 1$, and initial conditions $x_0 = 1$, $E_0 = 0.5$.

The first and the second time derivatives of (19) are

$$\dot{x} = -2B\omega \ \text{cn} \ \text{sn} \ \text{dn}, \tag{20}$$

$$\ddot{x} = 2B\omega^2[(1-k^2) - 2(1-2k^2)\,\text{cn}^2 - 3k^2\,\text{cn}^4], \tag{21}$$

where $\text{sn} \equiv \text{sn}(\omega t + \alpha, k^2)$ and $\text{dn} \equiv \text{dn}(\omega t + \alpha, k^2)$ are also the Jacobi elliptic functions [14]. Substituting (19)–(21) into (1) and separating the terms with the same order of cn function up to $O(\text{cn}^6)$, the following system of algebraic equations is obtained

$$\begin{aligned}
\text{cn}^0 &: & A + 2B\omega^2(1-k^2) &= 0,\\
\text{cn}^2 &: & 1 + 2(\pm 4\delta)B\omega^2(1-k^2) - 4\omega^2(1-2k^2) &= 0,\\
\text{cn}^4 &: & 3k^2 + (\pm 4\delta)B(1-2k^2) &= 0,
\end{aligned}$$

whose solutions as functions of B are

$$\begin{aligned}
\omega^2 &= \frac{3 - 2(\pm 4\delta)B}{2(16\delta^2 B^2 - 3(\pm 4\delta)B + 6)}, \qquad k^2 = \frac{(\pm 4\delta)B}{2(\pm 4\delta)B - 3},\\
A &= \frac{(\pm 4\delta)B^2 - 3B}{16\delta^2 B^2 - 3(\pm 4\delta)B + 6}.
\end{aligned} \tag{22}$$

The unknown constants B and α are obtained by solving the relations for the initial conditions

$$\begin{aligned}
x(t_0) &= A + B\,\text{cn}^2(\omega t_0 + \alpha, k^2)\\
&= \frac{(\pm 4\delta)B^2 - 3B}{16\delta^2 B^2 - 3(\pm 4\delta)B + 6} + B\,\text{cn}_0^2\left(\alpha, \frac{(\pm 4\delta)B}{2(\pm 4\delta)B - 3}\right),\\
\dot{x}(t_0) &= -\sqrt{2}B\sqrt{\frac{3 - 2(\pm 4\delta)B}{16\delta^2 B^2 - 3(\pm 4\delta)B + 6}}\ \text{cn}_0\,\text{sn}_0\,\text{dn}_0
\end{aligned} \tag{23}$$

where t_0 stands for the initial time, and

$$\begin{aligned}
\text{cn}_0 &\equiv \text{cn}\left(t_0\sqrt{\frac{3 - 2(\pm 4\delta)B}{2(16\delta^2 B^2 - 3(\pm 4\delta)B + 6)}} + \alpha, \frac{(\pm 4\delta)B}{2(\pm 4\delta)B - 3}\right),\\
\text{sn}_0 &\equiv \text{sn}\left(t_0\sqrt{\frac{3 - 2(\pm 4\delta)B}{2(16\delta^2 B^2 - 3(\pm 4\delta)B + 6)}} + \alpha, \frac{(\pm 4\delta)B}{2(\pm 4\delta)B - 3}\right),\\
\text{dn} &\equiv \text{dn}\left(t_0\sqrt{\frac{3 - 2(\pm 4\delta)B}{2(16\delta^2 B^2 - 3(\pm 4\delta)B + 6)}} + \alpha, \frac{(\pm 4\delta)B}{2(\pm 4\delta)B - 3}\right).
\end{aligned} \tag{24}$$

According to the consideration given in Section 2 the motion is divided into intervals where the velocity is positive or negative. The displacement interval from left to right or right to left is (x_{i-1}, x_i) and the time interval (T_{i-1}, T_i), where $i = 1, 2, \ldots$. For every interval of motion the parameter properties of the system (ω_i, k_i, A_i, B_i and α_i) have to be calculated. Depending on the direction of the motion, the parameters are determined according to (22)–(24), where the upper sign corresponds to the motion from left to right, and the lower sign for the motion from right to left. The initial conditions for each interval (except the first (2)) are determined

$$T_{i-1} = \sum_{0}^{i-1} \frac{K(k_n^2)}{\omega_n}, \qquad x_{i-1} = A_{i-1}, \qquad \dot{x}_{i-1} = 0, \tag{25}$$

where K is the complete elliptic integral of the first kind. Substituting (25) into (23) and (24), the constants of integration α_i and B_i are calculated

$$16\delta^2 B_i^3 + B_i^2(2(\pm 4\delta) - 16\delta^2 A_{i-1}) + 3B_i(1 - (\pm 4\delta)A_{i-1}) - 6A_{i-1} = 0,$$
$$-\omega_{i-1}T_{i-1} = \alpha_i.$$

Introducing α_i and B_i into (22) the parameters ω_i, k_i, A_i are as follows

$$\begin{aligned} \omega_i^2 &= \frac{3 - 2(\pm 4\delta)B_i}{2(16\delta^2 B_i^2 - 3(\pm 4\delta)B_i + 6)}, \qquad k_i^2 = \frac{(\pm 4\delta)B_i}{2(\pm 4\delta)B_i - 3}, \\ A &= \frac{(\pm 4\delta)B_i^2 - 3B_i}{16\delta^2 B_i^2 - 3(\pm 4\delta)B_i + 6}. \end{aligned} \tag{26}$$

The substitution of the obtained parameters in (19) gives the solution for all the separate intervals of motion.

Analyzing relation (26) it can be concluded that the frequency and the modulus of the Jacobi elliptic function depend on the initial conditions. The initial conditions have a significant influence on the frequency and the period of vibrations of a system with quadratic damping.

4.1. Example. For the initial conditions $x(0) = 0.5$, $\dot{x}(0) = 0$, and the value of the damping parameter $\delta = 0.5$, an approximate analytical solution of the differential equation with quadratic damping is calculated. The first motion interval is from right to left according to

$$x_{\leftarrow} = -0.29804 + 0.79804\,\mathrm{cn}^2(0.48183t, 0.25776).$$

The motion stops at $T_1 = 3.4867$, $x(T_1) = -0.29804$, $\dot{x}(T_1) = 0$. Then, the motion is in the opposite direction

$$x_{\rightarrow} = 0.19916 - 0.4972\,\mathrm{cn}^2(0.50014(t - 3.4867), 0.98339),$$

and lasts $t_2 = 3.1391$ when the motion stops at the position $x(T_2) = 0.198407$. The initial conditions for the next interval of motion are $T_2 = 6.6258$, $x(T_2) = 0.198407$, $\dot{x}(T_2) = 0$. The motion continues from right to left

$$x_{\leftarrow} = -0.168 + 0.366407\,\mathrm{cn}^2(0.50393(t - 6.6258), 0.17741).$$

At the end of the interval the velocity is zero and the displacement is $x(T_3) = -0.17233$. The values $T_3 = 9.8937$, $x(T_3) = -0.168$, $\dot{x}(T_3) = 0$ are the initial values for the motion in the next interval.

In Fig. 6 the exact numerical solution and the approximate solution of (1) are compared. They are in good agreement.

In Table 1 the exact maximal displacements obtained in Example 1 of Section 2 (x_{ie}) and the approximate values (x_{ia}) are compared.

The differences between the dates in the table are quite small.

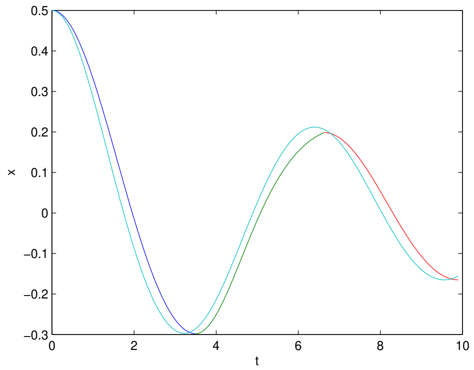

FIGURE 6. Comparison of the analytical solution (x_a) with exact numerical result (x_e) for the system with quadratic damping, damping coefficient $\delta = 0.5$ and initial conditions $x_0 = 0.5$, $dx_0/dt = 0$.

TABLE 1. Maximal displacements: exact (x_{ie}) and approximate (x_{ia}).

x_{ie}	0.50000	−0.29681	0.21197	−0.16504	0.13515	−0.11448	0.09929
x_{ia}	0.50000	−0.29804	0.19916	−0.16800	0.13105	−0.11601	0.09666

5. Conclusion

From the previous consideration the following is concluded:

1. For an oscillator with linear elastic properties and quadratic damping the first integral of energy type exists. The exact analytic form of energy-displacement function depends on the motion direction. For every interval of motion the new integration constant η_i, based on the maximal displacement in the previous motion interval, has to be determined. In each position, where the direction is changed, the new first integral is formed, which insures the continuity of the energy dissipation.

2. The maximal displacement x_i achieves for zero velocity ($\dot{x}_i = 0$), when the energy is $E_i = x_i^2/2$. In the E–x plane the energy curve is a quadratic parabola. For all the oscillators with linear elastic force and linear frequency $\omega_0 = 1$, the maximal displacement has value which lies on that parabola, independently of the order of a damping (linear $\alpha = 1$, or quadratic $\alpha = 2$ or fractal $1 < \alpha < 2$) and the damping coefficient (δ).

3. The order of damping has a significant influence on the decrease of energy: for quadratic damping the energy decrease is slower than for linear damping.

4. For the same order of damping function, the energy decrease is faster for higher values of the damping coefficient.

5. For obtaining more accurate approximate solution for (1) with the initial conditions (2), the number of terms with cn function have to be increased. In

general, an approximate solution is

$$x = A + \sum_{n=1}^{N} B_n \operatorname{cn}^{2n}(\omega t + \alpha_n, k^2).$$

6. An approximate analytic solution in the form of a quadratic Jacobi elliptic function cn is in good agreement with the exact solution of the vibrations of the oscillator with quadratic damping obtained numerically.

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