PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 85(99) (2009), 111–118 DOI:10.2298/PIM0999111I

SPACELIKE AND TIMELIKE NORMAL CURVES IN MINKOWSKI SPACE-TIME

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Communicated by Vladimir Dragović

ABSTRACT. We define normal curves in Minkowski space-time E_1^4. In particular, we characterize the spacelike normal curves in E_1^4 whose Frenet frame contains only non-null vector fields, as well as the timelike normal curves in E_1^4, in terms of their curvature functions. Moreover, we obtain an explicit equation of such normal curves with constant curvatures.

1. Introduction

In the Euclidean space E^3, it is well known that to each unit speed curve $\alpha : I \subset R \to E^3$, whose successive derivatives $\alpha'(s)$, $\alpha''(s)$ and $\alpha'''(s)$ are linearly independent vectors, one can associate the moving orthonormal Frenet frame $\{T, N, B\}$, consisting of the tangent, the principal normal and the binormal vector field respectively. Moreover, the planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are respectively known as the osculating, the rectifying and the normal plane. The rectifying curve in E^3 is defined in [2] as a curve whose position vector always lies in its rectifying plane. In analogy with the Euclidean case, the normal curve in Minkowski 3-space E_1^3 is defined in [4] as a curve whose position vector always lies in its normal plane. Some characterizations of spacelike, timelike and null normal curves, lying fully in the Minkowski 3-space, are given in [3,4].

In this paper, we firstly define the normal space of an arbitrary curve in the Minkowski space-time E_1^4, and then we define the normal curve in E_1^4 as a curve whose the position vector always lies in its normal space. We restrict our investigation of normal curves in E_1^4 to timelike normal curves, as well as to spacelike normal curves whose Frenet frame $\{T, N, B_1, B_2\}$ contains only non-null vector fields. We characterize such normal curves in terms of their curvature functions and find the necessary and the sufficient conditions for such curves to be the normal curves. Furthermore, we prove that every timelike W-curve or spacelike W-curve with non-null

2000 *Mathematics Subject Classification*: Primary 53C50; Secondary 53C40.
Key words and phrases: Minkowski space-time, normal curve, curvature.

vector fields N, B_1 and B_2, is a normal curve and obtain the explicit equation of such normal curves in E_1^4.

2. Preliminaries

The Minkowski space-time E_1^4 is the Euclidean 4-space E^4 equipped with the indefinite flat metric given by $g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$, where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of E_1^4. Recall that an arbitrary vector $v \in E_1^4 \setminus \{0\}$ can be *spacelike*, *timelike* or *null* (*lightlike*), if respectively holds $g(v,v) > 0$, $g(v,v) < 0$ or $g(v,v) = 0$. In particular, the vector $v = 0$ is spacelike. The norm of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$, and two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in E_1^4, can be locally *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null. A spacelike or timelike curve $\alpha(s)$ has a unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$. Recall that the pseudosphere, the pseudohyperbolic space and the lightcone are hyperquadrics in E_1^4, respectively defined by

$$\begin{aligned} S_1^3(m,r) &= \big\{x \in E_1^4 : g(x-m, x-m) = r^2\big\}, \\ H_0^3(m,r) &= \big\{x \in E_1^4 : g(x-m, x-m) = -r^2\big\}, \\ C^3(m) &= \big\{x \in E_1^4 : g(x-m, x-m) = 0\big\}, \end{aligned}$$

where $r > 0$ is the radius and $m \in E_1^4$ is the center (or vertex) of hyperquadric.

Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along a unit speed non-null curve α in E_1^4, consisting of the tangent, the principal normal, the first binormal and the second binormal vector field, respectively. If α is a spacelike curve, let us assume that its Frenet frame contains only non-null vector fields. On the other hand, if α is a timelike curve, its Frenet frame contains only non-null vector fields. Therefore, $\{T, N, B_1, B_2\}$ is an orthonormal frame. Accordingly, let us put

$$g(T,T) = \epsilon_1, \quad g(N,N) = \epsilon_2, \quad g(B_1,B_1) = \epsilon_3, \quad g(B_2,B_2) = \epsilon_4, \tag{2.1}$$

whereby $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \{-1, 1\}$. Moreover, when $\epsilon_i = -1$, then $\epsilon_j = 1$ for all $j \neq i$ $(i, j \in \{1,2,3,4\})$, and consequently $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = -1$. Recall that with respect to the orthonormal frame $\{T, N, B_1, B_2\}$, the vector fields T', N', B_1', B_2' have the following decompositions [6]:

$$\begin{aligned} T' &= \epsilon_1 g(T',T)T + \epsilon_2 g(T',N)N + \epsilon_3 g(T',B_1)B_1 + \epsilon_4 g(T',B_2)B_2, \\ N' &= \epsilon_1 g(N',T)T + \epsilon_2 g(N',N)N + \epsilon_3 g(N',B_1)B_1 + \epsilon_4 g(N',B_2)B_2, \\ B_1' &= \epsilon_1 g(B_1',T)T + \epsilon_2 g(B_1',N)N + \epsilon_3 g(B_1',B_1)B_1 + \epsilon_4 g(B_1',B_2)B_2, \\ B_2' &= \epsilon_1 g(B_2',T)T + \epsilon_2 g(B_2',N)N + \epsilon_3 g(B_2',B_1)B_1 + \epsilon_4 g(B_2',B_2)B_2. \end{aligned}$$

Since the curvature functions $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ of α can be defined by

$$\kappa_1(s) = g\big(T'(s), N(s)\big), \quad \kappa_2(s) = g\big(N'(s), B_1(s)\big), \quad \kappa_3(s) = g\big(B_1'(s), B_2(s)\big),$$

the Frenet equations read (see [5])

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1\epsilon_2 & 0 & 0 \\ -\kappa_1\epsilon_1 & 0 & \kappa_2\epsilon_3 & 0 \\ 0 & -\kappa_2\epsilon_2 & 0 & -\kappa_3\epsilon_1\epsilon_2\epsilon_3 \\ 0 & 0 & -\kappa_3\epsilon_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.2}$$

where the following conditions are satisfied

$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(N,B_2) = g(B_1,B_2) = 0.$$

The curve α lies fully in E_1^4, if $\kappa_3(s) \neq 0$ for each s.

Let α be an arbitrary (non-null or null) curve in E_1^4. We define the normal space of α as the orthogonal complement $T^\perp$ of its tangent vector field T. Hence the normal space is given by $T^\perp = \{w \in E_1^4 \mid g(w,T) = 0\}$. Next, we define a normal curve in E_1^4 as a curve whose position vector always lies in its normal space. In particular, if α is a spacelike curve with Frenet frame containing non-null vector fields, the normal space $T^\perp$ of α is the timelike hyperplane of E_1^4, spanned by $\{N, B_1, B_2\}$. On the other hand, if α is a timelike curve, the normal space $T^\perp$ is the spacelike hyperplane of E_1^4, spanned by $\{N, B_1, B_2\}$. Consequently, the position vector of timelike normal curve or spacelike normal curve with non-null vector fields N, B_1, and B_2, satisfies the equation

$$\alpha(s) = \lambda(s)N(s) + \mu(s)B_1(s) + \nu(s)B_2(s), \tag{2.3}$$

for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ in arclength function s.

3. Some characterizations of non-null normal curves in $\mathbf{E}_1^4$

Timelike normal curves as well as spacelike normal curves (with non-null vector fields N, B_1, B_2) in E_1^4, with the third curvature $\kappa_3(s) = 0$, lie fully in the Minkowski 3-space and their characterization is given in [3, 4]. It can be easily proved that timelike and spacelike normal curves with the second curvature $\kappa_2(s) = 0$ (and non-null vector fields N, B_1, B_2) are circles lying in a timelike or spacelike plane of E_1^4.

In this section, we characterize the timelike and the spacelike normal curves with non-null vector fields N, B_1, B_2 and the third curvature $\kappa_3(s) \neq 0$ for each s.

Let $\alpha(s)$ be a unit speed timelike or spacelike normal curve with non-null vector fields N, B_1 and B_2, lying fully in E_1^4. Then its position vector satisfies the equation (2.3). By taking the derivative of (2.3) with respect to s and using the Frenet equations (2.2), we obtain

$$T = -\kappa_1\epsilon_1\lambda T + (\lambda' - \kappa_2\epsilon_2\mu)N + (\kappa_2\epsilon_3\lambda + \mu' - \kappa_3\epsilon_3\nu)B_1 + (\nu' - \kappa_3\epsilon_1\epsilon_2\epsilon_3\mu)B_2,$$

and therefore

$$-\kappa_1\epsilon_1\lambda = 1,\ \lambda' - \kappa_2\epsilon_2\mu = 0,\ \kappa_2\epsilon_3\lambda + \mu' - \kappa_3\epsilon_3\nu = 0,\ \nu' - \kappa_3\epsilon_1\epsilon_2\epsilon_3\mu = 0. \tag{3.1}$$

From the first three equations we find

$$(3.2)\qquad \begin{aligned} &\lambda(s) = -\frac{\epsilon_1}{\kappa_1(s)}, \qquad \mu(s) = -\frac{\epsilon_1\epsilon_2}{\kappa_2(s)}\Big(\frac{1}{\kappa_1(s)}\Big)', \\ &\nu(s) = -\frac{\epsilon_1}{\kappa_3(s)}\left[\frac{\kappa_2(s)}{\kappa_1(s)} + \epsilon_2\epsilon_3\Big(\Big(\frac{1}{\kappa_1(s)}\Big)'\frac{1}{\kappa_2(s)}\Big)'\right]. \end{aligned}$$

Substituting relation (3.2) into (2.3), we get that the position vector of the normal curve α is given by

$$(3.3)\qquad \begin{aligned} \alpha(s) = &-\frac{\epsilon_1}{\kappa_1(s)}N(s) - \frac{\epsilon_1\epsilon_2}{\kappa_2(s)}\Big(\frac{1}{\kappa_1(s)}\Big)'B_1(s) \\ &-\frac{\epsilon_1}{\kappa_3(s)}\left[\frac{\kappa_2(s)}{\kappa_1(s)} + \epsilon_2\epsilon_3\Big(\Big(\frac{1}{\kappa_1(s)}\Big)'\frac{1}{\kappa_2(s)}\Big)'\right]B_2(s). \end{aligned}$$

Then we have the following theorem.

THEOREM 3.1. *Let $\alpha(s)$ be a unit speed timelike or spacelike curve with non-null vector fields N, B_1 and B_2, lying fully in E_1^4. Then α is congruent to a normal curve if and only if*

$$(3.4)\qquad \frac{\epsilon_3\kappa_3}{\kappa_2}\Big(\frac{1}{\kappa_1}\Big)' = \epsilon_1\left[\frac{1}{\kappa_3}\left(\frac{\kappa_2}{\kappa_1} + \epsilon_2\epsilon_3\Big(\Big(\frac{1}{\kappa_1}\Big)'\frac{1}{\kappa_2}\Big)'\right)\right]'.$$

PROOF. Let us first assume that α is congruent to a normal curve. Then relations (3.1) and (3.2) imply that (3.4) holds.

Conversely, assume that relation (3.4) holds. Let us consider the vector $m \in E_1^4$ given by

$$(3.5)\qquad \begin{aligned} m(s) = \alpha(s) &+ \frac{\epsilon_1}{\kappa_1(s)}N(s) + \frac{\epsilon_1\epsilon_2}{\kappa_2(s)}\Big(\frac{1}{\kappa_1(s)}\Big)'B_1(s) \\ &+ \frac{\epsilon_1}{\kappa_3(s)}\left[\frac{\kappa_2(s)}{\kappa_1(s)} + \epsilon_2\epsilon_3\Big(\Big(\frac{1}{\kappa_1(s)}\Big)'\frac{1}{\kappa_2(s)}\Big)'\right]B_2(s). \end{aligned}$$

Differentiating (3.5) with respect to s and by applying (2.2), we get

$$m' = -\frac{\epsilon_3\kappa_3}{\kappa_2}\Big(\frac{1}{\kappa_1}\Big)'B_2 + \epsilon_1\left[\frac{1}{\kappa_3}\left(\frac{\kappa_2}{\kappa_1} + \epsilon_2\epsilon_3\Big(\Big(\frac{1}{\kappa_1}\Big)'\frac{1}{\kappa_2}\Big)'\right)\right]'B_2.$$

From relation (3.4) it follows that m is a constant vector, which means that α is congruent to a normal curve. $\square$

THEOREM 3.2. *Let $\alpha(s)$ be a unit speed timelike or spacelike curve with non-null vector fields N, B_1 and B_2, lying fully in E_1^4. If α is a normal curve, then the following statements hold:*

(i) *the principal normal and the first binormal component of the position vector α are respectively given by*

$$g(\alpha, N) = -\frac{\epsilon_1\epsilon_2}{\kappa_1}, \quad g(\alpha, B_1) = -\frac{\epsilon_1\epsilon_2\epsilon_3}{\kappa_2}\Big(\frac{1}{\kappa_1}\Big)';$$

(ii) *the first binormal and the second binormal component of the position vector α are respectively given by*

$$g(\alpha, B_1) = -\frac{\epsilon_1\epsilon_2\epsilon_3}{\kappa_2}\Big(\frac{1}{\kappa_1}\Big)', \quad g(\alpha, B_2) = \frac{1}{\kappa_3}\left[\frac{\epsilon_2\epsilon_3\kappa_2}{\kappa_1} + \Big(\Big(\frac{1}{\kappa_1}\Big)'\frac{1}{\kappa_2}\Big)'\right].$$

Conversely, if $\alpha(s)$ is a unit speed timelike or spacelike curve with non-null vector fields N, B_1, B_2, lying fully in E_1^4, and one of statements (i) *or* (ii) *holds, then α is a normal curve.*

Proof. If $\alpha(s)$ is a normal curve, it is easy to check that relation (3.3) implies statements (i) and (ii).

Conversely, if statement (i) holds, differentiating equation $g(\alpha, N) = -\epsilon_1\epsilon_2/\kappa_1$ with respect to s and by applying (2.2), we find $g(\alpha, T) = 0$ which means that α is a normal curve. If statement (ii) holds, in a similar way we conclude that α is a normal curve. □

In the next theorem, we obtain interesting geometric characterization of non-null normal curves.

Theorem 3.3. *Let $\alpha(s)$ be a unit speed timelike or spacelike curve, lying fully in E_1^4, with non-null vector fields N, B_1, and B_2. Then α is congruent to a normal curve if and only if α lies in some hyperquadric in E_1^4.*

Proof. First assume that α is congruent to a normal curve. It follows, by straightforward calculations using Theorem 3.1, that

$$\begin{aligned}&\frac{2\epsilon_2}{\kappa_1}\Big(\frac{1}{\kappa_1}\Big)' + \frac{2\epsilon_3}{\kappa_2}\Big(\frac{1}{\kappa_1}\Big)'\Big(\Big(\frac{1}{\kappa_1}\Big)'\frac{1}{\kappa_2}\Big)' \\ &\quad - \frac{2\epsilon_1\epsilon_2\epsilon_3}{\kappa_3}\left[\frac{\kappa_2}{\kappa_1} + \epsilon_2\epsilon_3\Big(\Big(\frac{1}{\kappa_1}\Big)'\frac{1}{\kappa_2}\Big)'\right]\left[\frac{1}{\kappa_3}\left(\frac{\kappa_2}{\kappa_1} + \epsilon_2\epsilon_3\Big(\Big(\frac{1}{\kappa_1}\Big)'\frac{1}{\kappa_2}\Big)'\right)\right]' = 0.\end{aligned}$$

On the other hand, the previous equation is differential of the equation

$$\epsilon_2\Big(\frac{1}{\kappa_1}\Big)^2 + \epsilon_3\left[\Big(\frac{1}{\kappa_1}\Big)'\frac{1}{\kappa_2}\right]^2 - \epsilon_1\epsilon_2\epsilon_3\Big(\frac{1}{\kappa_3}\Big)^2\left[\frac{\kappa_2}{\kappa_1} + \epsilon_2\epsilon_3\Big(\Big(\frac{1}{\kappa_1}\Big)'\frac{1}{\kappa_2}\Big)'\right]^2 = r, \quad r \in R. \tag{3.6}$$

By using (2.1) and (3.5), it is easy to check that

$$g(\alpha-m, \alpha-m) = \epsilon_2\Big(\frac{1}{\kappa_1}\Big)^2 + \epsilon_3\left[\Big(\frac{1}{\kappa_1}\Big)'\frac{1}{\kappa_2}\right]^2 - \epsilon_1\epsilon_2\epsilon_3\Big(\frac{1}{\kappa_3}\Big)^2\left[\frac{\kappa_2}{\kappa_1} + \epsilon_2\epsilon_3\Big(\Big(\frac{1}{\kappa_1}\Big)'\frac{1}{\kappa_2}\Big)'\right]^2,$$

which together with (3.6) gives $g(\alpha - m, \alpha - m) = r$. Consequently, α lies in some hyperquadric in E_1^4.

Conversely, if α lies in some hyperquadric in E_1^4, then $g(\alpha - m, \alpha - m) = r$, $r \in R$, where $m \in E_1^4$ is a constant vector. By taking the derivative of the previous equation with respect to s, we easily obtain $g(\alpha - m, T) = 0$ which proves the theorem. □

Recall that arbitrary curve α in E_1^4 is called a W-curve (or a helix), if it has constant curvature functions (see [7]). The following theorem gives the characterization of non-null W-curves in E_1^4, in terms of normal curves.

THEOREM 3.4. *Every unit speed timelike or spacelike W-curve, with non-null vector fields N, B_1, B_2, lying fully in E_1^4, is congruent to a normal curve.*

PROOF. By assumption we have $\kappa_1(s) = c_1$, $\kappa_2(s) = c_2$, $\kappa_3(s) = c_3$, where $c_1, c_2, c_3 \in R_0$. Since the curvature functions obviously satisfy relation (3.4), according to Theorem 3.1, α is congruent to a normal curve. $\square$

Note that Theorem 3.4 allows us to find the explicit parametric equation of non-null normal curve with constant curvature functions. Let $\alpha(s)$ be a unit speed curve in E_1^4 with Frenet equations (2.2) and curvature functions $\kappa_1(s) = c_1$, $\kappa_2(s) = c_2$, $\kappa_3(s) = c_3$, whereby $c_1, c_2, c_3 \in R_0$. By using (2.2), we easily obtain differential equation with constant coefficients

$$T'''' + (\epsilon_1\epsilon_2 c_1^2 + \epsilon_2\epsilon_3 c_2^2 - \epsilon_1\epsilon_2 c_3^2)T'' - c_1^2 c_3^2 T = 0.$$

The solution of the previous equation is given by

$$T(s) = \cosh(\lambda_1 s)V_1 + \sinh(\lambda_1 s)V_2 + \cos(\lambda_2 s)V_3 + \sin(\lambda_2 s)V_4, \tag{3.7}$$

where $V_1, V_2, V_3, V_4 \in E_1^4$ are constant vectors and

$$\lambda_1^2 = \frac{1}{2}\Big(-K + \sqrt{K^2 + 4c_1^2c_3^2}\Big), \quad \lambda_2^2 = \frac{1}{2}\Big(K + \sqrt{K^2 + 4c_1^2c_3^2}\Big),$$
$$K = \epsilon_1\epsilon_2(c_1^2 - c_3^2) + \epsilon_2\epsilon_3 c_2^2.$$

Integrating (3.7), we get that the normal curve α has the parametric equation of the form

$$\alpha(s) = \frac{1}{\lambda_1}\big(\sinh(\lambda_1 s)V_1 + \cosh(\lambda_1 s)V_2\big) + \frac{1}{\lambda_2}\big(\sin(\lambda_2 s)V_3 - \cos(\lambda_2 s)V_4\big).$$

Moreover, by using equations $g(T,T) = \epsilon_1$ and $g(T',T') = \epsilon_2 c_1^2$, we may chose constant vectors V_1,V_2,V_3 and V_4 in the following way

$$V_1 = \Big(\sqrt{|\epsilon_1\lambda_2^2 - \epsilon_2 c_1^2|/(\lambda_1^2+\lambda_2^2)}, 0, 0, 0\Big),\ V_2 = \Big(0, \sqrt{|\epsilon_1\lambda_2^2 - \epsilon_2 c_1^2|/(\lambda_1^2+\lambda_2^2)}, 0, 0\Big),$$
$$V_3 = \Big(0, 0, \sqrt{(\epsilon_1\lambda_2^2 + \epsilon_2 c_1^2)/(\lambda_1^2+\lambda_2^2)}, 0\Big),\ V_4 = \Big(0, 0, 0, \sqrt{(\epsilon_1\lambda_2^2 + \epsilon_2 c_1^2)/(\lambda_1^2+\lambda_2^2)}\Big).$$

The next lemma is direct consequence of Theorem 3.1.

LEMMA 3.1. *A unit speed timelike or spacelike curve $\alpha(s)$, with non-null vector fields N, B_1, B_2, lying fully in E_1^4, is congruent to a normal curve if and only if there exists a differentiable function $f(s)$ such that*

$$f(s)\,\kappa_3(s) = \frac{\kappa_2(s)}{\kappa_1(s)} + \epsilon_2\epsilon_3\left(\Big(\frac{1}{\kappa_1(s)}\Big)'\frac{1}{\kappa_2(s)}\right)', \quad f'(s) = \frac{\epsilon_1\epsilon_3\kappa_3(s)}{\kappa_2(s)}\Big(\frac{1}{\kappa_1(s)}\Big)'. \tag{3.8}$$

By using the similar methods as in [1] and [8], as well as Lemma 3.1, we obtain the following two theorems which give the necessary and the sufficient conditions for non-null curves in E_1^4 to be the normal curves.

THEOREM 3.5. *Let $\alpha(s)$ be a unit speed spacelike curve in E_1^4, with spacelike principal normal N. Then α is congruent to a normal curve if and only if there exist constants $a_0, b_0 \in R$ such that*

$$\begin{aligned}\frac{\epsilon_3}{\kappa_2(s)}\Big(\frac{1}{\kappa_1(s)}\Big)' = &\left(a_0 + \int \frac{\kappa_2(s)}{\kappa_1(s)}\sinh\theta(s)\,ds\right)\sinh\theta(s) \\ &- \left(b_0 + \int \frac{\kappa_2(s)}{\kappa_1(s)}\cosh\theta(s)\,ds\right)\cosh\theta(s),\end{aligned} \tag{3.9}$$

where $\theta(s) = \int_0^s \kappa_3(s)\,ds$.

PROOF. If $\alpha(s)$ is congruent to a normal curve, according to Lemma 3.1 there exists a differentiable function $f(s)$ such that relation (3.8) holds, whereby $\epsilon_1 = \epsilon_2 = 1$. Let us define differentiable functions $\theta(s), a(s)$ and $b(s)$ by

$$\begin{aligned}\theta(s) &= \int_0^s \kappa_3(s)\,ds,\\ a(s) &= -\frac{\epsilon_3}{\kappa_2(s)}\Big(\frac{1}{\kappa_1(s)}\Big)'\sinh\theta(s) + f(s)\cosh\theta(s) - \int \frac{\kappa_2(s)}{\kappa_1(s)}\sinh\theta(s)\,ds,\\ b(s) &= -\frac{\epsilon_3}{\kappa_2(s)}\Big(\frac{1}{\kappa_1(s)}\Big)'\cosh\theta(s) + f(s)\sinh\theta(s) - \int \frac{\kappa_2(s)}{\kappa_1(s)}\cosh\theta(s)\,ds.\end{aligned} \tag{3.10}$$

By using (3.8), we easily find $\theta'(s) = \kappa_3(s)$, $a'(s) = 0$, $b'(s) = 0$ and thus

$$a(s) = a_0, \quad b(s) = b_0, \quad a_0, b_0 \in R. \tag{3.11}$$

Multiplying the second and the third equations in (3.10), respectively with $\sinh\theta(s)$ and $-\cosh\theta(s)$, adding the obtained equations and using (3.11), we conclude that relation (3.9) holds.

Conversely, assume that there exist constants $a_0, b_0 \in R$ such that relation (3.9) holds. By taking the derivative of (3.9) with respect to s, we find

$$\begin{aligned}&\frac{\kappa_2(s)}{\kappa_1(s)} + \epsilon_3\Big(\frac{1}{\kappa_2(s)}\Big(\frac{1}{\kappa_1(s)}\Big)'\Big)' \\ &= \kappa_3(s)\left[\left(a_0+\int\frac{\kappa_2(s)}{\kappa_1(s)}\sinh\theta(s)\,ds\right)\cosh\theta(s)-\left(b_0+\int\frac{\kappa_2(s)}{\kappa_1(s)}\cosh\theta(s)\,ds\right)\sinh\theta(s)\right].\end{aligned} \tag{3.12}$$

Let us define the differentiable function $f(s)$ by

$$f(s) = \frac{1}{\kappa_3(s)}\left[\frac{\kappa_2(s)}{\kappa_1(s)} + \epsilon_3\Big(\Big(\frac{1}{\kappa_1(s)}\Big)'\frac{1}{\kappa_2(s)}\Big)'\right]. \tag{3.13}$$

Next, relations (3.12) and (3.13) imply

$$f(s) = \left(a_0+\int\frac{\kappa_2(s)}{\kappa_1(s)}\sinh\theta(s)\,ds\right)\cosh\theta(s)-\left(b_0+\int\frac{\kappa_2(s)}{\kappa_1(s)}\cosh\theta(s)\,ds\right)\sinh\theta(s).$$

By using this and (3.9), we obtain $f'(s) = (\epsilon_3\kappa_3(s)/\kappa_2(s))(1/\kappa_1(s))'$. Finally, Lemma 3.1 implies that α is congruent to a normal curve. □

For timelike curves and spacelike curves with timelike principal normal we obtain the following theorem, which can be proved in a similar way as Theorem 3.5.

Theorem 3.6. *Let $\alpha(s)$ be a unit speed timelike curve or spacelike curve with timelike principal normal N in E_1^4. Then α is congruent to a normal curve if and only if there exist constants $a_0, b_0 \in R$ such that*

$$\frac{\epsilon_2\epsilon_3}{\kappa_2}\Big(\frac{1}{\kappa_1(s)}\Big)' = \bigg(a_0 - \int \frac{\kappa_2(s)}{\kappa_1(s)}\sin\theta(s)\,ds\bigg)\sin\theta(s) - \bigg(b_0 + \int \frac{\kappa_2(s)}{\kappa_1(s)}\cos\theta(s)\,ds\bigg)\cos\theta(s),$$

where $\theta(s) = \int_0^s \kappa_3(s)ds$.

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(Received 04 08 2007)
(Revised 07 07 2008)

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