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DISTRIBUTION GROUPS

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Abstract. We introduce distribution groups and $[B_0, \ldots, B_n, C_0, \ldots, C_{n-1}]$-groups with not necessarily densely defined generators and systematically analyze relations between them.

1. Introduction

Distribution semigroups and their generators were introduced by Lions in the pioneering paper [**31**] and almost four decades after that, Kunstmann [**28**] and Wang [**42**] analyzed distribution semigroups with non-densely defined generators. Balabane and Emami-Rad [**4**]–[**5**] were the first who defined smooth distribution groups and applied them in the analysis of Schrödinger evolution equations in $L^p(\mathbb{R}^n)$-type spaces. On the other hand, global integrated groups were introduced and investigated by El-Mennaoui in his doctoral dissertation [**13**]. We refer the reader to [**3**]–[**6**], [**12**]–[**13**], [**16**], [**18**]–[**20**], [**26**] and, especially, to the paper [**33**] where Miana analyzed global α-times integrated groups and smooth distribution groups in the framework of fractional calculus. It is also meaningful to accent that Keyantuo [**20**] briefly considered an abstract Laplacian in $L^p(\mathbb{R}^n)$-type spaces and proved several relations between exponentially bounded integrated cosine functions and global integrated groups. For further information, see [**20**, Theorem 1.2, Propositions 2.1–2.2, Theorem 2.6 and Proposition 4.2]. The class of (local) convoluted C-groups extending the well known classes of integrated groups and regularized groups has been recently introduced in [**26**].

In a series of papers, many authors relate global integrated groups to functional calculi and proved, in such a way, different generalizations of Stone's theorem. For various aspects in this direction, we refer to [**6**], [**8**]–[**12**], [**14**] and [**16**]. Further on, Galé and Miana [**18**] have recently introduced one-parameter groups of regular quasimultipliers within Esterle's theory of quasimultipliers [**15**] and applied them

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in the study of regularized, distribution, integrated groups as well as holomorphic semigroups and functional calculi.

Operator-valued distribution groups considered in this article do not fall under the scope of [**18**, Definition 3.4] since our concept does not contain any density and growth assumptions. The assertions which link distribution groups of [**18**] to *global* integrated groups with the corresponding growth order established in [**18**, Propositions 3.7–3.8] with the help of the Riesz functions and the Weyl homomorphisms are no longer applicable and this is the main reason why we analyze *local* integrated groups. Furthermore, we focus our attention to the following system of convolution type equations (the notions and terminology are explained below):

$$G * (\delta' \otimes I - \delta \otimes A) = 0 \otimes I_{[D(A)]} \text{ and } (\delta' \otimes I - \delta \otimes A) * G = 0 \otimes I_E, \tag{1.1}$$

where A is a closed linear operator acting on a Banach space E, $\delta' \otimes I - \delta \otimes A \in \mathcal{D}'\big(L([D(A)], E)\big)$, $G \in \mathcal{D}'\big(L(E, [D(A)])\big)$ and I denotes the inclusion $D(A) \to E$. Contrary to the case of distribution semigroups and distribution cosine functions (cf. [**28**, Theorem 3.10, pp. 844–845] and [**23**, Theorem 3.3]), the uniqueness of solutions of (1.1) is not satisfied. Here we stress that every operator-valued distribution G satisfying, for every $\varphi \in \mathcal{D}$ and $x \in E$:

$$G \in \mathcal{D}'(L(E)),\ G(\varphi)x \in D(A),\ AG(\varphi)x = G(-\varphi')x,\ G(\varphi)A \subseteq AG(\varphi), \tag{1.2}$$

can be viewed as an element of the space $\mathcal{D}'\big(L(E, [D(A)])\big)$ which solves (1.1) (cf. also [**33**]). It turns out that the introduced class of $[B_0, \ldots, B_n, C_0, \ldots, C_{n-1}]$-groups presents a natural framework for investigation of equations involving operators satisfying (1.2). Roughly speaking, such a concept enables one to consider in a unified treatment the notions of integrated groups and regularized groups ([**9**]–[**12**]) as well as to get through to the new important relations between distribution groups and local integrated groups.

The paper is organized as follows. In Section 2, we characterize the basic structural properties of (degenerate) distribution groups, connect local integrated groups to analytic integrated semigroups, global differentiable regularized groups and establish a complex variable characterization of generators of local integrated groups. In this section, it is also proved that every generator of a local integrated group is also the generator of a distribution group. The third section is devoted to the study of (exponentially bounded) $[B_0, \ldots, B_n, C_0, \ldots, C_{n-1}]$-groups and their subgenerators. The composition property of a $[B_0, \ldots, B_n, C_0, \ldots, C_{n-1}]$-group is proved only in the case when a subgenerator of such a group commutes with $B_1, \ldots, B_n, C_0, \ldots, C_{n-1}$. The loss of commutativity is disagreeable and additionally hinders our work. Section 4 is the systematic exposition of distribution groups. Our main results are Theorem 4.1 and Theorem 4.2; concerning these theorems, we would like to point out that the order of the operator-valued distribution G solving (1.1) plays a crucial role. In such a way, we notice the remarkable differences between once integrated groups and n-times integrated groups, where $n \in \mathbb{N}$ and $n > 1$. Theorem 4.1 describes solutions of (1.1) which fulfill the condition $(DG)_4$ stated below. The fundamental relationship between distribution groups and local integrated groups is established in Theorem 4.2(v) and says that the generator A of a distribution group is also the generator of a local integrated group, if $\rho(A) \neq \emptyset$. In the present situation, the author does not know whether there exists a distribution group whose generator possesses the empty resolvent set.

The analysis of ultradistribution and (Fourier) hyperfunction groups [**25**] is an open problem since the argumentation presented in this paper becomes quite inoperative and cannot be employed anymore.

By E and $L(E)$ are denoted a complex Banach space and the Banach algebra of bounded linear operators on E. For a closed linear operator A on E, $D(A)$, $N(A)$, $R(A)$, $\rho(A)$ denote its domain, kernel, range and resolvent set, respectively, while $[D(A)]$ stands for the Banach space $D(A)$ equipped with the graph norm. Put $D_\infty(A) := \bigcap_{n=0}^{\infty} D(A^n)$ and $\|x\|_n := \sum_{i=0}^{n} \|A^i x\|$, $n \in \mathbb{N}$, $x \in D(A^n)$. Further, let us recall that A is stationary dense [**27**] if

$$n(A) := \inf\bigl\{k \in \mathbb{N}_0 : D(A^m) \subseteq \overline{D(A^{m+1})} \text{ for all } m \geqslant k\bigr\} < \infty.$$

If Y is a subspace of E, denote by A_Y the part of A in Y, i.e., $A_Y = \{(x,y) \in A : x \in Y,\ y \in Y\}$. We assume henceforth $C \in L(E)$ and C is injective.

Schwartz spaces of test functions on the real line $\mathbb{R}$ are denoted by $\mathcal{D} = C_0^\infty$, $\mathcal{E} = C^\infty$ and $\mathcal{S}$. Their strong duals are $\mathcal{D}'$, $\mathcal{E}'$ and $\mathcal{S}'$, respectively. By $\mathcal{D}_0$ we denote the subspace of $\mathcal{D}$ which consists of the elements supported by $[0,\infty)$. Further on, $\mathcal{D}'(L(E)) = L(\mathcal{D}, L(E))$, $\mathcal{E}'(L(E)) = L(\mathcal{E}, L(E))$ and $\mathcal{S}'(L(E)) = L(\mathcal{S}, L(E))$ are the spaces of continuous linear functions $\mathcal{D} \to L(E)$, $\mathcal{E} \to L(E)$ and $\mathcal{S} \to L(E)$, respectively, equipped with the topology of uniform convergence on bounded subsets of $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{S}$, respectively; $\mathcal{D}_0'(L(E))$, $\mathcal{E}_0'(L(E))$ and $\mathcal{S}_0'(L(E))$ are the subspaces of $\mathcal{D}'(L(E))$, $\mathcal{E}'(L(E))$ and $\mathcal{S}'(L(E))$, respectively, containing the elements supported by $[0,\infty)$. Let $\rho \in \mathcal{D}$ satisfy $\int_{-\infty}^{\infty} \rho(t)\,dt = 1$ and $\operatorname{supp}\rho \subseteq [0,1]$. By a regularizing sequence we mean a sequence (ρ_n) in $\mathcal{D}_0$ obtained by $\rho_n(t) := n\rho(nt)$, $t \in \mathbb{R}$, $n \in \mathbb{N}$. If $K \subseteq \mathbb{R}$, put $\mathcal{D}_K := \{\varphi \in \mathcal{D} : \operatorname{supp}\varphi \subseteq K\}$. In this paper, the convolution of operator-valued distributions is taken in the sense of [**28**, Proposition 1.1]. Suppose $t \in \mathbb{R}$. A distribution δ_t is defined by $\delta_t(\varphi) := \varphi(t)$, $\varphi \in \mathcal{D}$. Further, if $\varphi \in \mathcal{D}$ and $G \in \mathcal{D}'(L(E))$, we define $\check{\varphi}(\cdot) := \varphi(-\cdot)$ and $\check{G}(\cdot) := G(\check{\cdot})$. Clearly, $(\varphi * \psi)\check{} = \check{\varphi} * \check{\psi}$ and $\check{\varphi}^{(n)} = (-1)^n \bigl(\varphi^{(n)}\bigr)\check{}$, $\varphi, \psi \in \mathcal{D}$, $n \in \mathbb{N}$.

Let $a > 0$ and $b > 0$. The exponential region $E(a,b)$ is defined in [**1**] by

$$E(a,b) := \bigl\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geqslant b,\ |\operatorname{Im}\lambda| \leqslant e^{a\operatorname{Re}\lambda}\bigr\}.$$

Finally, if $f : \mathbb{R} \to \mathbb{C}$ and $t \in \mathbb{R}$, put $\tau_t f(s) := f(s-t)$, $s \in \mathbb{R}$.

2. Structural properties of distribution groups

We need the following definition of a C-distribution semigroup.

Definition 2.1. [**22**] Let $\mathcal{G} \in \mathcal{D}_0'(L(E))$ and $C\mathcal{G} = \mathcal{G}C$. If

$$\text{(C.D.S.1)} \qquad \mathcal{G}(\varphi *_0 \psi)C = \mathcal{G}(\varphi)\,\mathcal{G}(\psi), \quad \varphi, \psi \in \mathcal{D},$$

where $\varphi *_0 \psi(t) = \int_0^t \varphi(t-u)\psi(u)\,du$, $t \in \mathbb{R}$, then $\mathcal{G}$ is called a pre-(C-DSG) and if, additionally,

(C.D.S.2)
$$\mathcal{N}(\mathcal{G}) := \bigcap_{\varphi\in\mathcal{D}_0} N(\mathcal{G}(\varphi)) = \{0\},$$

then $\mathcal{G}$ is called a C-distribution semigroup, (C-DSG) in short.

Let $\mathcal{G}$ be a (C-DSG) and let $T \in \mathcal{E}'_0$. Define $G(T)$ on a subspace of E by

$$y = G(T)x \text{ iff } \mathcal{G}(T * \varphi)x = \mathcal{G}(\varphi)y \text{ for all } \varphi \in \mathcal{D}_0.$$

Denote its domain by $D(G(T))$. By (C.D.S.2), $G(T)$ is a function. Moreover, $G(T)$ is a closed linear operator. If $\varphi \in \mathcal{D}$, put $\varphi_+(t) := \varphi(t)H(t)$ and $\varphi_-(t) := \varphi(t)H(-t)$, $t \in \mathbb{R}$, where $H(\cdot)$ is the Heaviside function. Then $\varphi_+, \varphi_- \in \mathcal{E}'$ and the definitions of $G(\varphi_+)$ and $G(\varphi_-)$ are clear. We know that $\mathcal{G}(\varphi) = 0$, $\varphi \in \mathcal{D}_{(-\infty,0]}$ and that $G(\varphi_+)C = \mathcal{G}(\varphi)$, $\varphi \in \mathcal{D}$ [**22**].

The infinitesimal generator of a (C-DSG) $\mathcal{G}$ is defined by $A := G(-\delta')$.

Finally, if $C = I$, then we also say that $\mathcal{G}$ is a distribution semigroup, (DSG) shortly; if this is the case, then there is no risk for confusion and we also write G for $\mathcal{G}$.

Definition 2.2. An element $G \in \mathcal{D}'(L(E))$ is called a pre-distribution group, pre-(DG) in short, if the following condition holds:

$(DG)_1$
$$G(\varphi * \psi) = G(\varphi)\,G(\psi) \ \text{ for all } \varphi, \psi \in \mathcal{D}.$$

If G additionally satisfies:

$(DG)_2$
$$\mathcal{N}(G) := \bigcap_{\varphi\in\mathcal{D}} N(G(\varphi)) = \{0\},$$

then G is called a distribution group, (DG) shortly. A pre-(DG) G is dense iff:

$(DG)_3$
$$\text{The set } \mathcal{R}(G) := \bigcup_{\varphi\in\mathcal{D}} R(G(\varphi)) \text{ is dense in } E.$$

Suppose $G \in \mathcal{D}'(L(E))$ satisfies $(DG)_2$ and $T \in \mathcal{E}'$. We define $G(T)$ by

$$G(T) := \big\{(x,y) \in E^2 \mid G(T * \varphi)x = G(\varphi)y \text{ for all } \varphi \in \mathcal{D}\big\}.$$

Due to $(DG)_2$, $G(T)$ is a function and it is straightforward to see that $G(T)$ is a closed linear operator in E.

The generator A of a (DG) G is defined by $A := G(-\delta')$. Notice, if G is a (DG) generated by A, then (1.2) holds.

Further on, an element $G \in \mathcal{D}'(L(E))$ is called regular (representable) if the following holds:

$(DG)_4$ For every $x \in \mathcal{R}(G)$, there is a function $t \mapsto u(t;x)$, $t \in \mathbb{R}$ satisfying:

$$u(\cdot;x) \in C(\mathbb{R} : E),\ u(0;x) = x \text{ and } G(\psi)x = \int_{-\infty}^{\infty} \psi(t)\,u(t;x)\,dt, \quad \psi \in \mathcal{D}.$$

It is checked at once that the function $u(\cdot;x)$ is unique.

Example 2.1. (i) Suppose $\pm A$ are generators of C-distribution semigroups $G_{\pm}$. Put $G(\varphi) := G_+(\varphi) + G_-(\check{\varphi})$, $\varphi \in \mathcal{D}$. Then A and G fulfill (1.2). Indeed, $G \in \mathcal{D}'(L(E))$, $G(\varphi)A \subseteq AG(\varphi)$, $\varphi \in \mathcal{D}$, $AG_+(\varphi)x = G_+(-\varphi')x - \varphi(0)Cx$ and $-AG_-(\varphi)x = G_-(-\varphi')x - \varphi(0)Cx$, $\varphi \in \mathcal{D}$, $x \in E$ (cf. [**22**]). So, $AG(\varphi)x = G_+(-\varphi')x - \varphi(0)Cx + G_-(\check{\varphi}')x + \check{\varphi}(0)Cx = G_+(-\varphi')x + G_-(-\check{\varphi'})x = G(-\varphi')x$, $x \in E$, $\varphi \in \mathcal{D}$. Furthermore, it can be proved the following: $G(\varphi*\psi)C = G(\varphi)G(\psi)$, $\varphi, \psi \in \mathcal{D}$ (cf. [**22**], [**26**] and [**33**]), $\bigcap\limits_{\varphi \in \mathcal{D}_0} N(G(\varphi)) = \{0\}$ and $\bigcap\limits_{\varphi \in \mathcal{D}_0} N(G(\check{\varphi})) = \{0\}$.

(ii) Suppose G is a (DG), $P \in L(E)$, $P^2 = P$ and $GP = PG$. Set $G_P(\varphi)x := G(\varphi)Px$, $\varphi \in \mathcal{D}$, $x \in E$. Then G_P is a pre-(DG) and $\mathcal{N}(G_P) = N(P)$.

(iii) Suppose A and G fulfill (1.2). Define G_T ($T \in \mathcal{E}'$) by $G_T(\varphi)x := G(T*\varphi)x$, $\varphi \in \mathcal{D}$, $x \in E$. Then (1.2) holds for A and G_T.

(iv) [9, Example 16.3] Let $E := \{f : \mathbb{R} \to \mathbb{C}$ is continuous $\mid \lim_{|x|\to\infty} e^{x^2} f(x) = 0\}$, $\|f\| := \sup_{x \in \mathbb{R}} |e^{x^2} f(x)|$, $f \in E$ and $A := \frac{d}{dx}$ with maximal domain. Put $(S(t)f)(x) := e^{-(x+t)^2} f(x+t)$, $x \in \mathbb{R}$, $t \in \mathbb{R}$, $f \in E$. Then $S(t)f \in E$, $\|S(t)\| \leqslant e^{2t^2}$, $\int_0^t S(s)f\,ds \in D(A)$ and $A\int_0^t S(s)f\,ds = S(t)f - S(0)f$, $t \in \mathbb{R}$, $f \in E$. Let $f \in E$ and $\varphi \in \mathcal{D}$ be fixed. Set $G(\varphi)f := \int_{-\infty}^{\infty} \varphi(t)S(t)f\,dt$. Clearly, $G \in \mathcal{D}'(L(E))$ and the partial integration shows $G(\varphi)f \in D(A)$, $AG(\varphi)f = G(-\varphi')f$ and

$$\big(G(\varphi)Af - AG(\varphi)f\big)(x) = 2\int_{-\infty}^{\infty} \varphi(t)(x+t)e^{-(x+t)^2} f(x+t)\,dt, \quad x \in \mathbb{R}.$$

Consequently, A does not commute with $G(\cdot)$ and (1.2) does not hold. Furthermore, it can be checked directly that G fulfills $(DG)_2$ and that G is not regular.

(v) Let $\mathcal{F}$ denote the Fourier transform on the real line,

$$\mathcal{F}(f)(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi t} f(t)dt, \quad \xi \in \mathbb{R}.$$

Suppose that $\mathcal{E}$ is a quasi-spectral distribution in the sense of [**12**, Definition 2.2] and that $\mathcal{E}$ can be continuously extended to $\mathcal{S}$. Put $\mathcal{F}(\mathcal{D}) := \{\mathcal{F}(\varphi) : \varphi \in \mathcal{D}\}$ and $G(\varphi) := \mathcal{E}(\mathcal{F}^{-1}(\varphi))$, $\varphi \in \mathcal{S}$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Then $G \in \mathcal{S}'(L(E))$, $G(\varphi*\psi) = G(\varphi)G(\psi)$, $\varphi, \psi \in \mathcal{S}$ and $\bigcap_{\varphi \in \mathcal{F}(\mathcal{D})} N(G(\varphi)) = \{0\}$. Suppose, additionally, that for every $x \in E$ and $\phi \in \mathcal{S}$ with $\phi(0) = 1$:

$$\lim_{n\to\infty} \mathcal{E}(\phi_n)x = x, \text{ where } \phi_n(t) = \phi(t/n), \quad t \in \mathbb{R},\ n \in \mathbb{N}. \tag{2.1}$$

Notice that (2.1) implies that $\mathcal{E}$ is a spectral distribution in the sense of [**12**, Definition 2.4] (cf. also [**6**, Definition 1.1]). We will show that $\bigcap_{\varphi \in \mathcal{D}_0} N(G(\varphi)) = \{0\}$. Indeed, suppose $\rho \in \mathcal{D}$, $\int_{-\infty}^{\infty} \rho(t)\,dt = 1$, $\operatorname{supp}\rho \subseteq [0,1]$ and $G(\varphi)x = 0$, $\varphi \in \mathcal{D}_0$, i.e., $\mathcal{E}(\mathcal{F}^{-1}(\varphi))x = 0$, $\varphi \in \mathcal{D}_0$. Let $\phi(t) = \mathcal{F}^{-1}(\rho)(t) = \int_{-\infty}^{\infty} e^{i\xi t}\rho(\xi)\,d\xi$, $t \in \mathbb{R}$. Then $\phi \in \mathcal{S}$ and $\phi(0) = 1$. Put $\rho_n(t) = n\rho(nt)$ and $\phi_n(t) = \mathcal{F}^{-1}(\rho_n)(t)$, $t \in \mathbb{R}$, $n \in \mathbb{N}$. Clearly, $\phi_n(t) = \phi(t/n)$, $t \in \mathbb{R}$, $n \in \mathbb{N}$ and (2.1) implies $x = \lim_{n\to\infty} \mathcal{E}(\phi_n)x =$

$\lim_{n\to\infty} \mathcal{E}(\mathcal{F}^{-1}(\rho_n))x = 0$. Analogously, $\bigcap_{\varphi\in\mathcal{D}_0} N(G(\check{\varphi})) = \{0\}$ and this implies $(DG)_2$ for G.

Further on, a closed linear operator A satisfying (1.2) need not be the generator of a (DG) and this implies that relations between distribution groups and convolution type equations are, at least, quite unclear.

The proofs of the following assertions are omitted.

LEMMA 2.1. *Suppose G is a pre-(DG). Then $\check{G}$ is a pre-(DG). If, in addition, G is a (DG) generated by A, then $\check{G}$ is a (DG) generated by $-A$.*

PROPOSITION 2.1. [**28**] *Let G be a pre-(DG), $F := E/\mathcal{N}(G)$ and q be the corresponding canonical mapping $q : E \to E/\mathcal{N}(G)$.*

(a) *Let $H \in L(\mathcal{D}, L(F))$ be defined by $qG(\varphi) := H(\varphi)q$ for all $\varphi \in \mathcal{D}$. Then H is a (DG) in F.*
(b) *$\overline{\langle\mathcal{R}(G)\rangle} = \overline{\mathcal{R}(G)}$, where $\langle\mathcal{R}(G)\rangle$ is the linear span of $\mathcal{R}(G)$.*
(c) *Assume that G is not dense. Set $R := \overline{\mathcal{R}(G)}$ and $H := G_{|R}$. Then H is a dense pre-(DG) in R. Moreover, if G is a (DG) generated by A, then H is a (DG) in R generated by A_R.*
(d) *The adjoint G^* of G is a pre-(DG) in E^* with $\mathcal{N}(G^*) = \overline{\mathcal{R}(G)}^{\circ}$. ($\overline{\mathcal{R}(G)}^{\circ}$ is the polar of $\overline{\mathcal{R}(G)}$.)*
(e) *If E is reflexive, then $\mathcal{N}(G) = \overline{\mathcal{R}(G^*)}^{\circ}$.*
(f) *G^* is a (DG) in E^* iff G is a dense pre-(DG). If E is reflexive, then G^* is a dense pre-(DG) in E^* iff G is a (DG).*
(h) *$\mathcal{N}(G) \cap \langle\mathcal{R}(G)\rangle = \{0\}$.*
(j) *Suppose $x = G(\varphi)y$, for some $\varphi \in \mathcal{D}$ and $y \in E$. Set $u(t;x) := G(\tau_t\varphi)y$, $t \in \mathbb{R}$. Then $u(0;x) = x$, $u(\cdot;x) \in C^\infty(\mathbb{R} : E)$, $\frac{d^n}{dt^n}u(t;x) = A^n u(t;x)$, $t \in \mathbb{R}$, $n \in \mathbb{N}_0$ and $G(\psi)x = \int_{-\infty}^{\infty} \psi(t)\, u(t;x)\, dt$, $\psi \in \mathcal{D}$. Hence, G is regular.*

PROPOSITION 2.2. [**28**] *Let G be a (DG) and let $S, T \in \mathcal{E}'$, $\varphi \in \mathcal{D}$ and $x \in E$. Then:*

(a) $\big(G(\varphi)x,\, G(\underbrace{T * \cdots * T}_{m} * \varphi)x\big) \in G(T)^m$, $m \in \mathbb{N}$.
(b) *$G(S)G(T) \subseteq G(S*T)$, $D(G(S)G(T)) = D(G(S*T)) \cap D(G(T))$ and $G(S) + G(T) \subseteq G(S+T)$. In general, $G(S)G(T) \neq G(S*T)$.*
(c) *$G(\varphi)G(T) \subseteq G(T)G(\varphi)$.*
(d) *If G is dense, its generator is densely defined.*

Suppose, for the time being, that $\mathcal{D}'(L(E)) \ni G$ fulfills $(DG)_3$ and $(DG)_4$. Then G is a pre-(DG) iff:

$$\bigcup_{\substack{t\in\mathbb{R},\\ x\in\mathcal{R}(G)}} u(t;x) \subseteq \mathcal{R}(G) \text{ and } u(t+s;x) = u(t;u(s;x)), \quad t,s \in \mathbb{R},\ x \in \mathcal{R}(G). \tag{2.2}$$

The necessity of (2.2) follows directly from Proposition 2.1(j). To prove the sufficiency, notice that

$$\begin{aligned}G(\varphi*\psi)x &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\varphi(t-s)\,\psi(s)\,ds]u(t;x)\,ds\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(t)\,\psi(s)\,u(t+s;x)\,ds\,dt\\ &= \int_{-\infty}^{\infty}\varphi(t)\int_{-\infty}^{\infty}\psi(s)\,u(s;u(t;x))\,ds\,dt = \int_{-\infty}^{\infty}\varphi(t)\,G(\psi)\,u(t;x)\,dt\\ &= G(\psi)\int_{-\infty}^{\infty}\varphi(t)\,u(t;x)\,dt = G(\psi)\,G(\varphi)x, \quad x\in\mathcal{R}(G).\end{aligned}$$

The denseness of $\mathcal{R}(G)$ in E automatically implies $(DG)_1$.

From now on, we employ the following definition of an α-times integrated C-semigroup.

DEFINITION 2.3. Let A be a closed operator, $0<\tau\leqslant\infty$ and $\alpha>0$. If there exists a strongly continuous operator family $(S(t))_{t\in[0,\tau)}$ such that $S(t)A\subseteq AS(t)$, $t\in[0,\tau)$, $S(t)C=CS(t)$, $t\in[0,\tau)$, $\int_0^t S(s)x\,ds\in D(A)$, $x\in E$, $t\in[0,\tau)$ and

$$A\int_0^t S(s)x\,ds = S(t)x-\frac{t^\alpha}{\Gamma(\alpha+1)}Cx, \quad x\in E,\ t\in[0,\tau),$$

then $(S(t))_{t\in[0,\tau)}$ is called a (local, if $\tau<\infty$) α-times integrated C-semigroup with a subgenerator A. If $\tau=\infty$, then we say that $(S(t))_{t\geqslant 0}$ is an exponentially bounded, α-times integrated C-semigroup with a subgenerator A if, additionally, there exist $M>0$ and $\omega\in\mathbb{R}$ such that $\|S(t)\|\leqslant Me^{\omega t}$, $t\geqslant 0$.

We know (cf. [**24**] and [**29**]–[**30**]) that $(S(t))_{t\in[0,\tau)}$ satisfies $S(t)S(s)=S(s)S(t)$, $0\leqslant t,\ s<\tau$ and

$$S(t)S(s)x=\Big[\int_0^{t+s}-\int_0^t-\int_0^s\Big]\frac{(t+s-r)^{\alpha-1}}{\Gamma(\alpha)}S(r)Cx\,dr, \quad x\in E,\ 0\leqslant t,s,t+s<\tau.$$

In general, a subgenerator of $(S(t))_{t\in[0,\tau)}$ is not unique but, in the case $C=I$, every subgenerator is unique and coincides with the (integral) generator of $(S(t))_{t\in[0,\tau)}$, defined by

$$\Big\{(x,y)\in E^2 : S(t)x-\frac{t^\alpha}{\Gamma(\alpha+1)}Cx=\int_0^t S(s)y\,ds \text{ for all } t\in[0,\tau)\Big\}.$$

We refer the reader to [**43**]–[**44**] for the definition of a local regularized semigroup and its generator. Suppose $n\in\mathbb{N}$ and $\tau\in(0,\infty)$; then it is well known [**44**] that a closed linear operator A generates a local n-times integrated semigroup on $[0,\tau)$ if and only if $\rho(A)\neq\emptyset$ and A generates a local $R(\lambda:A)^n$-semigroup on $[0,\tau)$, where $\lambda\in\rho(A)$.

Definition 2.4. [**26**] Let A and B be closed linear operators, $\tau \in (0,\infty]$ and $\alpha > 0$. A strongly continuous operator family $(S(t))_{t\in(-\tau,\tau)}$ is called a (local, if $\tau < \infty$) α-times integrated group generated by A if:

(i) $(S_+(t) := S(t))_{t\in[0,\tau)}$ and $(S_-(t) := S(-t))_{t\in[0,\tau)}$ are (local) α-times integrated semigroups generated by A and B, respectively, and

(ii) for every $x \in E$ and $t, s \in (-\tau,\tau)$ with $t < 0 < s$:

$$S(t)S(s)x = S(s)S(t)x = \begin{cases} \int\limits_{t+s}^{s} \frac{(r-t-s)^{\alpha-1}}{\Gamma(\alpha)} S(r)x\,dr + \int\limits_{t}^{0} \frac{(t+s-r)^{\alpha-1}}{\Gamma(\alpha)} S(r)x\,dr, & t+s \geqslant 0,\\ \int\limits_{t}^{t+s} \frac{(t+s-r)^{\alpha-1}}{\Gamma(\alpha)} S(r)x\,dr + \int\limits_{0}^{s} \frac{(r-t-s)^{\alpha-1}}{\Gamma(\alpha)} S(r)x\,dr, & t+s < 0. \end{cases}$$

Lemma 2.2. [**26**] (i) *Let $(S(t))_{t\in(-\tau,\tau)}$ be an α-times integrated group generated by A, for some $\alpha > 0$ and $\tau \in (0,\infty]$. Put $\check{S}(t) := S(-t)$, $t \in (-\tau,\tau)$. Then $(\check{S}(t))_{t\in(-\tau,\tau)}$ is an α-times integrated group generated by B.*

(ii) *Suppose $\tau \in (0,\infty]$, $\alpha > 0$ and A is the generator of an α-times integrated group $(S(t))_{t\in(-\tau,\tau)}$. Then there exist $a > 0$ and $b > 0$ so that:*

(ii.1) *$E(a,b) \subseteq \rho(A) \cap \rho(B)$, $R(\lambda : A)S(t) = S(t)R(\lambda : A)$, $\lambda \in E(a,b)$, $t \in (-\tau,0]$ and $R(\lambda : B)S(s) = S(s)R(\lambda : B)$, $\lambda \in E(a,b)$, $s \in [0,\tau)$.*

(ii.2) *$S(t)A \subseteq AS(t)$, $t \in (-\tau,0]$ and $S(s)B \subseteq BS(s)$, $s \in [0,\tau)$.*

(iii) *Suppose $\beta > \alpha > 0$ and A is the generator of an α-times integrated group $(S_\alpha(t))_{t\in(-\tau,\tau)}$. Put*

$$S_\beta(t) := \begin{cases} \int\limits_{0}^{t} \frac{(t-s)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} S_\alpha(s)x\,ds, & t \in [0,\tau),\ x \in E\\ \int\limits_{0}^{-t} \frac{(-t-s)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} S_\alpha(-s)x\,ds, & t \in (-\tau,0),\ x \in E. \end{cases}$$

Then $(S_\beta(t))_{t\in(-\tau,\tau)}$ is a β-times integrated group generated by A.

(iv) *Let $\alpha > 0$, $\tau \in (0,\infty]$ and let $(S(t))_{t\in(-\tau,\tau)}$ be an α-times integrated group generated by A. Then $B = -A$.*

(v) *Suppose $0 < \tau \leqslant \infty$, $\alpha > 0$ and $\pm A$ generate α-times integrated semigroups $(S_\pm(t))_{t\in[0,\tau)}$. Then A generates an α-times integrated group $(S(t))_{t\in(-\tau,\tau)}$ given by: $S(t) =: S_+(t)$, $t \in [0,\tau)$, $S(t) := S_-(-t)$, $t \in (-\tau,0)$.*

(vi) *Suppose $0 < \tau \leqslant \infty$ and $\alpha > 0$. A strongly continuous operator family $(S(t))_{t\in(-\tau,\tau)}$ is an α-times integrated group generated by A if and only if $(S_\pm(t))_{t\in[0,\tau)}$ are α-times integrated semigroups generated by $\pm A$.*

Proposition 2.3. *Suppose $\pm A$ generate distribution semigroups $G_\pm$ and put $G(\varphi) := G_+(\varphi) + G_-(\check{\varphi})$, $\varphi \in \mathcal{D}$. Then G is a (DG) generated by A.*

Proof. By the standard arguments, we have that there exists $n \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$, $\pm A$ generate $(2^k n)$-times integrated semigroups $(S_\pm^k(t))_{t\in[0,2^k\tau)}$.

Then one obtains

$$G_+(\varphi)x = (-1)^{2^k n} \int_0^\infty \varphi^{(2^k n)}(t) S_+^k(t)x\, dt \quad \text{and}$$

$$G_-(\varphi)x = (-1)^{2^k n} \int_0^\infty \varphi^{(2^k n)}(t) S_-^k(t)x\, dt, \quad \varphi \in \mathcal{D}_{(-\infty, 2^k\tau)}.$$

In order to prove that G is a (DG) generated by A, suppose $x \in \mathcal{N}(G)$. Then, for every $\varphi \in \mathcal{D}_0$, $G(\varphi)x = 0$ and this implies $G_+(\varphi)x = 0$, $\varphi \in \mathcal{D}_0$. Since G_+ is a (DSG) generated by A, we have $x = 0$ and $(DG)_2$ holds for G. Note that Lemma 2.2(vi) implies that, for every $k \in \mathbb{N}$, A generates a local $(2^k n)$-times integrated group $(S^k(t))_{t \in (-2^k\tau, 2^k\tau)}$. Now one can repeat literally the arguments given in the proof of [**33**, Theorem 6] in order to conclude that $(DG)_1$ holds for all $\varphi, \psi \in \mathcal{D}_{(-2^{k-1}\tau, 2^{k-1}\tau)}$. Hence, G satisfies $(DG)_1$. It remains to prove that $B = A$, where B is the generator of G. Suppose $(x, y) \in B$. Then $G(-\varphi')x = G(\varphi)y$, $\varphi \in \mathcal{D}$, i.e., $G_+(-\varphi')x + G_-(-\check{\varphi'})x = G_+(\varphi)x + G_-(\check{\varphi})x$, $\varphi \in \mathcal{D}$. This, in particular, holds for every $\varphi \in \mathcal{D}_0$ and one obtains $G_+(-\varphi')x = G_+(\varphi)x$, $\varphi \in \mathcal{D}_0$. In other words, $B \subseteq A$. Assume now $(x, y) \in A$. Then the definition of G and [**28**, Lemma 3.6] imply:

$$\begin{aligned} G(\varphi)y = G(\varphi)Ax &= G_+(\varphi)Ax + G_-(\check{\varphi})Ax \\ &= G_+(-\varphi')x - \varphi(0)x - G_-(-\check{\varphi}')x + \varphi(0)x \\ &= G_+(-\varphi')x + G_-(-\check{\varphi'})x = G(-\varphi')x, \quad \varphi \in \mathcal{D}. \end{aligned}$$

This gives $A \subseteq B$ and ends the proof of proposition. □

The previous theorem implies that a wide class of multiplication operators acting on $L^p(\mathbb{R}^n)$-type spaces can be used for the construction of distribution groups. In particular, several examples presented in [**1**] offers one to construct local once integrated groups which can be explicitly calculated.

The following corollary is an immediate consequence of Lemma 2.2, [**30**, Theorems 2.1–2.2] and [**43**, Corollary 2.7].

COROLLARY 2.1. (a) *Suppose* $\alpha > 0$, $\tau \in (0, \infty]$ *and* A *generates an* α*-times integrated group* $(S_\alpha(t))_{t \in (-\tau, \tau)}$. *Then, for every* $a \in (0, \frac{\tau}{\alpha})$, *there exist* $b > 0$ *and* $M > 0$ *so that:*

(2.3) $$E(a, b) \subseteq \rho(\pm A) \text{ and } \|R(\lambda : \pm A)\| \leqslant M|\lambda|^\alpha, \quad \lambda \in E(a, b).$$

(b) *Suppose there exist* $a > 0$, $b > 0$, $M > 0$ *and* $\alpha > -1$ *so that* (2.3) *holds. Then, for every* $\beta > \alpha + 1$ *and* $\tau = a(\beta - \alpha - 1)$, A *generates a local* β*-times integrated group* $(S_\beta(t))_{t \in (-\tau, \tau)}$.

(c) *Suppose* $n \in \mathbb{N}$, $k \in \mathbb{N}$, $\tau \in (0, \infty)$ *and* A *generates a local* n*-times integrated group* $(S_n(t))_{t \in (-\tau, \tau)}$. *Then* A *generates a local* (kn)*-times integrated group* $(S_{kn}(t))_{t \in (-k\tau, k\tau)}$.

Suppose $\alpha \in (0,\infty)$, $\alpha \notin \mathbb{N}$ and $f \in \mathcal{S}$. Put $n = \lceil \alpha \rceil := \inf\{k \in \mathbb{Z} : k \geqslant \alpha\}$. Recall [**35**], the *Weyl fractional derivatives* W_+^α and W_-^α of order α are defined by:

$$W_+^\alpha f(t) := \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^\infty (s-t)^{n-\alpha-1} f(s)\, ds, \quad t \in \mathbb{R}$$

$$W_-^\alpha f(t) := \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{-\infty}^t (t-s)^{n-\alpha-1} f(s)\, ds, \quad t \in \mathbb{R}.$$

If $\alpha = n \in \mathbb{N}$, put $W_+^n := (-1)^n \frac{d^n}{dt^n}$ and $W_-^n := \frac{d^n}{dt^n}$. Then we know [**33**] that $W_\pm^{\alpha+\beta} = W_\pm^\alpha W_\pm^\beta$, $\alpha > 0$, $\beta > 0$.

The following result can be attributed to Miana.

Theorem 2.1. [**33**] *Suppose $\alpha > 0$ and $(S(t))_{t\in\mathbb{R}}$ is an α-times integrated group generated by A. Put $G(\varphi)x := \int_0^\infty W_+^\alpha \varphi(t) S(t)x\, dt + \int_0^\infty W_+^\alpha \check{\varphi}(t) S(-t)x\, dt$, $\varphi \in \mathcal{D}$, $x \in E$. Then G is a (DG) generated by A.*

Notice that, in the case $\alpha = n \in \mathbb{N}$, we have the following equality:

$$G(\varphi)x = (-1)^n \int_0^\infty \varphi^{(n)}(t) S(t)x\, dt + \int_{-\infty}^0 \varphi^{(n)}(t) S(t)x\, dt.$$

We refer the reader to [**9**, Section XXI] and [**24**]–[**25**] for the basic material concerning analytic integrated semigroups.

Remark 2.1. Let $\alpha > 0$, $\omega \geqslant 0$ and let A be the generator of an α-times integrated group $(S_\alpha(t))_{t\in\mathbb{R}}$ with $\|S_\alpha(t)\| = O(e^{\omega|t|})$, $t \in \mathbb{R}$. Due to [**33**, Theorem 8], A^2 generates an exponentially bounded, $(\frac{\alpha}{2})$-times integrated semigroup $(V_\alpha(t))_{t\geqslant 0}$ given by $V_\alpha(t) := \frac{1}{\sqrt{4t\pi}} \int_{-\infty}^\infty e^{-s^2/4t} S_\alpha(s)\, ds$. If $\operatorname{Re} z > 0$, then one can define $V_\alpha(z)$ by $V_\alpha(z) := \frac{1}{\sqrt{4z\pi}} \int_{-\infty}^\infty e^{-s^2/4z} S_\alpha(s)\, ds$ $(\sqrt{1} = 1)$. Arguing as in the proof of [**25**, Theorem 11] (cf. also [**3**, p. 220], [**25**, Proposition 8] and [**26**, Proposition 2.4]), we have that $(V_\alpha(t))_{t\geqslant 0}$ is an exponentially bounded, *analytic* $(\frac{\alpha}{2})$-times integrated semigroup of angle $\frac{\pi}{2}$.

The next theorem clarifies an interesting relation between integrated groups and global differentiable regularized groups.

Theorem 2.2. *Suppose $\alpha > 0$, $\tau \in (0,\infty]$, $b \in (0,1)$ and A generates an α-times integrated group $(S_\alpha(t))_{t\in(-\tau,\tau)}$. Then, for every $\gamma \in (0, \arctan(\cos(b\frac{\pi}{2})))$, there exist two analytic operator families $(T_{b,+}(t))_{t\in\Sigma_\gamma} \subseteq L(E)$ and $(T_{b,-}(t))_{t\in\Sigma_\gamma} \subseteq L(E)$ so that:*

(a) *For every $t \in \Sigma_\gamma$, $T_{b,+}(t)$ and $T_{b,-}(t)$ are injective operators.*

(b) *For every $t_1 \in \Sigma_\gamma$ and $t_2 \in \Sigma_\gamma$, A generates a global $(T_{b,+}(t_1)T_{b,-}(t_2))$-group $(V_{b,t_1,t_2}(s))_{s\in\mathbb{R}}$.*

(c) *For every $x \in E$, $t_1 \in \Sigma_\gamma$ and $t_2 \in \Sigma_\gamma$, the mapping $s \mapsto V_{b,t_1,t_2}(s)x$, $s \in \mathbb{R}$ is infinitely differentiable in $(-\infty,0) \cup (0,\infty)$.*

PROOF. Due to Corollary 2.1, there exist $c > 0$, $d > 0$ and $M > 0$ so that $E(c, d) \subseteq \rho(\pm A)$ and that $\|R(\lambda : \pm A)\| \leqslant M|\lambda|^{\alpha}$, $\lambda \in E(c, d)$. Choose a number $a \in (0, \frac{\pi}{2})$ such that $b \in \bigl(0, \frac{\pi}{2(\pi - a)}\bigr)$ and that $\gamma \in \bigl(0, \arctan(\cos(b(\pi - a)))\bigr)$. It is clear that there are numbers $d \in (0, 1]$ and $\omega \in (d + 1, \infty)$ so that

$$\Omega_{a,d} := \bigl\{z \in \mathbb{C} : |z| \leqslant d\bigr\} \cup \bigl\{re^{i\theta} : r > 0,\ \theta \in [-a, a]\bigr\} \subseteq \rho(A - \omega) \cap \rho(-A - \omega).$$

Let the curve $\Gamma_{a,d} = \partial\Omega_{a,d}$ be oriented upwards. Define $T_{b,\pm}(t)$, $t \in \Sigma_\gamma$ by:

$$T_{b,\pm}(t)x := \frac{1}{2\pi i} \int\limits_{\Gamma_{a,d}} e^{-t(-\lambda)^b} R(\lambda : \pm A - \omega)x \, d\lambda, \quad x \in E.$$

Applying the arguments given in Section 2 of [**39**], one can deduce that $(T_{b,\pm}(t))_{t\in\Sigma_\gamma}$ are analytic operator families and that, for every $t \in \Sigma_\gamma$, $T_{b,+}(t)$ and $T_{b,-}(t)$ are injective operators. Clearly,

$$\begin{aligned} T_{b,+}(t_1)(-A - \omega) &\subseteq (-A - \omega)T_{b,+}(t_1), \\ T_{b,-}(t_2)(A - \omega) &\subseteq (A - \omega)T_{b,-}(t_2), \quad t_1, t_2 \in \Sigma_\gamma. \end{aligned}$$

It is straightforward to prove that $T_{b,+}(t_1)T_{b,-}(t_2) = T_{b,-}(t_2)T_{b,+}(t_1)$, $t_1, t_2 \in \Sigma_\gamma$ and the arguments given in [**21**] shows that $\pm A - \omega$ are generators of global $T_{b,\pm}(t)$-semigroups $(U_{b,t,\pm}(s))_{s\geqslant 0}$. Suppose $t_1, t_2 \in \Sigma_\gamma$ and $x \in E$. Then one obtains

$$\begin{aligned} &T_{b,-}(t_2)(U_{b,t_1,+}(s)x - T_{b,+}(t_1)x) \\ &\quad = T_{b,-}(t_2)(A - \omega) \int\limits_0^s U_{b,t_1,+}(v)x \, dv = (A - \omega)T_{b,-}(t_2) \int\limits_0^s U_{b,t_1,+}(v)x \, dv. \end{aligned}$$

Hence,

$$(A - \omega) \int\limits_0^s \bigl(T_{b,-}(t_2)U_{b,t_1,+}(v)\bigr)x \, dv = T_{b,-}(t_2)U_{b,t_1,+}(s)x - T_{b,+}(t_1)T_{b,-}(t_2)x, \ s \geqslant 0.$$

Furthermore, $[T_{b,-}(t_2)U_{b,t_1,+}(s)]T_{b,+}(t_1) = T_{b,+}(t_1)[T_{b,-}(t_2)U_{b,t_1,+}(s)]$, $s \geqslant 0$, and $[T_{b,-}(t_2)U_{b,t_1,+}(s)](A - \omega) \subseteq (A - \omega)[T_{b,-}(t_2)U_{b,t_1,+}(s)]$, $s \geqslant 0$. This simply implies that $(T_{b,-}(t_2)U_{b,t_1,+}(s))_{s\geqslant 0}$ is a global $(T_{b,+}(t_1)T_{b,-}(t_2))$-semigroup generated by $(T_{b,+}(t_1)T_{b,-}(t_2))^{-1}(A - \omega)(T_{b,+}(t_1)T_{b,-}(t_2)) = A - \omega$. So, $(e^{\omega s}T_{b,-}(t_2)U_{b,t_1,+}(s))_{s\geqslant 0}$ is a global $(T_{b,+}(t_1)T_{b,-}(t_2))$-semigroup generated by A. Analogically, for every $t_1, t_2 \in \Sigma_\gamma$, $(e^{\omega s}T_{b,+}(t_1)U_{b,t_2,-}(s))_{s\geqslant 0}$ is a global $(T_{b,+}(t_1)T_{b,-}(t_2))$-semigroup generated by $-A$. Hence, for every $t_1 \in \Sigma_\gamma$ and $t_2 \in \Sigma_\gamma$, A generates a global $(T_{b,+}(t_1)T_{b,-}(t_2))$-group $(V_{b,t_1,t_2}(s))_{s\in\mathbb{R}}$ given by:

$$V_{b,t_1,t_2}(s) := \begin{cases} e^{\omega s}T_{b,-}(t_2)\, U_{b,t_1,+}(s), & s \geqslant 0, \\ e^{-\omega s}T_{b,+}(t_1)\, U_{b,t_2,-}(-s), & s < 0. \end{cases}$$

The mapping $s \mapsto V_{b,t_1,t_2}(s)x$, $s \in \mathbb{R}$ is infinitely differentiable in $(-\infty, 0) \cup (0, \infty)$ since the mappings $s \mapsto T_{b,-}(t_2)U_{b,t_1,+}(s)x$ and $s \mapsto T_{b,+}(t_1)U_{b,t_2,-}(s)x$ are infinitely differentiable in $s > 0$ [**21**]. The proof is completed. $\square$

3. $[B_0,\dots,B_n,C_0,\dots,C_{n-1}]$-groups

We introduce $[B_0,\dots,B_n,C_0,\dots,C_{n-1}]$-groups as follows.

Definition 3.1. Let A be a closed linear operator. Suppose, further, $0<\tau\leqslant\infty$, $n\in\mathbb{N}$ and $B_0,\dots,B_n,C_0,\dots,C_{n-1}\in L(E)$. A strongly continuous operator family $(S(t))_{t\in(-\tau,\tau)}$ is a $[B_0,\dots,B_n,C_0,\dots,C_{n-1}]$-group with a subgenerator A if and only if:

(a) $A\int\limits_0^t S(s)x\,ds=S(t)x+\sum\limits_{j=0}^{n}t^jB_jx,\ t\in(-\tau,\tau),\ x\in E$ and

(b) $AS(t)x-S(t)Ax=\sum\limits_{j=0}^{n-1}t^jC_jx,\ t\in(-\tau,\tau),\ x\in D(A).$

It is said that $(S(t))_{t\in(-\tau,\tau)}$ is non-degenerate if the assumption $S(t)x=0$, for all $t\in(-\tau,\tau)$, implies $x=0$. Define the integral generator of a non-degenerate $[B_0,\dots,B_n,C_0,\dots,C_{n-1}]$-group $(S(t))_{t\in(-\tau,\tau)}$ by:

$$\hat{A}=\left\{(x,y)\in E^2:S(t)x+\sum_{j=0}^{n}t^jB_jx-\sum_{j=0}^{n-1}\frac{t^{j+1}}{j+1}C_jx=\int\limits_0^t S(s)yds,\ t\in(-\tau,\tau)\right\}.$$

The integral generator $\hat{A}$ of a non-degenerate $[B_0,\dots,B_n,C_0,\dots,C_{n-1}]$-group $(S(t))_{t\in(-\tau,\tau)}$ is a function and it is straightforward to see that $\hat{A}$ is a closed linear operator which is an extension of any subgenerator of $(S(t))_{t\in(-\tau,\tau)}$. Further on, the injectiveness of B_i for some $i\in\{0,\dots,n\}$ implies that $(S(t))_{t\in(-\tau,\tau)}$ is non-degenerate. In general, a subgenerator A of $(S(t))_{t\in(-\tau,\tau)}$ does not commute with $S(\cdot)$ and the set of all subgenerators of $(S(t))_{t\in(-\tau,\tau)}$ need not be monomial. Let us show this by the following illustrative example.

Example 3.1. (i) Let

$$E:=\mathbb{R}^2,\ A(x_1,x_2):=(x_1-x_2,0),$$
$$B_0(x_1,x_2):=(0,-x_2),\ B_1(x_1,x_2):=(-x_1-x_2,-x_1),\ B_2(x_1,x_2):=(0,0),$$
$$C_0(x_1,x_2):=(-x_2,0),\ C_1(x_1,x_2):=(-x_1+x_2,-x_1+x_2),$$
$$S(t)(x_1,x_2):=(tx_1,tx_1+x_2),\ t\in\mathbb{R},\ (x_1,x_2)\in E.$$

It is straightforward to verify that $(S(t))_{t\in\mathbb{R}}$ is a $[B_0,B_1,B_2,C_0,C_1]$-group with a subgenerator A and that: $S(t)S(s)=S(s)S(t)$ iff $t=s$, $S(t)D\neq DS(t)$, $t\in\mathbb{R}$, $D\in\{B_1,C_0,C_1\}$ and $D_0D_1\neq D_1D_0$, $D_i\in\{B_i,C_i\}$, $i=1,2$.

(ii) Suppose $C_j=0$, $j=0,\dots,n-1$ and the bounded linear operators B_j, $j=0,\dots,n$ fulfill $E\neq\overline{\sum_{i=0}^{n}R(B_i)}$. Put $S(t)x:=-\sum_{j=0}^{n}t^jB_jx$, $t\in(-\tau,\tau)$, $x\in E$ and denote by Λ the family of all closed subspaces of E containing $\overline{\sum_{i=0}^{n}R(B_i)}$. If $F\in\Lambda$, define a closed linear operator A_F by $D(A_F):=F$ and $A_Fx:=0$, $x\in D(A_F)$. Then A_F is a subgenerator of a $[B_0,\dots,B_n,\underbrace{0,\dots,0}_{n}]$-group $(S(t))_{t\in(-\tau,\tau)}$.

Remark 3.1. (i) Assume $n\in\mathbb{N}$, $\tau\in(0,\infty]$ and A generates an n-times integrated group $(S(t))_{t\in(-\tau,\tau)}$. Put $\overline{S}(t):=S(t)$, $t\in[0,\tau)$ and $\overline{S}(t):=(-1)^nS(t)$,

$t \in (-\tau, 0)$. Then $(\overline{S}(t))_{t\in(-\tau,\tau)}$ is a $\big[\underbrace{0,\ldots,0}_{n}, \frac{(-1)}{n!}I, \underbrace{0,\ldots,0}_{n}\big]$-group having A as a subgenerator.

(ii) Let A be a subgenerator of a $[B_0,\ldots,B_n,C_0,\ldots,C_{n-1}]$-group $(S(t))_{t\in(-\tau,\tau)}$. Put $\check{S}(t) := S(-t)$, $t \in (-\tau,\tau)$, $\check{B}_j := (-1)^j B_j$ and $\check{C}_j := (-1)^{j+1}C_j$. Then $(\check{S}(t))_{t\in(-\tau,\tau)}$ is a $[\check{B}_0,\ldots,\check{B}_n,\check{C}_0,\ldots,\check{C}_{n-1}]$-group with a subgenerator $-A$.

(iii) Let $k \in \mathbb{N}$ and let $D_1,\ldots,D_k \in L(E)$. For a given $i \in \{1,\ldots,k\}$, put $\overline{D_i} := \prod_{j=1}^{i} D_j$. Define $S_i(\cdot)$, $i \in \{0,\ldots,k\}$ recursively by:

$$S_0(t)x := S(t)x, \ldots, S_i(t)x := \int_0^t S_{i-1}(s)D_i x\,ds, \quad x \in E,\ t \in (-\tau,\tau),$$

and suppose, additionally, that $D_iA \subseteq AD_i$, $i \in \{1,\ldots,k\}$. By a simple induction argument, one can deduce that, for every $i \in \{1,\ldots,k\}$, $(S_i(t))_{t\in(-\tau,\tau)}$ is a $\big[\underbrace{0,\ldots,0}_{i}, \frac{0!B_0\overline{D_i}}{i!},\ldots,\frac{n!B_n\overline{D_i}}{(i+n)!}, \underbrace{0,\ldots,0}_{i}, \frac{0!C_0\overline{D_i}}{i!}, \frac{(n-1)!C_{n-1}\overline{D_i}}{(i+n-1)!}\big]$-group with a subgenerator A.

(iv) Suppose A generates a C-regularized group $(T(t))_{t\in\mathbb{R}}$ in the sense of [**9**, Definition 7.2]. Put $T_k(t)x := \int_0^t \frac{(t-s)^{k-1}}{(k-1)!}T(s)x\,ds$, $t \in \mathbb{R}$, $x \in E$, $k \in \mathbb{N}$. Then $(T_k(t))_{t\in\mathbb{R}}$ is a $[\underbrace{0,\ldots,0}_{k}, -\frac{1}{k!}C, \underbrace{0,\ldots,0}_{k}]$-group having A as a subgenerator.

Suppose A is closed, $B_0,\cdots,B_n \in L(E)$ and define

$$\rho_{B_0,\ldots,B_n}(A) := \left\{\lambda \in \mathbb{C} : R\biggl(\sum_{j=0}^{n}\frac{j!}{\lambda^j}B_j\biggr) \subseteq R(\lambda - A)\right\}.$$

The following profiling of exponentially bounded $[B_0,\ldots,B_n,C_0,\ldots,C_{n-1}]$-groups can be simply proved ([**24**]).

Proposition 3.1. (i) *Let A be a subgenerator of a $[B_0^+,\ldots,B_n^+,C_0^+,\ldots,C_{n-1}^+]$-group $(S(t))_{t\in\mathbb{R}}$ satisfying $\|S(t)\| \leqslant Me^{\omega|t|}$, $t \in \mathbb{R}$, for some $M > 0$ and $\omega \geqslant 0$. Set $B_j^- := (-1)^j B_j^+$ and $C_j^- := (-1)^{j+1}C_j^+$. Then:*

(i1) $\rho_{B_0^+,\ldots,B_n^+}(A) \cap \rho_{B_0^-,\ldots,B_n^-}(-A) \supseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\}$,

(i2) $\int_0^\infty e^{-\lambda t}S(\pm t)x\,dt = -(\lambda \mp A)^{-1}\sum_{j=0}^{n}\frac{j!}{\lambda^j}B_j^{\pm}x$, $\operatorname{Re}\lambda > \omega$, $x \in E$ *and*

(i3) $\pm A\int_0^\infty e^{-\lambda t}S(\pm t)x\,dt - \int_0^\infty e^{-\lambda t}S(\pm t) \pm Axdt = \sum_{j=0}^{n-1}\frac{j!}{\lambda^{j+1}}C_j^{\pm}x$, $\operatorname{Re}\lambda > \omega$, $x \in D(A)$.

(ii) *Suppose A is a closed operator and $(S(t))_{t\in\mathbb{R}}$ is a strongly continuous operator family satisfying $\|S(t)\| \leqslant Me^{\omega|t|}$, $t \in \mathbb{R}$, for some $M > 0$ and $\omega \geqslant 0$. If* (i1), (i2) *and* (i3) *hold, then $(S(t))_{t\in\mathbb{R}}$ is a $[B_0^+,\ldots,B_n^+,C_0^+,\ldots,C_{n-1}^+]$-group with a subgenerator A.*

Suppose $n \in \mathbb{N}$. If A is a closed operator and $B_0, \cdots, B_n \in L(E)$, then we define linear operators Y_i, $i \in \{0, \ldots, n\}$ recursively by:

$$Y_0 := B_0, \;\; Y_{i+1} := (i+1)! B_{i+1} + AY_i, \quad i \in \{0, \ldots, n-1\}.$$

Note that Y_1 is closed and that the assumption $0 \in \rho(A)$ simply implies the closedness of Y_i, $i \in \{0, \ldots, n\}$.

PROPOSITION 3.2. *Suppose $\tau \in (0, \infty]$, $n \in \mathbb{N} \setminus \{1\}$ and A is a subgenerator of a $[B_0, \ldots, B_n, C_0, \ldots, C_{n-1}]$-group $(S(t))_{t \in (-\tau, \tau)}$. Then:*

(i) *$iB_i x - C_{i-1} x \in D(A)$, $x \in D(A)$, $i \in \{1, \ldots, n\}$, $A(iB_i x - C_{i-1} x) = iB_i Ax - iC_i x$, $i \in \{1, \ldots, n-1\}$ and $A(nB_n x - C_{n-1} x) = nB_n Ax$, $x \in D(A)$.*

(ii) *$D(A^k) \subseteq \bigcap_{j=0}^{k} D(Y_j)$, $k \in \{0, \ldots, n\}$ and $Y_k x = -\big(\frac{d^k}{dt^k} S(t)x\big)_{t=0}$, $x \in D(A^k)$, $k \in \{0, \ldots, n\}$.*

(iii) *For every $k \in \{0, \ldots, n-1\}$ and $x \in D(A^{k+1})$:*

$$C_k x + \frac{1}{k!} AY_k(x) = \frac{1}{k!} Y_k(Ax). \tag{3.1}$$

(iv) *If $R(B_0) \subseteq D(A)$, then Y_2 is closed, $D(A^k) \subseteq \bigcap_{j=0}^{k+1} D(Y_j)$, $k \in \{0, \ldots, n-1\}$, (3.1) holds for every $k \in \{1, \ldots, n-1\}$ and $x \in D(A^k)$ and there exists an appropriate constant $M > 0$ so that $\|Y_{k+1} x\| \leqslant M \|x\|_k$, $k \in \{0, \ldots, n-1\}$, $x \in D(A^k)$.*

(v) *$A(-Y_n x + Y_{n-1} Ax) = -n! B_n Ax$, $x \in D(A^n)$; if $R(B_0) \subseteq D(A)$, then $AY_n x = Y_n Ax$, $x \in D(A^n)$.*

PROOF. Suppose $x \in D(A)$. Clearly, $\frac{d}{dt} S(t)x = AS(t)x - \sum_{i=1}^{n} it^{i-1} B_i x$, $t \in (-\tau, \tau)$ and

$$\sum_{i=0}^{n-1} t^i C_i x = AS(t)x - S(t)Ax = AS(t)x - \Bigg[A \int_0^t S(s)Ax \, ds - \sum_{i=0}^{n} t^i B_i Ax \Bigg], \; t \in (-\tau, \tau).$$

Hence,

$$\sum_{i=0}^{n-1} t^i C_i x - \sum_{i=0}^{n} t^i B_i Ax = A \Bigg[S(t)x - \int_0^t S(s)Ax \, ds \Bigg], \; t \in (-\tau, \tau).$$

Since

$$\begin{aligned} \frac{d}{dt} \Bigg[S(t)x - \int_0^t S(s)Ax \, ds \Bigg] &= AS(t)x - \sum_{i=1}^{n} it^{i-1} B_i x - S(t)Ax \\ &= \sum_{i=0}^{n-1} t^i C_i x - \sum_{i=1}^{n} it^{i-1} B_i x, \quad t \in (-\tau, \tau), \end{aligned}$$

the closedness of A implies

$$\sum_{i=0}^{n-1} t^i C_i x - \sum_{i=1}^{n} it^{i-1} B_i x = \frac{d}{dt} \Bigg[S(t)x - \int_0^t S(s)Ax \, ds \Bigg] \in D(A), \quad t \in (-\tau, \tau)$$

and:

$$(3.2)\quad A\Big[\sum_{i=0}^{n-1} t^i C_i x-\sum_{i=1}^{n} it^{i-1}B_i x\Big]=\sum_{i=1}^{n-1} it^{i-1}C_i x-\sum_{i=1}^{n} it^{i-1}B_i Ax,\quad t\in(-\tau,\tau).$$

Differentiate (3.2) sufficiently many times in order to see that $iB_ix-C_{i-1}x\in D(A)$, $i\in\{1,\dots,n\}$ and that (i) holds. To prove (ii), notice that the closedness of A and argumentation used in the proof of (i) enable one to conclude that the mapping $t\mapsto \frac{d^k}{dt^k}S(t)x$, $t\in(-\tau,\tau)$ is k-times continuously differentiable for every $k\in\{0,\dots,n\}$ and $x\in D(A^k)$. Fix a $k\in\{0,\dots,n\}$; then we obtain:

$$(3.3)\qquad \frac{d^{l+1}}{dt^{l+1}}S(t)x=A\frac{d^l}{dt^l}S(t)x-\sum_{j=l+1}^{n} j\cdots(j-l)t^{j-l-1}B_jx,\quad t\in(-\tau,\tau),$$

for every $l\in\{0,\dots,k-1\}$.

Since $Y_0=B_0$, the proof of (ii) follows by induction.

Suppose now $x\in D(A^{k+1})$. Then the mapping $t\mapsto S(t)Ax$, $t\in(-\tau,\tau)$ is k-times continuously differentiable. Since $C_kx=\frac{1}{k!}\frac{d^k}{dt^k}[AS(t)x-S(t)Ax]$, $t\in(-\tau,\tau)$, we have that the mapping $t\mapsto AS(t)x$, $t\in(-\tau,\tau)$ is k-times continuously differentiable and the closedness of A implies that $\frac{d^k}{dt^k}AS(t)x=A\frac{d^k}{dt^k}S(t)x$, $t\in(-\tau,\tau)$ and that $C_kx=\frac{1}{k!}\big[A\frac{d^k}{dt^k}S(t)x-\frac{d^k}{dt^k}S(t)Ax\big]$, $\in(-\tau,\tau)$. Put $t=0$ in the last equality to finish the proof of (iii).

To prove (iv), notice that $R(B_0)\subseteq D(A)$ and that the Closed Graph Theorem implies $Y_1=AB_0+B_1\in L(E)$; the closedness of Y_2 simply follows from this fact. Suppose now $x\in D(A^k)$. Since $\frac{d}{dt}S(t)x-S(t)Ax=\sum_{i=0}^{n-1}t^iC_ix-\sum_{i=1}^{n}it^{i-1}B_ix$, $t\in(-\tau,\tau)$, one concludes that

$$\frac{d^k}{dt^k}S(t)x-\frac{d^{k-1}}{dt^{k-1}}S(t)Ax=\frac{d^{k-1}}{dt^{k-1}}\Big[\sum_{i=0}^{n-1}t^iC_ix-\sum_{i=1}^{n}it^{i-1}B_ix\Big],\quad t\in(-\tau,\tau).$$

This implies $-Y_kx+Y_{k-1}Ax=(k-1)!C_{k-1}x-k!B_kx$ and an employment of (i) gives $-Y_kx+Y_{k-1}Ax\in D(A)$ and:

$$(3.4)\quad A[-Y_kx+Y_{k-1}Ax]=k!C_kx-k!B_kAx,\quad k\in\{1,\dots,n-1\},\ x\in D(A^k).$$

Because $R(B_0)\subseteq D(A)$, one concludes inductively from (3.4) that $Y_kx\in D(A)$, $x\in D(A^k)$, $k\in\{0,\dots,n-1\}$, i.e., $D(A^k)\subseteq\bigcap_{i=0}^{k+1}D(Y_i)$, $k\in\{0,\dots,n-1\}$ and (3.4) implies $k!C_kx+AY_kx=k!B_kAx+AY_{k-1}Ax=Y_kAx$, $k\in\{1,\dots,n-1\}$, $x\in D(A^k)$. The existence of a constant $M>0$ satisfying $\|Y_{k+1}x\|\leqslant M\|x\|_k$, $k\in\{0,\dots,n-1\}$, $x\in D(A^k)$ essentially follows from an application of (3.1) and an induction argument. This ends the proof of (iv) while the proof of (v) follows simply from that of (iv). □

Remark 3.2. (a) Suppose $\tau\in(0,\infty]$ and A is a subgenerator of a $[B_0,B_1,C_0]$-group $(S(t))_{t\in[0,\tau)}$. Arguing as in the proof of Proposition 3.2, one obtains $A(B_1x-C_0x)=B_1Ax$, $x\in D(A)$ and $AY_1x=Y_1Ax$, $x\in D(A^2)$. Furthermore, if $R(B_0)\subseteq D(A)$, then $AY_1x=Y_1Ax$, $x\in D(A)$.

(b) The next question is motivated by the analysis of Arendt, El-Mennaoui and Keyantuo [**1**]: If A is a subgenerator of a $[\underbrace{0,\dots,0}_{n}, \frac{-1}{n!}I, C_0,\dots,C_{n-1}]$-group $(S(t))_{t\in[0,\tau)}$, $n\in\mathbb{N}$, $0<\tau\leqslant\infty$, does $S(t)A\subseteq AS(t)$, $t\in(-\tau,\tau)$? The answer is affirmative and we will show this only in the non-trivial case $n>1$. Indeed, $S(0)=0$ and this implies $C_0x=0$, $x\in D(A)$. By Proposition 3.2(i), we have $AC_{i-1}x=iC_ix$, $i\in\{1,\dots,n-1\}$, $x\in D(A)$. Inductively, $C_ix=0$, $i\in\{1,\dots,n-1\}$, $x\in D(A)$ and an immediate consequence is $S(t)A\subseteq AS(t)$, $t\in(-\tau,\tau)$.

(c) Let A be a subgenerator of a $[B_0,\dots,B_n,C_0,\dots,C_{n-1}]$-group $(S(t))_{t\in(-\tau,\tau)}$, $n\geqslant 2$, $(S(t))_{t\in(-\tau,\tau)}$ non-degenerate and $\hat{A}$ the integral generator of $(S(t))_{t\in(-\tau,\tau)}$. Then $iB_ix-C_{i-1}x\in D(A)$, $x\in D(\hat{A})$, $i\in\{1,\dots,n\}$, $A(iB_ix-C_{i-1}x)=iB_i\hat{A}x-iC_ix$, $i\in\{1,\dots,n-1\}$ and $A(nB_nx-C_{n-1}x)=nB_n\hat{A}x$, $x\in D(\hat{A})$. To prove this, suppose $(x,y)\in\hat{A}$. Clearly,

$$A\int_0^t S(s)x\,ds=\int_0^t\Big(S(s)y+\sum_{j=0}^{n-1}s^jC_jx\Big)ds,\quad t\in(-\tau,\tau).$$

Differentiate this equality to obtain that $S(t)x\in D(A)$ and that $AS(t)x=S(t)y+\sum_{j=0}^{n-1}t^jC_jx$, $t\in(-\tau,\tau)$. Hence,

$$A\Big[\int_0^t S(s)y\,ds+\sum_{j=0}^{n-1}\frac{t^{j+1}}{j+1}C_jx-\sum_{j=0}^{n}t^jB_jx\Big]=S(t)y+\sum_{j=0}^{n-1}t^jC_jx,$$

$$A\Big[\sum_{j=0}^{n-1}\frac{t^{j+1}}{j+1}C_jx-\sum_{j=0}^{n}t^jB_jx\Big]=\sum_{j=0}^{n-1}t^jC_jx-\sum_{j=0}^{n}t^jB_jy,\quad t\in(-\tau,\tau).$$

Differentiation of the previous equality leads us to the desired assertion. Notice that (c) extends Proposition 3.2(i) to non-degenerate groups and that, in the case $n=1$, $A(B_1x-C_0x)=B_1Ax$, $x\in D(\hat{A})$.

PROPOSITION 3.3. *Suppose $0<\tau\leqslant\infty$ and A is a subgenerator of a $[B_0,\dots,B_n,C_0,\dots,C_{n-1}]$-group $(S(t))_{t\in(-\tau,\tau)}$. If*

$$C_jA\subseteq AC_j,\ j=0,\dots,n-1\ \text{and}\ B_jA\subseteq AB_j,\ j=1,\dots,n, \tag{3.5}$$

then, for every $x\in E$,

$$\begin{aligned}S(t)S(s)x&=\sum_{j=1}^{n}\Big[\int_0^s j(t+s-r)^{j-1}B_jS(r)x\,dr-\int_t^{t+s}j(t+s-r)^{j-1}S(r)B_jx\,dr\Big]\\&\quad-\sum_{j=0}^{n-1}\Big[\int_0^s(t+s-r)^jC_jS(r)x\,dr+t^jC_j\int_0^sS(r)x\,dr\Big]\\&\quad-S(t+s)B_0x-\sum_{j=0}^{n-1}\sum_{i=0}^{n}\int_0^s(t+s-r)^jr^i\,dr\,C_jB_ix,\quad t,s\in(-\tau,\tau),\ |t+s|<\tau.\end{aligned}\tag{3.6}$$

PROOF. Suppose, for a moment, $y \in D(A)$ and $t \in (-\tau, \tau)$. Then $\int_0^t AS(s)y\,ds = S(t)y + \sum_{j=0}^{n} t^j B_j y$, i.e., $\int_0^t \big[S(s)Ay + \sum_{j=0}^{n-1} s^j C_j y\big]ds = S(t)y + \sum_{j=0}^{n} t^j B_j y$. Hence,

$$\frac{d}{dt}S(t)y = S(t)Ay + \sum_{j=0}^{n-1} t^j C_j y - \sum_{j=1}^{n} j t^{j-1} B_j y, \quad t \in (-\tau, \tau).$$

Fix an $x \in E$ and $t, s \in (-\tau, \tau)$ with $|t+s| < \tau$. Define afterwards the function $f : (t+s-\tau, t+s+\tau) \cap (-\tau, \tau) \to E$ by $f(r) := S(t+s-r) \int_0^r S(s)x\,ds$. Then we obtain:

$$\begin{aligned}
\frac{d}{dr}f(r) &= \frac{d}{dr}\Big[S(t+s-r)\int_0^r S(v)x\,dv\Big] = S(t+s-r)S(r)x - \Big[S(t+s-r)A\int_0^r S(v)x\,dv \\
&\quad + \sum_{j=0}^{n-1}(t+s-r)^j C_j \int_0^r S(v)x\,dv - \sum_{j=1}^{n} j(t+s-r)^{j-1}B_j \int_0^r S(v)x\,dv\Big] \\
&= S(t+s-r)S(r)x - S(t+s-r)\Big[S(r)x + \sum_{j=0}^{n} r^j B_j x\Big] \\
&\quad - \sum_{j=0}^{n-1}(t+s-r)^j C_j \int_0^r S(v)x\,dv + \sum_{j=1}^{n} j(t+s-r)^{j-1}B_j \int_0^r S(v)x\,dv \\
&= -\sum_{j=0}^{n-1}(t+s-r)^j C_j \int_0^r S(v)x\,dv + \sum_{j=1}^{n} j(t+s-r)^{j-1}B_j \int_0^r S(v)x\,dv \\
&\quad - \sum_{j=0}^{n} r^j S(t+s-r)B_j x,
\end{aligned}$$

for all $r \in (t+s-\tau, t+s+\tau) \cap (-\tau, \tau)$. Integrate the last equality with respect to r from 0 to s to conclude that:

$$\begin{aligned}
(3.7)\quad S(t)\int_0^s S(v)x\,dv &= -\sum_{j=0}^{n-1} C_j \int_0^s (t+s-r)^j \int_0^r S(v)x\,dv\,dr \\
&\quad + \sum_{j=1}^{n} jB_j \int_0^s (t+s-r)^{j-1} \int_0^r S(v)x\,dv\,dr - \sum_{j=0}^{n} \int_0^s r^j S(t+s-r)B_j x\,dr.
\end{aligned}$$

Thereby,

$$\begin{aligned}
S(t)S(s)x &= S(t)\Big[A\int_0^s S(v)x\,dv - \sum_{j=0}^{n} s^j B_j x\Big] \\
&= \Big[AS(t)\int_0^s S(v)x\,dv - \sum_{j=0}^{n-1} t^j C_j \int_0^s S(v)x\,dv\Big] - \sum_{j=0}^{n} s^j S(t)B_j x
\end{aligned}$$

$$= A\Big[-\sum_{j=0}^{n-1} C_j \int_0^s (t+s-r)^j \int_0^r S(v)x\,dv\,dr$$

$$+ \sum_{j=1}^{n} jB_j \int_0^s (t+s-r)^{j-1} \int_0^r S(v)x\,dv\,dr - \sum_{j=0}^{n} \int_0^s r^j S(t+s-r)B_j x\,dr\Big]$$

$$- \sum_{j=0}^{n-1} t^j C_j \int_0^s S(v)x\,dv - \sum_{j=0}^{n} s^j S(t)B_j x.$$

Taking into consideration (3.5), we get:

$$= -\sum_{j=0}^{n-1} C_j \int_0^s (t+s-r)^j \Big[S(r)x + \sum_{i=0}^{n} r^i B_i x\Big] dr$$

$$+ \sum_{j=1}^{n} jB_j \int_0^s (t+s-r)^{j-1} \Big[S(r)x + \sum_{i=0}^{n} r^i B_i x\Big] dr$$

$$- A\sum_{j=0}^{n} \int_0^s r^j S(t+s-r)B_j x\,dr - \sum_{j=0}^{n-1} t^j C_j \int_0^s S(v)x\,dv - \sum_{j=0}^{n} s^j S(t)B_j x. \tag{3.8}$$

Observing that:

$$A\int_0^s S(t+s-r)B_0 x\,dr = A\int_t^{t+s} S(v)B_0 x\,dv = A\Big[\int_0^{t+s} S(v)B_0 x\,dv - \int_0^t S(v)B_0 x\,dv\Big]$$

$$= S(t+s)B_0 x + \sum_{i=0}^{n} (t+s)^i B_i B_0 x - S(t)B_0 x - \sum_{i=0}^{n} t^i B_i B_0 x$$

and that

$$A\int_0^s r^j S(t+s-r)B_j x\,dr = A\int_t^{t+s} (t+s-v)^j S(v)B_j x\,dv$$

$$= A\Big[-s^j \int_0^t S(\sigma)B_j x\,d\sigma + \int_t^{t+s} j(t+s-v)^{j-1} \int_0^v S(\sigma)B_j x\,d\sigma\,dv\Big]$$

$$= -s^j\Big[S(t)B_j x + \sum_{i=0}^{n} t^i B_i B_j x\Big] + \int_t^{t+s} j(t+s-v)^{j-1}\Big[S(v)B_j x + \sum_{i=0}^{n} v^i B_i B_j x\Big] dv,$$

for all $j = 1, \dots, n$, (3.8) implies:

$$S(t)S(s)x = -\sum_{j=0}^{n-1} \int_0^s (t+s-r)^j C_j S(r)x\,dr - \sum_{j=0}^{n-1} \int_0^s (t+s-r)^j \sum_{i=0}^{n} r^i C_j B_i x\,dr$$

$$+\sum_{j=1}^{n} j\int_{0}^{s}(t+s-r)^{j-1}B_jS(r)x\,dr+\sum_{j=1}^{n} j\int_{0}^{s}(t+s-r)^{j-1}\sum_{i=0}^{n} r^iB_jB_ix\,dr$$

$$-S(t+s)B_0x-\sum_{i=0}^{n}(t+s)^iB_iB_0x+S(t)B_0x+\sum_{i=0}^{n}t^iB_iB_0x$$

$$+\sum_{j=1}^{n}s^j\Big[S(t)B_jx+\sum_{i=0}^{n}t^iB_iB_jx\Big]-\sum_{j=1}^{n}j\int_{t}^{t+s}(t+s-v)^{j-1}\Big[S(v)B_jx+\sum_{i=0}^{n}v^iB_iB_jx\Big]dv$$

$$-\sum_{j=0}^{n-1}t^jC_j\int_{0}^{s}S(v)x\,dv-\sum_{j=0}^{n}s^jS(t)B_jx. \tag{3.9}$$

Clearly, $S(t)B_0x+\sum_{j=1}^{n}s^jS(t)B_jx-\sum_{j=0}^{n}s^jS(t)B_jx=0$ and:

$$\begin{aligned}-\sum_{j=1}^{n}\sum_{i=0}^{n}\int_{t}^{t+s}j(t+s-v)^{j-1}v^i dvB_iB_jx&=-\sum_{j=1}^{n}\sum_{i=0}^{n}\int_{0}^{s}(t+s-r)^i jr^{j-1}drB_iB_jx\\ &=-\sum_{j=1}^{n}\sum_{i=0}^{n}t^is^jB_iB_jx-\sum_{j=1}^{n}\sum_{i=1}^{n}\int_{0}^{s}i(t+s-r)^{i-1}r^jdrB_iB_jx\\ &=-\sum_{j=1}^{n}\sum_{i=0}^{n}t^is^jB_iB_jx-\sum_{j=1}^{n}\sum_{i=1}^{n}\int_{0}^{s}j(t+s-r)^{j-1}r^idrB_jB_ix.\end{aligned}$$

Therefore,

$$\begin{aligned}-\sum_{j=1}^{n}\sum_{i=0}^{n}\int_{t}^{t+s}j(t+s-v)^{j-1}v^idvB_iB_jx&+\sum_{j=1}^{n}\sum_{i=0}^{n}s^jt^iB_iB_jx\\ &+\sum_{j=1}^{n}\sum_{i=0}^{n}\int_{0}^{s}j(t+s-r)^{j-1}r^idrB_jB_ix\\ =\sum_{j=1}^{n}\int_{0}^{s}j(t+s-r)^{j-1}drB_jB_0x&=\sum_{j=1}^{n}[(t+s)^j-t^j]B_jB_0x.\end{aligned} \tag{3.10}$$

Finally, (3.6) follows from an application of (3.9) and (3.10). □

REMARK 3.3. The composition property does not remain true if the condition (3.5) is neglected. Namely, let $A, B_0, B_1, B_2, C_0, C_1$ and $(S(t))_{t\in\mathbb{R}}$ possess the same meaning as in Example 3.1(i). Then $(S(t))_{t\in\mathbb{R}}$ is a $[B_0, B_1, B_2, C_0, C_1]$-group with a subgenerator A and a tedious matrix computation shows that (3.5) and (3.6) are not valid. Moreover, $\rho_{B_0,B_1,B_2}(A)\supseteq\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0\}$ and $R(B_0)+R(B_1)+R(B_2)\nsubseteq R(1-A)$ (see Proposition 3.1).

4. Connections between distribution groups and local integrated groups

In order to establish a satisfactory relationship of distribution groups with local integrated groups, we need the following definition introduced by Tanaka and Okazawa in [**40**] (cf. [**40**, Definition 4]):

($\triangle$) Suppose $n \in \mathbb{N}$ and $\tau \in (0,\infty]$. A strongly continuous operator family $(S(t))_{t\in[0,\tau)}$ is called a (local) n-times integrated semigroup if:

(i) $S(t)S(s)x = \left[\int_0^{t+s} - \int_0^t - \int_0^s\right] \frac{(t+s-r)^{n-1}}{(n-1)!} S(r)x\,dr$, $x \in E$, $0 \leqslant t$, $s, t+s < \tau$, and $S(0) = 0$,

(ii) $S(t)x = 0$ for every $t \in [0,\tau)$ implies $x = 0$.

Suppose $(S(t))_{t\in[0,\tau)}$ is an n-times integrated semigroup in the sense of ($\triangle$). The infinitesimal generator A_0 of $(S(t))_{t\in[0,\tau)}$ is defined in [**40**] by

$$D(A_0) := \left\{x \in \bigcup_{\sigma\in(0,\tau]} C^n(\sigma) : \lim_{h\to 0+} \frac{S^{(n)}(h)x - x}{h} \text{ exists}\right\},$$

$$A_0 x := \lim_{h\to 0+} \frac{S^{(n)}(h)x - x}{h}, \quad x \in D(A_0),$$

where $C^n(\sigma) := \{x \in E \mid S(\cdot)x : [0,\sigma) \to E$ is n-times continuously differentiable$\}$.

The infinitesimal generator A_0 of $(S(t))_{t\in[0,\tau)}$ is a closable linear operator and the closure of A_0, $\overline{A_0}$, is said to be the complete infinitesimal generator, c.i.g in short, of $(S(t))_{t\in[0,\tau)}$.

Suppose $(S(t))_{t\in[0,\tau)}$ is a (local) n-times integrated semigroup in the sense of Definition 2.3. Then $(S(t))_{t\in[0,\tau)}$ is an n-times integrated semigroup in the sense of ($\triangle$); in general, the converse statement does not hold (see [**1**], [**24**, Proposition 2.1], [**28**] and [**40**, Proposition 4.5]).

THEOREM 4.1. (a) *Suppose $G \in \mathcal{D}'(L(E))$ and A is a closed linear operator so that* (1.2) *holds. Then, for every $\tau \in (0,\infty)$, there exist $n_0 = n_0(\tau) \in \mathbb{N}$ and $B_0, \dots, B_{n_0}, C_0, \dots, C_{n_0-1} \in L(E)$ such that A is a subgenerator of a $[B_0, \dots, B_{n_0}, C_0, \dots, C_{n_0-1}]$-group $(S_\tau(t))_{t\in(-\tau,\tau)}$ satisfying $S_\tau(t)x \in D(A)$ for all $x \in E$ and $t \in (-\tau,\tau)$.*

(b) *Let G and A be as in the formulation of (a) and let $A_1 = A_{\overline{\mathcal{R}(G)}}$. Suppose, in addition, that G is regular and put $S(t) := S_\tau(t)$, $t \in (-\tau,\tau)$, where $(S_\tau(t))_{t\in(-\tau,\tau)}$ is a $[B_0, \dots, B_{n_0}, C_0, \dots, C_{n_0-1}]$-group constructed in* (a). *Then:*

(b1) *$\mathcal{R}(G) \subseteq \bigcap_{i=0}^{n_0} D(Y_i)$, $Y_{n_0}x = -x$, $x \in \mathcal{R}(G)$, the function $t \mapsto u(t;x)$, $t \in \mathbb{R}$ is infinitely differentiable, $u(t;x) \in D_\infty(A)$ and $\frac{d^n}{dt^n}u(t;x) = u(t;A^n x)$, $t \in \mathbb{R}$, $x \in \mathcal{R}(G)$, $n \in \mathbb{N}$.*

(b2) *If $n_0 = 1$, then $\pm A_1$ generate local once integrated semigroups $(S^1_\pm(t))_{t\in[0,\tau)}$ in $\mathcal{R}(G)$ given by $S^1_\pm(t)x := S(\pm t)(\pm x) \pm B_0 x$, $x \in \overline{\mathcal{R}(G)}$, $t \in [0,\tau)$. Furthermore, A_1 generates a C_0-group in $\overline{\mathcal{R}(G)}$.*

(b3) *Suppose* $n_0 = 2$ *and put* $S^2_\pm(t)x := S(\pm t)x + B_0x + t(\pm AB_0 \pm B_1)x$, $t \in [0,\tau)$, $x \in \overline{\mathcal{R}(G)}$. *Then* $S^2_\pm(t) \in L(\overline{\mathcal{R}(G)})$,

$$\Big(\int_0^t S^2_\pm(s)x\,ds, S^2_\pm(t)x - \frac{t^2}{2}x\Big) \in \pm A_1, \quad x \in \overline{\mathcal{R}(G)},\ t \in [0,\tau),$$

$S^2_\pm(t)A_1x = A_1S^2_\pm(t)x$, $t \in [0,\tau)$, $x \in \mathcal{R}(G)$, $S^2_\pm(t)x \in D(A)$, $x \in \overline{\mathcal{R}(G)}$, *the mapping* $t \mapsto \frac{d}{dt}S^2_\pm(t)x$, $t \in [0,\tau)$ *is continuously differentiable for every* $x \in \overline{\mathcal{R}(G)}$, $\overline{\mathcal{R}(G)} \subseteq \bigcap_{i=0}^2 D(Y_i)$ *and* $Y_2x = -x$, $x \in \overline{\mathcal{R}(G)}$. *Furthermore,* $\pm A_1$ *are generators of local once integrated semigroups* $(\frac{d}{dt}S^2_\pm(t))_{t\in[0,\tau)}$.

(b4) *Suppose* $n_0 \geqslant 3$,

$$\overline{\mathcal{R}(G)} \subseteq \bigcap_{i=2}^{n_0-1} D(Y_i) \tag{4.1}$$

and there exists $M > 0$ *with*

$$\|Y_ix\| \leqslant M\|x\|, \quad x \in \overline{\mathcal{R}(G)},\ i = 2,\dots,n_0-1. \tag{4.2}$$

The following holds: $\overline{\mathcal{R}(G)} \subseteq D(Y_{n_0})$ *and* $Y_{n_0}x = -x$, $x \in \overline{\mathcal{R}(G)}$. *Set* $S^{n_0}_+(t)x := S(t)x + \sum_{i=0}^{n_0-1}\frac{t^i}{i!}Y_ix$ *and* $S^{n_0}_-(t)x := (-1)^{n_0}S(-t)x + \sum_{i=0}^{n_0-1}\frac{(-1)^{n_0+i}t^i}{i!}Y_ix$, $x \in \overline{\mathcal{R}(G)}$, $t \in [0,\tau)$. *Then:* $S^{n_0}_\pm(t) \in L(\overline{\mathcal{R}(G)})$, $\big(\int_0^t S^{n_0}_\pm(s)x\,ds, S^{n_0}_\pm(t)x - \frac{t^{n_0}}{n_0!}x\big) \in \pm A_1$, $x \in \overline{\mathcal{R}(G)}$, $t \in [0,\tau)$, $S^{n_0}_\pm(t)A_1x = A_1S^{n_0}_\pm(t)x$, $t \in [0,\tau)$, $x \in \mathcal{R}(G)$ *and* $S^{n_0}_\pm(t)x \in D(A)$, $x \in \overline{\mathcal{R}(G)}$. *Set*

$$A'_{n_0-1,\pm} := \Big\{(x,y) \in \pm A_1 : C_ix + \frac{1}{i!}AY_ix = \pm\frac{1}{i!}Y_iy,\ i = 2,\dots,n_0-1\Big\}.$$

Then $A'_{n_0-1,\pm}$ *are generators of local* (n_0-1)*-times integrated semigroups* $(\frac{d}{dt}S^{n_0}_\pm(t))_{t\in[0,\tau)}$.

(b5) *Suppose* $n_0 \geqslant 3$ *and* $\rho(A) \neq \emptyset$. *Then, for every* $\tau_0 \in (0,\infty)$, *there exists* $n(\tau_0) \in \mathbb{N}$ *such that* A_1 *generates a local* $n(\tau_0)$*-times integrated group on* $(-\tau_0,\tau_0)$.

Proof. (a) Let $\tau \in (0,\infty)$ be chosen arbitrarily. Since $AG(\varphi)x = G(-\varphi')x$, $\varphi \in \mathcal{D}$, $x \in E$ we have $G \in \mathcal{D}'\big(L\big(E,[D(A)]\big)\big)$. An employment of [**32**, Theorem 2.1.1] implies that there exist an integer $n_0 = n_0(\tau)$ and a continuous function $S_\tau : [-\tau,\tau] \to L(E,[D(A)])$ such that $G(\varphi)x = (-1)^{n_0}\int_{-\tau}^\tau \varphi^{(n_0)}(t)S_\tau(t)x\,dt$, $\varphi \in \mathcal{D}_{(-\tau,\tau)}$, $x \in E$. We obtain:

$$\begin{aligned} AG(\varphi)x &= (-1)^{n_0}\int_{-\tau}^{\tau}\varphi^{(n_0)}(t)AS_\tau(t)x\,dt = (-1)^{n_0+1}\int_{-\tau}^{\tau}\varphi^{(n_0+1)}(t)\int_0^t AS_\tau(s)x\,ds\,dt \\ &= G(-\varphi')x = (-1)^{n_0+1}\int_{-\tau}^{\tau}\varphi^{(n_0+1)}(t)S_\tau(t)x\,dt, \quad \varphi \in \mathcal{D}_{(-\tau,\tau)},\ x \in E. \end{aligned}$$

An immediate consequence is:

$$\int_{-\tau}^{\tau} \varphi^{(n_0+1)}(t)\Bigg[\int_0^t AS_\tau(s)x\,ds - S_\tau(t)x\Bigg]dt = 0, \quad \varphi \in \mathcal{D}_{(-\tau,\tau)},\ x \in E.$$

The well-known arguments of distribution theory (cf. for instance [**17**, Lemma 8.1.1] or apply the Hanh–Banach theorem and [**36**, Theorem 5.10, p. 80]) imply that there exist $B_0, \dots, B_{n_0} \in L(E)$ which satisfy (a) of Definition 3.1. Similarly, if $x \in D(A)$, then $G(\varphi)Ax = AG(\varphi)x$, $\varphi \in \mathcal{D}$ and we get:

$$\int_{-\tau}^{\tau} \varphi^{(n_0)}(t)[AS_\tau(t)x - S_\tau(t)Ax]\,dt = 0, \quad \varphi \in \mathcal{D}_{(-\tau,\tau)},\ x \in E.$$

Thus, there exist $C_0, \dots, C_{n_0-1} \in L(E)$ which satisfy

$$AS_\tau(t)x - S_\tau(t)Ax = \sum_{j=0}^{n-1} t^j C_j x,$$

for all $t \in (-\tau, \tau)$ and $x \in D(A)$. To prove (b1), we need the following notion from [**23**]. Suppose $\zeta \in \mathcal{D}$ and $\int_{-\infty}^{\infty} \zeta(t)\,dt = 1$. Given $\varphi \in \mathcal{D}$, we define $I_\zeta(\varphi)$ by:

$$I_\zeta(\varphi)(t) := \int_{-\infty}^{t} \Bigg[\varphi(u) - \zeta(u)\int_{-\infty}^{\infty} \varphi(v)\,dv\Bigg]du, \quad t \in \mathbb{R}.$$

Then we have: $I_\zeta(\varphi) \in \mathcal{D}$, $I_\zeta(\varphi') = \varphi$ and $\frac{d}{dt}I_\zeta(\varphi)(t) = \varphi(t) - \zeta(t)\int_{-\infty}^{\infty} \varphi(v)\,dv$, $t \in \mathbb{R}$ [**23**]. Suppose $x \in \mathcal{R}(G)$. Since $AG(\varphi)x = G(-\varphi')x$, $\varphi \in \mathcal{D}$ one concludes $-\int_{-\infty}^{\infty} \varphi'(t)\,u(t;x)\,dt = A\int_{-\infty}^{\infty} \varphi(t)\,u(t;x)\,dt$, $\varphi \in \mathcal{D}$ and the partial integration gives:

$$A\int_{-\infty}^{\infty} \varphi'(t)\int_0^t u(s;x)\,ds\,dt = \int_{-\infty}^{\infty} \varphi'(t)\,u(t;x)\,dt, \quad \varphi \in \mathcal{D}. \tag{4.3}$$

Suppose (ρ_n) is a regularizing sequence and put $\varphi_n = I_\zeta(\rho_n)$ in (4.3) in order to see that:

$$A\int_{-\infty}^{\infty} [\rho_n(t) - \zeta(t)]\int_0^t u(s;x)\,ds\,dt = \int_{-\infty}^{\infty} [\rho_n(t) - \zeta(t)]u(t;x)\,dt.$$

The closedness of A and $u(0;x) = x$ imply, for every $\zeta \in \mathcal{D}$ with $\int_{-\infty}^{\infty} \zeta(t)\,dt = 1$:

$$A\int_{-\infty}^{\infty} \zeta(t)\int_0^t u(s;x)\,ds\,dt = \int_{-\infty}^{\infty} \zeta(t)\,u(t;x)\,dt - x. \tag{4.4}$$

It is evident that, for every $t \in \mathbb{R}$, there exists a sequence (ζ_n) in $\mathcal{D}$ so that $\int_{-\infty}^{\infty} \zeta_n(t)\,dt = 1$, $n \in \mathbb{N}$ and that $\lim_{n\to\infty} \zeta_n = \delta_t$, in the sense of distributions.

Put ζ_n in (4.4). As above, the closedness of A implies $\int_0^t u(s;x)ds \in D(A)$ and $A\int_0^t u(s;x)\,ds = u(t;x) - x$, $t \in \mathbb{R}$. Inductively,

$$A\int_0^t \frac{(t-s)^k}{k!}\,u(s;x)\,ds = \int_0^t \frac{(t-s)^{k-1}}{(k-1)!}\,u(s;x)\,ds - \frac{t^k}{k!}x, \quad t \in \mathbb{R},\ k \in \mathbb{N}. \tag{4.5}$$

Clearly, $Ax \in \mathcal{R}(G)$ and A commutes with $G(\cdot)$. Hence,

$$A\int_{-\infty}^{\infty} \varphi(t)u(t;x)\,dt = \int_{-\infty}^{\infty} \varphi(t)\,u(t;Ax)\,dt, \quad \varphi \in \mathcal{D}. \tag{4.6}$$

An application of (4.6) gives $u(t;x) \in D(A)$, $Au(t;x) = u(t;Ax)$, $t \in \mathbb{R}$ and this implies $u(t;x) \in D_\infty(A)$, $t \in \mathbb{R}$. Since $A\int_0^t u(s;x)\,ds = u(t;x) - x$, $t \in \mathbb{R}$ one obtains by induction that the function $t \mapsto u(t;x)$, $t \in \mathbb{R}$ is infinitely differentiable and that $\frac{d^n}{dt^n}u(t;x) = u(t;A^n x)$, $t \in \mathbb{R}$, $x \in \mathcal{R}(G)$, $n \in \mathbb{N}$. Furthermore,

$$A\int_0^t \frac{(t-s)^{n_0-1}}{(n_0-1)!}\,u(s;x)\,ds = \int_0^t \frac{(t-s)^{n_0-1}}{(n_0-1)!}\,u(s;Ax)\,ds. \tag{4.7}$$

Since

$$\begin{aligned} G(\varphi)x &= (-1)^{n_0}\int_{-\infty}^{\infty} \varphi^{(n_0)}(t)S(t)x\,dt = \int_{-\infty}^{\infty} \varphi(t)\,u(t;x)\,dt \\ &= (-1)^{n_0}\int_{-\infty}^{\infty} \varphi^{(n_0)}(t)\int_0^t \frac{(t-s)^{n_0-1}}{(n_0-1)!}\,u(s;x)\,ds\,dt, \quad \varphi \in \mathcal{D}_{(-\tau,\tau)}, \end{aligned}$$

there is a subset $\{y_0(x), \dots, y_{n_0-1}(x)\}$ of E such that:

$$S(t)x - \int_0^t \frac{(t-s)^{n_0-1}}{(n_0-1)!}\,u(s;x)\,ds = -\sum_{i=0}^{n_0-1} t^i y_i(x), \quad t \in (-\tau,\tau). \tag{4.8}$$

Put $t = 0$ to obtain $y_0(x) = B_0 x$. A consequence of (4.8) is:

$$\int_0^t S(s)x\,ds - \int_0^t \frac{(t-s)^{n_0}}{n_0!}\,u(s;x)\,ds = -\sum_{i=0}^{n_0-1} \frac{t^{i+1}}{i+1}y_i(x), \quad t \in (-\tau,\tau). \tag{4.9}$$

Due to (4.5), one can apply A on both sides of (4.9) in order to see that, for every $t \in (-\tau,\tau)$:

$$\left[S(t)x + \sum_{i=0}^{n_0} t^i B_i x\right] - \int_0^t \frac{(t-s)^{n_0-1}}{(n_0-1)!}\,u(s;x)\,ds + \frac{t^{n_0}}{n_0!}\,x = -A\sum_{i=0}^{n_0-1} \frac{t^{i+1}}{i+1}\,y_i(x).$$

Returning to (4.8) implies:

$$(4.10)\quad -\sum_{i=0}^{n_0-1} t^i y_i(x)+\sum_{i=0}^{n_0} t^i B_i x+\frac{t^{n_0}}{n_0!}\,x=-A\sum_{i=0}^{n_0-1}\frac{t^{i+1}}{i+1}\,y_i(x),\quad t\in(-\tau,\tau).$$

Since A is closed, one can differentiate (4.10) sufficiently many times to obtain that: $x\in\bigcap_{i=0}^{n_0}D(Y_i)$, $Y_{n_0}x=-x$ and $y_i(x)=\frac{1}{i!}Y_ix$, $i\in\{0,\dots,n_0-1\}$. This completes the proof of (b1).

To prove (b2), fix an $x\in\mathcal{R}(G)$. Put $S_+^{n_0}(t)x=S(t)x+\sum_{i=0}^{n_0-1}\frac{t^i}{i!}Y_i(x)$, $t\in[0,\tau)$. By (4.8), $S_+^{n_0}(t)x=\int_0^t\frac{(t-s)^{n_0-1}}{(n_0-1)!}\,u(s;x)\,ds$, $t\in[0,\tau)$ and an employment of (4.7) implies $AS_+^{n_0}(t)x=S_+^{n_0}(t)Ax$, $t\in[0,\tau)$. To prove (b2), suppose $n_0=1$. Then $S_+^1(t)x=S(t)x+B_0x$, $t\in[0,\tau)$, $x\in\overline{\mathcal{R}(G)}$. By the proof of (b1), one yields $S_+^1(t)x=\int_0^t u(s;x)\,ds$, $t\in[0,\tau)$, $x\in\mathcal{R}(G)$. Accordingly, $S_+^1(t)(\overline{\mathcal{R}(G)})\subseteq\overline{\mathcal{R}(G)}$, $t\in[0,\tau)$. By (4.5), $\big(\int_0^t S_+^1(s)x\,ds,S_+^1(t)x-tx\big)\in A_1$, $t\in[0,\tau)$, $x\in\mathcal{R}(G)$ and the closedness of A implies $\big(\int_0^t S_+^1(s)x\,ds,S_+^1(t)x-tx\big)\in A_1$, $t\in[0,\tau)$, $x\in\overline{\mathcal{R}(G)}$. Clearly, $S_+^1(t)A_1\subseteq A_1S_+^1(t)$, $t\in[0,\tau)$ and this proves that $(S_+^1(t))_{t\in[0,\tau)}$ is a once integrated semigroup generated by A_1. The similar arguments (see also the proof of (b3)) work for $-A_1$ and $(S_-^1(t))_{t\in[0,\tau)}$. To prove that A_1 generates a C_0-group in $\overline{\mathcal{R}(G)}$, we argue as follows. Since $\big(\int_0^t S_+^1(s)x\,ds,S_+^1(t)x-tx\big)\in A_1$, $t\in[0,\tau)$, $x\in\overline{\mathcal{R}(G)}$ and $R(S_+^1(t))\subseteq D(A)$, $t\in[0,\tau)$ one gets that the mapping $t\mapsto\frac{d}{dt}S_+^1(t)x$, $t\in[0,\tau)$ is continuously differentiable for every $x\in\overline{\mathcal{R}(G)}$ and that $\frac{d}{dt}S_+^1(t)x=AS_1(t)x+x$, $t\in[0,\tau)$, $x\in\overline{\mathcal{R}(G)}$. Moreover, it can be easily checked that, for every fixed $x\in\overline{\mathcal{R}(G)}$, the function $u(t)=S_+^1(t)x$, $t\in[0,\tau)$ is a unique solution of the problem:

$$C_1(\tau):\begin{cases} u\in C\big([0,\tau):[D(A_1)]\big)\cap C^1\big([0,\tau):\overline{\mathcal{R}(G)}\big),\\ u'(t)=A_1u(t)+x,\quad t\in[0,\tau),\\ u(0)=0.\end{cases}$$

An application of [**1**, Theorem 1.2] gives that A_1 generates a C_0-semigroup in $\overline{\mathcal{R}(G)}$. Similarly, $-A_1$ generates a C_0-semigroup in $\overline{\mathcal{R}(G)}$ and this clearly implies that A_1 generates a C_0-group in $\overline{\mathcal{R}(G)}$.

To prove (b3), note that the proof of (b1) implies that $S_+^2(t)x=\int_0^t(t-s)u(s;x)\,ds$, $t\in[0,\tau)$, $x\in\mathcal{R}(G)$. So, $S_+^2(t)(\overline{\mathcal{R}(G)})\subseteq\overline{\mathcal{R}(G)}$, $t\in[0,\tau)$. Note also that $S(0)=-B_0$ and that the Closed Graph Theorem gives $S_+^2(t)\in L(\overline{\mathcal{R}(G)})$, $t\in[0,\tau)$. Next, the closedness of A and (4.5) imply $\big(\int_0^t S_+^2(s)x\,ds,S_+^2(t)-\frac{t^2}{2}x\big)\in A_1$, $x\in\overline{\mathcal{R}(G)}$, $t\in[0,\tau)$. Since $\int_0^t S_+^2(s)x\,ds\in D(A)$, $x\in\overline{\mathcal{R}(G)}$, $t\in[0,\tau)$ and $R(B_0)\subseteq D(A)$, we immediately obtain $AB_0x+B_1x\in D(A)$, $x\in\overline{\mathcal{R}(G)}$. Further on,

$$\begin{aligned}A\int_0^t S_+^2(s)x\,ds&=S(t)x+B_0x+tB_1x+t^2B_2x+tAB_0x+\frac{t^2}{2}A(AB_0x+B_1x)\\&=S_+^2(t)x-\frac{t^2}{2}x,\quad x\in\overline{\mathcal{R}(G)},\ t\in[0,\tau).\end{aligned}$$

Therefore, $A(AB_0x+B_1x)=-x-2B_2x$, $x\in\overline{\mathcal{R}(G)}$, and in conclusion, one yields: $S_+^2(t)x\in D(A)$, $AS_+^2(t)x=A(S(t)x+B_0x)+t(-x-2B_2x)$, $x\in\overline{\mathcal{R}(G)}$, $t\in[0,\tau)$, $\overline{\mathcal{R}(G)}\subseteq\bigcap_{i=0}^2 D(Y_i)$ and $Y_2x=-x$, $x\in\overline{\mathcal{R}(G)}$. Suppose $x\in D(A_1)$. Since $R(B_0)\subseteq D(A)$ and $AS_+^2(t)x-S_+^2(t)Ax=\bigl[AS(t)x+AB_0x+tA(AB_0+B_1)x\bigr]-\bigl[S(t)Ax+B_0Ax+t(AB_0+B_1)Ax\bigr]=tC_1x+t\bigl[A(AB_0+B_1)x-(AB_0+B_1)Ax\bigr]$, $t\in[0,\tau)$, Proposition 3.2(iv) immediately implies $(S_+^2(t)x,S_+^2(t)A_1x)\in A_1$, $t\in[0,\tau)$. So, $(S_+^2(t))_{t\in[0,\tau)}$ is a twice integrated semigroup generated by A_1. Because $R(S_+^2(t))\subseteq D(A)$, $t\in[0,\tau)$, the mapping $t\mapsto S_+^2(t)x$ is continuously differentiable for every fixed $x\in\overline{\mathcal{R}(G)}$ and the following holds: $\frac{d}{dt}S_+^2(t)x=AS_+^2(t)x+tx=A(S(t)x+B_0x)-2tB_2x$, $t\in[0,\tau)$, $x\in\overline{\mathcal{R}(G)}$. Then it is straightforward to see that $\frac{d}{dt}S_+^2(t)\in L(\overline{\mathcal{R}(G)})$, $t\in[0,\tau)$ and that $\bigl(\int_0^t(\frac{d}{ds}S_+^2(s)x)\,ds,(\frac{d}{dt}S_+^2(t)x)-tx\bigr)\in A_1$, $t\in[0,\tau)$. Suppose now $x\in D(A_1)$. Then $\frac{d}{dt}S_+^2(t)x=AS_+^2(t)x+tx=S_+^2(t)Ax+tx\in D(A)$ and $A\frac{d}{dt}S_+^2(t)x=AS_+^2(t)Ax+tAx=\frac{d}{dt}S_+^2(t)Ax$, $t\in[0,\tau)$, $\bigl(\frac{d}{dt}S_+^2(t)x,\frac{d}{dt}S_+^2(t)A_1x\bigr)\in A_1$, $t\in[0,\tau)$, and consequently, $\bigl(\frac{d}{dt}S_+^2(t)\bigr)_{t\in[0,\tau)}$ is a once integrated semigroup generated by A_1. In order to obtain the corresponding statement for the operator $-A_1$ and $(S_-^2(t))_{t\in[0,\tau)}$, notice the following facts: (1.2) holds for $-A$ and $\check{G}$, $\check{G}$ fulfills $(DG)_4$ with $u(\check{\cdot};x)$, $\check{G}(\varphi)x=(-1)^{n_0}\int_{-\infty}^{\infty}\varphi^{(n_0)}(t)[(-1)^{n_0}\check{S}(t)]x\,dt$, $x\in E$, $\varphi\in\mathcal{D}$ and $((-1)^{n_0}\check{S}(t))_{t\in(-\tau,\tau)}$ is a $[(-1)^{n_0}B_0,\dots,(-1)^{n_0+n_0}B_{n_0},(-1)^{n_0+1}C_0,\dots,(-1)^{n_0+n_0}C_{n_0-1}]$-group with a subgenerator $-A$.

In order to prove (b4), assume $x\in\overline{\mathcal{R}(G)}$. Let (x_n) be a sequence in $\mathcal{R}(G)$ with $\lim_{n\to\infty}x_n=x$. Due to (4.2) and (b1), $\lim_{n\to\infty}Y_{n_0-1}(x_n)=Y_{n_0-1}x$ and $\lim_{n\to\infty}AY_{n_0-1}(x_n)=-x-n_0!B_{n_0}x$. Hence, $Y_{n_0-1}x\in D(A)$, $x\in D(Y_{n_0})$ and $Y_{n_0}x=-x$ as claimed. This yields $S_\pm^{n_0}(t)x\in D(A)$, $x\in\overline{\mathcal{R}(G)}$. As in the proofs of (b1), (b2) and (b3), one obtains $S_\pm^{n_0}(t)\in L(\overline{\mathcal{R}(G)})$, $\bigl(\int_0^t S_\pm^{n_0}(s)x\,ds,S_\pm^{n_0}(t)x-\frac{t^{n_0}}{n_0!}x\bigr)\in\pm A_1$, $x\in\overline{\mathcal{R}(G)}$, $t\in[0,\tau)$ and $S_\pm^{n_0}(t)A_1x=A_1S_\pm^{n_0}(t)x$, $t\in[0,\tau)$, $x\in\mathcal{R}(G)$. We will sketch the rest of the proof of (b4) only for A and $S(\cdot)$. Suppose t, $s\in[0,\tau)$ and $t+s<\tau$. Since $AS_+^{n_0}(\cdot)x=S_+^{n_0}(\cdot)Ax$, $x\in\mathcal{R}(G)$, one can repeat literally the arguments given in the proof of [**29**, Propostion 2.4] in order to conclude that:

$$(4.11)\quad S_+^{n_0}(t)S_+^{n_0}(s)x=\Biggl[\int_0^{t+s}-\int_0^t-\int_0^s\Biggr]\frac{(t+s-r)^{n_0-1}}{(n_0-1)!}S_+^2(r)x\,dr,\quad x\in\mathcal{R}(G).$$

The standard limit procedure implies that (4.11) remains true for every $x\in\overline{\mathcal{R}(G)}$ and t, $s\in[0,\tau)$ with $t+s<\tau$. It is straightforward to verify that $(S_+^{n_0}(t))_{t\in[0,\tau)}\subseteq L(\overline{\mathcal{R}(G)})$ is a local n_0-times integrated semigroup in the sense of $(\triangle)$. To prove that $A'_{n_0-1,+}$ is the generator of a local n_0-times integrated semigroup $(S_+^{n_0}(t))_{t\in[0,\tau)}$ in the sense of Definition 3.1, we argue as follows. First of all, let us observe that $A'_{n_0-1,+}$ is a closed operator and that the arguments employed in the proof of Proposition 3.2 also show that $D(A^{n_0})\cap D(A_1)\subseteq C^{n_0}(\tau)$. Suppose now $x\in D(A_0)$, where A_0 is the infinitesimal generator of $(S_+^{n_0}(t))_{t\in[0,\tau)}$. This implies the existence of a number $\sigma \in (0,\tau)$ so that the mapping $t \mapsto S_+^{n_0}(t)x$, $t \in [0,\sigma)$ is n_0-times continuously differentiable and that $A_0x = \lim_{t\to 0+}\bigl(\frac{d^{n_0}}{dt^{n_0}}(S_+^{n_0}(t)x) - x\bigr)/t$. On the other hand, the closedness of A offers one to show that $\frac{d^k}{dt^k}S(t)x \in D(A)$ and that $\frac{d^{k+1}}{dt^{k+1}}S(t)x = A\frac{d^k}{dt^k}S(t)x - \sum_{j=k+1}^{n} j\cdots(j-k)t^{j-k-1}B_jx$, for every $t \in [0,\sigma)$ and $k \in \{0,\dots,n_0-1\}$. Therefore, $x \in \bigcap_{i=0}^{n_0} D(Y_i)$ and $Y_kx = -\bigl(\frac{d^k}{dt^k}S(t)x\bigr)_{t=0}$, $k \in \{0,\dots,n_0\}$. Moreover,

$$\begin{aligned}\frac{d^{n_0}}{dt^{n_0}}\bigl(S_+^{n_0}(t)x\bigr) - x &= A\Bigl[\frac{d^{n_0-1}}{dt^{n_0-1}}S_+^{n_0}(t)x\Bigr] - n_0!B_{n_0}x - x\\ = A\Bigl[\frac{d^{n_0-1}}{dt^{n_0-1}}S_+^{n_0}(t)x\Bigr] + AY_{n_0-1}x &= A\Bigl[\frac{d^{n_0-1}}{dt^{n_0-1}}S_+^{n_0}(t)x - \Bigl(\frac{d^{n_0-1}}{dt^{n_0-1}}S(t)x\Bigr)_{t=0}x\Bigr].\end{aligned}$$

It is also evident that

$$x = \Bigl(\frac{d^{n_0}}{dt^{n_0}}S(t)x\Bigr)_{t=0} = \lim_{t\to 0+}\frac{1}{t}\Bigl(\frac{d^{n_0-1}}{dt^{n_0-1}}S_+^{n_0}(t)x - \Bigl(\frac{d^{n_0-1}}{dt^{n_0-1}}S(t)x\Bigr)_{t=0}x\Bigr).$$

The closedness of A implies $x \in D(A_1)$, $A_0x = A_1x$ and, because of that, $\overline{A_0} \subseteq A_1$. Further on, $\mathcal{R}(G) \subseteq D(A^{n_0}) \cap D(A_1) \subseteq C^{n_0}(\tau)$ and an application of [**40**, Proposition 4.5] gives $\bigl(\int_0^t S_+^{n_0}(s)x\,ds, S_+^{n_0}(t)x - \frac{t^{n_0}}{n_0!}x\bigr) \in \overline{A_0}$, $x \in \overline{\mathcal{R}(G)}$, $t \in [0,\tau)$ and $S_+^{n_0}(t)\overline{A_0}x = \overline{A_0}S_+^{n_0}(t)x$, $t \in [0,\tau)$, $x \in D(\overline{A_0})$. Suppose $(x,y) \in \overline{A_0}$. Then

$$0 = AS_+^{n_0}(t)x - S_+^{n_0}(t)y = \sum_{i=0}^{n_0-1} t^iC_ix + \sum_{i=0}^{n_0-1}\frac{t^i}{i!}AY_ix - \sum_{i=0}^{n_0-1}\frac{t^i}{i!}Y_iy, \quad t \in [0,\tau)$$

and this implies $\overline{A_0} \subseteq A'_{n_0-1,+}$. Further, fix an $x \in D(A'_{n_0-1,+})$ and notice that $AS_+^{n_0}(t)x = S_+^{n_0}(t)Ax$, $t \in [0,\tau)$ and that

$$\begin{aligned}A'_{n_0-1,+} \ni \Bigl(\int_0^t S_+^{n_0}(s)x\,ds, S_+^{n_0}(t)x - \frac{t^{n_0}}{n_0!}x\Bigr) &= \Bigl(\int_0^t S_+^{n_0}(s)x\,ds, A_1\int_0^t S_+^{n_0}(s)x\,ds\Bigr)\\ &= \Bigl(\int_0^t S_+^{n_0}(s)x\,ds, \int_0^t S_+^{n_0}(s)Ax\,ds\Bigr), \quad t \in [0,\tau).\end{aligned}$$

This implies

$$C_i\int_0^t S_+^{n_0}(s)x\,ds + \frac{1}{i!}AY_i\int_0^t S_+^{n_0}(s)x\,ds = \frac{1}{i!}Y_i\int_0^t S_+^{n_0}(s)Ax\,ds,$$
$$t \in [0,\tau),\ i \in \{2,\dots,n_0-1\}.$$

Differentiate this equality to obtain that $C_iS_+^{n_0}(t)x + \frac{1}{i!}AY_iS_+^{n_0}(t)x = \frac{1}{i!}Y_iS_+^{n_0}(t)Ax$, $t \in [0,\tau)$, $i \in \{2,\dots,n_0-1\}$. Thus, $S_+^{n_0}(t)A'_{n_0-1,+} \subseteq A'_{n_0-1,+}S_+^{n_0}(t)$, $t \in [0,\tau)$ and $A'_{n_0-1,+}$ is the generator of a local n_0-times integrated semigroup $(S_+^{n_0}(t))_{t\in[0,\tau)}$ in the sense of Definition 2.3. An application of the arguments given in the proof of [**24**, Proposition 2.1] gives $\overline{A_0} = A'_{n_0-1,+}$. Since $R(S_+^{n_0}(t)) \subseteq D(A)$, the mapping

$t \mapsto S_+^{n_0}(t)x$, $t \in [0,\tau)$ is continuously differentiable for every fixed $x \in \overline{\mathcal{R}(G)}$ and

$$\frac{d}{dt}S_+^{n_0}(t)x = AS_+^{n_0}(t)x + \frac{t^{n_0-1}}{(n_0-1)!}x = AS(t)x + \sum_{i=0}^{n_0-1}\frac{t^i}{i!}AY_i x + \frac{t^{n_0-1}}{(n_0-1)!}x,$$

$t \in [0,\tau)$, $x \in \overline{\mathcal{R}(G)}$. Then it can be easily verified that $(\frac{d}{dt}S_+^{n_0}(t))_{t\in[0,\tau)} \subseteq L(\overline{\mathcal{R}(G)})$ is a local (n_0-1)-times integrated semigroup in the sense of $(\triangle)$. The c.i.g of $(\frac{d}{dt}S_+^{n_0}(t))_{t\in[0,\tau)}$ is $\overline{A_0} = A'_{n_0-1,+}$ and an application of [**40**, Proposition 4.5] enables one to conclude that $A'_{n_0-1,+}$ is the generator of a local (n_0-1)-times integrated semigroup $(\frac{d}{dt}S_+^{n_0}(t))_{t\in[0,\tau)}$ in the sense of Definition 2.3.

To prove (b5), suppose $\lambda \in \rho(A)$ and set $A_\lambda = A - \lambda$, $G_\lambda = e^{-\lambda\cdot}G$ and $u_\lambda(\cdot;x) = e^{-\lambda\cdot}u(\cdot;x)$, $x \in \mathcal{R}(G) = \mathcal{R}(G_\lambda)$. It is straightforward to check that A_λ and G_λ fulfill (1.2) and that G_λ is regular with $G_\lambda(\varphi)x = \int_{-\infty}^{\infty}\varphi(t)u_\lambda(t;x)\,dt$, $\varphi \in \mathcal{D}$, $x \in \mathcal{R}(G_\lambda)$. Clearly,

$$G_\lambda(\varphi)x = G(e^{-\lambda\cdot}\varphi)x = (-1)^{n_0}\int_{-\tau}^{\tau}(e^{-\lambda\cdot}\varphi)^{(n_0)}(t)S(t)x\,dt$$

$$= (-1)^{n_0}\int_{-\tau}^{\tau}\sum_{i=0}^{n_0}(-1)^{n_0-i}\binom{n_0}{i}\lambda^{n_0-i}e^{-\lambda t}\varphi^{(i)}(t)S(t)x\,dt$$

$$= \sum_{i=0}^{n_0}(-1)^i\binom{n_0}{i}\lambda^{n_0-i}\int_{-\tau}^{\tau}\varphi^{(i)}(t)\bigl(e^{-\lambda t}S(t)x\bigr)\,dt = (-1)^{n_0}\int_{-\tau}^{\tau}\varphi^{(n_0)}(t)e^{-\lambda t}S(t)x\,dt$$

$$+ \sum_{i=1}^{n_0}(-1)^i\binom{n_0}{i}\lambda^{n_0-i}(-1)^{n_0-i}\int_{-\tau}^{\tau}\varphi^{(n_0)}(t)\int_0^t\frac{(t-s)^{n_0-i-1}}{(n_0-i-1)!}e^{-\lambda s}S(s)x\,ds\,dt$$

$$= (-1)^{n_0}\int_{-\tau}^{\tau}\varphi^{(n_0)}(t)\Bigl[e^{-\lambda t}S(t)x + \sum_{i=1}^{n_0}\binom{n_0}{i}\lambda^{n_0-i}\int_0^t\frac{(t-s)^{n_0-i-1}}{(n_0-i-1)!}e^{-\lambda s}S(s)x\,ds\Bigr]dt,$$

for every $\varphi \in \mathcal{D}_{(-\tau,\tau)}$ and $x \in E$. Put, for every $t \in (-\tau,\tau)$ and $x \in E$:

$$S_\lambda(t)x := e^{-\lambda t}S(t)x + \sum_{i=1}^{n_0}\binom{n_0}{i}\lambda^{n_0-i}\int_0^t\frac{(t-s)^{n_0-i-1}}{(n_0-i-1)!}e^{-\lambda s}S(s)x\,ds.$$

Then the mapping $S_\lambda : (-\tau,\tau) \to L(E,[D(A_\lambda)])$ is continuous and $G_\lambda(\varphi)x = (-1)^{n_0}\int_{-\tau}^{\tau}\varphi^{(n_0)}(t)S_\lambda(t)x\,dt$, $\varphi \in \mathcal{D}_{(-\tau,\tau)}$, $x \in E$. The proof of (a) implies that there exist bounded linear operators $B_0^\lambda,\ldots,B_{n_0}^\lambda,C_0^\lambda,\ldots,C_{n_0-1}^\lambda$ such that A_λ is a subgenerator of a $[B_0^\lambda,\ldots,B_{n_0}^\lambda,C_0^\lambda,\ldots,C_{n_0-1}^\lambda]$-group $(S_\lambda(t))_{t\in(-\tau,\tau)}$. Define Y_i^λ recursively by: $Y_0^\lambda = B_0^\lambda$ and $Y_{i+1}^\lambda := (i+1)!B_{i+1}^\lambda + A_\lambda Y_i^\lambda$, $i \in \{0,\ldots,n_0-1\}$. Since $0 \in \rho(A_\lambda)$, we have that Y_i^λ is closed, $i = 1,\ldots,n_0$. Suppose, for the time being, $x \in \overline{\mathcal{R}(G)}$ and (x_n) is a sequence in $\mathcal{R}(G)$ such that $\lim_{n\to\infty}x_n = x$. A consequence of $Y_{n_0}^\lambda x_n = -x_n$, $n \in \mathbb{N}$ is $\lim_{n\to\infty}A_\lambda Y_{n_0-1}^\lambda x_n = -x - n_0!B_{n_0}^\lambda x$ and the boundedness of A_λ^{-1} implies $\lim_{n\to\infty} Y^\lambda_{n_0-1}x_n = A_\lambda^{-1}(-x-n_0!B^\lambda_{n_0}x)$. Continuing this procedure enables one to establish that, for every $i=1,\dots,n_0-1$, $\lim_{n\to\infty} Y_i^\lambda x_n$ exists. The closedness of Y_i^λ yields $x\in\bigcap_{i=0}^{n_0} D(Y_i^\lambda)$ and $Y^\lambda_{n_0}x=-x$. Put $A_{1,\lambda}=(A_\lambda)_{\overline{\mathcal{R}(G_\lambda)}}$ and $Cx=A_\lambda^{-(n_0-1)}x$, $x\in\overline{\mathcal{R}(G)}$. Because $G_\lambda A_\lambda\subseteq A_\lambda G_\lambda$, we have $A_\lambda^{-k}G_\lambda=G_\lambda A_\lambda^{-k}$, $k\in\mathbb{N}$, $A_\lambda^{-k}(\mathcal{R}(G))\subseteq\mathcal{R}(G)$, $k\in\mathbb{N}$ and $A_\lambda^{-k}(\overline{\mathcal{R}(G)})\subseteq\overline{\mathcal{R}(G)}$, $k\in\mathbb{N}$. This offers one to see that $0\in\rho(A_{1,\lambda})$ and that $C\in L(\overline{\mathcal{R}(G)})$ is injective. Assume now $x\in D(A_{1,\lambda}^{n_0-1})$. Then $A_{1,\lambda}^{n_0-1}x\in\overline{\mathcal{R}(G)}$ and this gives $x=A_\lambda^{-(n_0-1)}(A_\lambda^{n_0-1}x)=C(A_\lambda^{n_0-1}x)\in R(C)$. Hence, $D(A_{1,\lambda}^{n_0-1})\subseteq R(C)$. Proceeding as in the proof of Proposition 3.2, one obtains that the mapping $t\mapsto S_\lambda(t)x$, $t\in(-\tau,\tau)$ is n_0-times continuously differentiable and that there exists a function $M:(-\tau,\tau)\to(0,\infty)$, independent of x, so that $\|\frac{d^{n_0}}{dt^{n_0}}S_\lambda(t)x\|\leqslant M(|t|)\sum_{i=0}^{n_0}\|A_\lambda^i x\|$, $t\in(-\tau,\tau)$. Put $u^\lambda(t;x)=\frac{d^{n_0}}{dt^{n_0}}S_\lambda(t)x$, $t\in[0,\tau)$, $x\in D(A_\lambda^{n_0-1})$ and $T(t)x=u^\lambda(t;Cx)$, $t\in[0,\tau)$, $x\in\overline{\mathcal{R}(G)}$. Due to Proposition 3.2, $D(A_\lambda^{n_0-1})\subseteq D(Y_\lambda^{n_0})$ and $u^\lambda(0;x)=-Y_\lambda^{n_0}x$, $x\in D(A_\lambda^{n_0-1})$. Moreover, $R(C)\subseteq\overline{\mathcal{R}(G)}\cap D(A_\lambda^{n_0-1})$ and this implies $u^\lambda(0;Cx)=-Y_\lambda^{n_0}Cx=Cx$. The mapping $t\mapsto T(t)x$, $t\in[0,\tau)$ is continuous for every fixed $x\in\overline{\mathcal{R}(G)}$ and $\|T(t)x\|=\|u^\lambda(t;A_\lambda^{-(n_0-1)}x)\|\leqslant M(t)\sum_{i=0}^{n_0-1}\|A_\lambda^{-1}\|^i\|x\|$, $t\in[0,\tau)$, $x\in\overline{\mathcal{R}(G)}$. The partial integration shows $G_\lambda(\varphi)x=\int_{-\tau}^{\tau}\varphi(t)u^\lambda(t;x)\,dt$, $\varphi\in\mathcal{D}_{[0,\tau)}$, $x\in D(A_\lambda^{n_0-1})$ and this implies $u^\lambda(t;x)\in\overline{\mathcal{R}(G)}$, $t\in[0,\tau)$, $x\in D(A_\lambda^{n_0-1})$. Therefore, $T(t)x\in\overline{\mathcal{R}(G)}$, $t\in[0,\tau)$, $x\in\overline{\mathcal{R}(G)}$ and $T(t)\in L(\overline{\mathcal{R}(G)})$, $t\in[0,\tau)$. As in the proof of (b1), one concludes $A_\lambda\int_0^t u^\lambda(s;Cx)\,ds=A_\lambda\int_0^t T(s)xds=u^\lambda(t;Cx)-u^\lambda(0;Cx)=T(t)x-Cx$, $t\in[0,\tau)$, $x\in\overline{\mathcal{R}(G)}$ and that $u^\lambda(t;A_\lambda x)=A_\lambda u^\lambda(t;x)$, $t\in[0,\tau)$, $x\in D(A_\lambda^{n_0})$. Due to the previous equality, we have $T(t)A_{1,\lambda}\subseteq A_{1,\lambda}T(t)$ and $T(t)C=CT(t)$, $t\in[0,\tau)$. Now it is straightforward to prove that the abstract Cauchy problem:

$$\begin{cases} v\in C\big([0,\tau):[D(A_{1,\lambda})]\big)\cap C^1\big([0,\tau):\overline{\mathcal{R}(G)}\big),\\ v'(t)=A_{1,\lambda}v(t)+Cx,\ t\in[0,\tau),\\ v(0)=0,\end{cases}$$

possesses a unique solution for every $x\in\overline{\mathcal{R}(G)}$, given by $v(t)=\int_0^t T(s)x\,ds$, $t\in[0,\tau)$, $x\in\overline{\mathcal{R}(G)}$. This simply implies that the abstract Cauchy problem:

$$(ACP,\tau):\begin{cases} f\in C\big([0,\tau):[D(A_{1,\lambda})]\big)\cap C^1\big([0,\tau):\overline{\mathcal{R}(G)}\big),\\ f'(t)=A_{1,\lambda}f(t),\ t\in[0,\tau),\\ f(0)=x,\end{cases}$$

has a unique solution for every $x\in C(D(A_{1,\lambda}))$ and that $A_{1,\lambda}$ is the integral generator of a local C-semigroup $(T(t))_{t\in[0,\tau)}$. As before, $D(A_\lambda^{n_0})\subseteq C(D(A_\lambda))$ and an application of [**44**, Theorem 4.4] shows that $A_{1,\lambda}$ generates a local (n_0-1)-times integrated semigroup on $[0,\tau)$. A rescaling result for local integrated semigroups (cf. for instance [**1**]) implies that A_1 generates a local (n_0-1)-times integrated semigroup on $[0,\tau)$. Analogously, $-A_1$ generates a local (n_0-1)-times integrated semigroup on $[0,\tau)$ and the proof ends an employment of Corollary 2.1. □

THEOREM 4.2. *Let G be a (DG) generated by A. Then the group $(S(t))_{t\in(-\tau,\tau)}$, constructed in Theorem* 4.1(a), *is non-degenerate. If $n_0 = 1$, then A generates a C_0-group. If $n_0 = 2$, then:*

(a) *$(S^1_{\pm}(t) := \pm A(S(\pm t)x + B_0x) - 2tB_2x)_{t\in[0,\tau)}$ are local once integrated semigroups in the sense of ($\triangle$).*

(b) *The c.i.g of $(S^1_{+}(t))_{t\in[0,\tau)}$, resp., $(S^1_{-}(t))_{t\in[0,\tau)}$ is $A_{\overline{\mathcal{R}(G)}}$, resp., $(-A)_{\overline{\mathcal{R}(G)}}$.*

(c) *Suppose A is densely defined or $\lambda - A$ is surjective for some $\lambda\in\mathbb{C}$. Then $\pm A$ are generators of local once integrated semigroups $(S^1_{\pm}(t))_{t\in[0,\tau)}$.*

Furthermore:

(i) *For every $x\in E$ and φ, $\psi\in\mathcal{D}_{(-\tau,\tau)}$ with $\operatorname{supp}\varphi + \operatorname{supp}\psi \subseteq (-\tau,\tau)$:*

$$G(\varphi)\,G(\psi)x = \sum_{i=0}^{n_0}(-1)^{i+1}i!\int_{-\infty}^{\infty}\varphi^{(n_0)}(t)\int_{-\infty}^{\infty}\psi^{(n_0-i)}(s)S(t+s)B_ix\,ds\,dt. \tag{4.12}$$

(ii) *$Y_{n_0}x = -x$, $x\in\bigcap_{i=0}^{n_0}D(Y_i)$.*

(iii) *Suppose $x\in D(A^{n_0-1})$. Then $\{x, Ax\}\subseteq\bigcap_{i=2}^{n_0}D(Y_i)$, $Y_{n_0}x = -x$, $Y_{n_0}Ax = -Ax$ and $D(A^{n_0-1})\subseteq\overline{\mathcal{R}(G)}$.*

(iv) *A is stationary dense with $n(A)\leqslant n_0 - 1$.*

(v) *If $\rho(A)\neq\emptyset$, then for every $\tau_0\in(0,\infty)$, there is an $n(\tau_0)\in\mathbb{N}$ so that A generates a local $n(\tau_0)$-times integrated group on $(-\tau_0,\tau_0)$.*

(vi) *G is dense iff $D_\infty(A)$ is dense in E. In the case $\rho(A)\neq\emptyset$, G is dense iff A is densely defined.*

(vii) *$\bigcap_{\varphi\in\mathcal{D}_0}N(G(\varphi)) = \{0\}$ and $\bigcap_{\varphi\in\mathcal{D}_0}N(G(\check{\varphi})) = \{0\}$.*

PROOF. Assume $S(t)x = 0$, $t\in(-\tau,\tau)$. This implies $G(\psi)x = 0$, $\psi\in\mathcal{D}_{(-\tau,\tau)}$ and $G(\rho)x = \lim_{n\to\infty}G(\rho*\rho_n)x = \lim_{n\to\infty}G(\rho)G(\rho_n)x = 0$, $\rho\in\mathcal{D}$, where (ρ_n) is a regularizing sequence. Owing to $(DG)_2$, one can deduce that $x\in\mathcal{N}(G)$ and that $(S(t))_{t\in(-\tau,\tau)}$ is non-degenerate. Put now $S_1(t)x = S(t)x + B_0x$, $t\in(-\tau,\tau)$, $x\in E$. We will prove that $(S_1(t))_{t\in[0,\tau)}$ is a once integrated semigroup generated by A. First of all, note that $S_1(t)A\subseteq AS_1(t)$, $t\in(-\tau,\tau)$ and that $S_1 : (-\tau,\tau)\to L(E,[D(A)])$ is continuous. This clearly implies $\frac{d}{dt}S_1(t)x = AS_1(t)x + Bx$, $t\in(-\tau,\tau)$, $x\in E$, where $B = -B_1 - AB_0\in L(E)$. Further, $\int_0^t S_1(s)x ds\in D(A)$, $t\in(-\tau,\tau)$, $x\in E$ and one gets that $A\int_0^t S_1(s)x\,ds = A\int_0^t(S(s)x + B_0x)\,ds = S(t)x + B_0x + tB_1x + tAB_0x = S_1(t)x - tBx$, $t\in(-\tau,\tau)$, $x\in E$ and that $(S_1(t))_{t\in(-\tau,\tau)}$ is a $[0,-B,0]$-group with a subgenerator A. We will prove that $B = I$. Suppose $\zeta,\eta\in\mathcal{D}_{(-\tau/4,\tau/4)}$ and (ρ_n) is a regularizing sequence. We know [**23**] that $\operatorname{supp}I_\zeta(\varphi)\subseteq\big[\min(-\tau/4,\inf(\operatorname{supp}\varphi)),\max(\tau/4,\sup(\operatorname{supp}\varphi))\big]$ and that there exists $k\in\mathbb{N}$ such that $\operatorname{supp}I_\zeta(\rho_n)\cup\operatorname{supp}I_\eta(\rho_n)\subseteq[-\tau/4,\tau/4]$, $n\geqslant k$. Fix an $x\in E$. By $(DG)_1$ (see also the equation (4.21) given below), one gets that, for every $\varphi,\psi\in\mathcal{D}_{(-\tau/4,\tau/4)}$:

$$\int_{-\infty}^{\infty}\varphi'(t)\int_{-\infty}^{\infty}\psi'(s)S_1(t)S_1(s)x\,ds\,dt = -\int_{-\infty}^{\infty}\varphi'(t)\int_{-\infty}^{\infty}\psi(s)S_1(t+s)x\,ds\,dt. \tag{4.13}$$

Put $\varphi = I_\zeta(\rho_n)$, $n\geqslant k$ in (4.13). Then one obtains, for every φ, $\psi\in\mathcal{D}_{(-\tau/4,\tau/4)}$:

$$\int_{-\infty}^{\infty}\big[\rho_n(t)-\zeta(t)\big]\int_{-\infty}^{\infty}\psi'(s)S_1(t)S_1(s)x\,ds\,dt \\ = -\int_{-\infty}^{\infty}\big[\rho_n(t)-\zeta(t)\big]\int_{-\infty}^{\infty}\psi(s)S_1(t+s)x\,ds\,dt.$$

Letting $n\to\infty$ and applying the partial integration, one concludes that, for every $\psi\in\mathcal{D}_{(-\tau/4,\tau/4)}$:

$$\text{(4.14)}\quad -\int_{-\infty}^{\infty}\zeta(t)\int_{-\infty}^{\infty}\psi'(s)S_1(t)S_1(s)x\,ds\,dt \\ = \int_{-\infty}^{\infty}\psi'(s)\int_0^s S_1(v)x\,dv\,ds-\int_{-\infty}^{\infty}\zeta(t)\int_{-\infty}^{\infty}\psi'(s)\int_0^s S_1(t+v)x\,dv\,ds\,dt.$$

Plug $\psi=I_\eta(\rho_n)$, $n\geqslant k$ into (4.14). We get, for every $\psi\in\mathcal{D}_{(-\tau/4,\tau/4)}$:

$$-\int_{-\infty}^{\infty}\zeta(t)\int_{-\infty}^{\infty}\big[\rho_n(s)-\eta(s)\big]S_1(t)S_1(s)x\,ds\,dt \\ =\int_{-\infty}^{\infty}\big[\rho_n(s)-\eta(s)\big]\int_0^s S_1(v)x\,dv\,ds-\int_{-\infty}^{\infty}\zeta(t)\int_{-\infty}^{\infty}\big[\rho_n(s)-\eta(s)\big]\int_0^s S_1(t+v)x\,dv\,dt\,ds.$$

The standard limit procedure leads us to the next equality:

$$\text{(4.15)}\quad \int_{-\infty}^{\infty}\zeta(t)\,\eta(s)S_1(t)S_1(s)x\,ds\,dt \\ = -\int_{-\infty}^{\infty}\eta(s)\int_0^s S_1(v)x\,dv\,ds+\int_{-\infty}^{\infty}\zeta(t)\,\eta(s)\int_0^s S_1(t+v)x\,dv\,dt\,ds.$$

Let $t,\ s\in(-\tau/4,\tau/4)$ be fixed and let $(\zeta_n)_{n\in\mathbb{N}}$ and $(\eta_n)_{n\in\mathbb{N}}$ be sequences in $\mathcal{D}_{(-\tau/4,\tau/4)}$ satisfying $\int_{-\infty}^{\infty}\zeta_n(t)\,dt=1$, $\int_{-\infty}^{\infty}\eta_n(t)\,dt=1$, $n\in\mathbb{N}$, $\lim_{n\to\infty}\zeta_n=\delta_t$ and $\lim_{n\to\infty}\eta_n=\delta_s$, in the sense of distributions. By virtue of (4.15), we have:

$$\text{(4.16)}\qquad S_1(t)S_1(s)x=\Big[\int_0^{t+s}-\int_0^t-\int_0^s\Big]S_1(r)x\,dr.$$

Notice that (4.16) implies

$$S_1(t)\Big(\frac{d}{dr}S_1(r)x\Big)_{|r=s}=S_1(t+s)x-S_1(s)x,$$
$$S_1(t)[AS_1(s)x+Bx]=S_1(t+s)x-S_1(s)x.$$

Since $S_1(t)A \subseteq AS_1(t)$, $t \in (-\tau, \tau)$, one yields:

$$A\Big[\int\limits_0^{t+s} - \int\limits_0^{t} - \int\limits_0^{s}\Big] S_1(r)x\,dr + S_1(t)Bx = S_1(t+s)x - S_1(s)x, \text{ i.e.,}$$

$$S_1(t+s)x - (t+s)Bx - S_1(t)x + tBx - S_1(s)x + sBx + S_1(t)Bx$$
$$= S_1(t+s)x - S_1(s)x.$$

So, $S_1(v)[Bx - x] = 0$, $v \in (-\tau/4, \tau/4)$. Since $G(\varphi) = -\int_{-\infty}^{\infty} \varphi'(v)S_1(v)x\,dv$, $\varphi \in \mathcal{D}_{(-\tau/4,\tau/4)}$, one can easily conclude that $(S_1(t))_{t\in(-\tau/4,\tau/4)}$ is a non-degenerate operator family. Hence, $B = I$ and $(S_1(t))_{t\in[0,\tau)}$ is a once integrated semigroup generated by A. Analogously, $(-S(-t) - B_0)_{t\in[0,\tau)}$ is a once integrated semigroup generated by $-A$ and one can repeat literally the arguments given in the proof of Theorem 4.1(b2) in order to see that A generates a C_0-group. Suppose now $n_0 = 2$ and denote $A_1 = A_{\overline{\mathcal{R}(G)}}$. We will only prove that A_1 is the c.i.g of $(S_+^1(t))_{t\in[0,\tau)}$. Evidently, $AB_0 + B_1 \in L(E)$, $G(\varphi)x = \int_{-\infty}^{\infty} \varphi''(t)[S(t)x + B_0x + t(AB_0 + B_1)x]dt$, $\varphi \in \mathcal{D}_{(-\tau,\tau)}$, $x \in E$ and the mapping $t \mapsto S(t)x + B_0x + t(AB_0 + B_1)x$, $t \in [0, \tau)$ is continuously differentiable with

$$\frac{d}{dt}[S(t)x + B_0x + t(AB_0 + B_1)x] = AS(t)x - B_1x - 2tB_2x + (AB_0 + B_1)x,$$

$t \in [0, \tau)$, $x \in E$. Therefore,

$$G(\varphi)x = -\int\limits_{-\infty}^{\infty} \varphi'(t)S_+^1(t)x\,dt, \quad \varphi \in \mathcal{D}_{[0,\tau)},\ x \in E. \tag{4.17}$$

Suppose $x \in E$, $\varphi, \psi \in \mathcal{D}_{[0,\tau)}$ and $\operatorname{supp}\varphi + \operatorname{supp}\psi \subseteq [0, \tau)$. Since G satisfies $(DG)_1$ (see also (4.21)), we obtain

$$\int\limits_{-\infty}^{\infty} \varphi'(t) \int\limits_{-\infty}^{\infty} \psi'(s)S_+^1(t)S_+^1(s)x\,dt\,ds = -\int\limits_{-\infty}^{\infty} \varphi'(t) \int\limits_{-\infty}^{\infty} \psi(s)S_+^1(t+s)x\,dt\,ds.$$

Arguing as in the case $n_0 = 1$, one gets, for every $t, s \in [0, \tau)$ with $t + s < \tau$:

$$S_+^1(t)S_+^1(s)x = \Big[\int\limits_0^{t+s} - \int\limits_0^{t} - \int\limits_0^{s}\Big] S_+^1(r)x\,dr.$$

Further on, $S_+^1(0) = 0$ and the mapping $t \mapsto S_+^1(t)x$, $t \in [0, \tau)$ is continuous. It can be simply verified that $(S_+^1(t))_{t\in[0,\tau)}$ is a non-degenerate operator family, and consequently, $(S_+^1(t))_{t\in[0,\tau)}$ is a local once integrated semigroup in the sense of $(\triangle)$. Suppose $x \in D(A_0)$. Then there exists $\sigma \in (0, \tau]$ such that the mapping $t \mapsto S_+^1(t)x$, $t \in [0, \sigma)$ is continuously differentiable and that $A_0x = \lim_{t\to 0+} \big(\frac{d}{dt}(S_+^1(t)x) - x\big)/t$. The partial integration and (4.17) yield:

$$G(\varphi)x = \int\limits_{-\infty}^{\infty} \varphi(t)\Big(\frac{d}{dt}S_+^1(t)x\Big)t, \quad \varphi \in \mathcal{D}_{[0,\sigma)}. \tag{4.18}$$

Owing to (4.18) and Theorem 4.1(b3), we get $\lim_{n\to\infty} G(\rho_n)x = x \in \overline{\mathcal{R}(G)}$, $S^2_+(t)x = S(t)x + B_0x + t(AB_0 + B_1)x \in \overline{\mathcal{R}(G)}$, $t \in [0,\tau)$ and $\overline{\mathcal{R}(G)} \ni \frac{d}{dt}S^2_+(t)x = AS(t)x - B_1x - 2tB_2x + (AB_0+B_1)x = S^1_+(t)x$, $t \in [0,\tau)$. Consequently, $\frac{d}{dt}S^1_+(t)x \in \overline{\mathcal{R}(G)}$, $t \in [0,\sigma)$, $A_0x = \lim_{t\to 0+}\big(\frac{d}{dt}(S^1_+(t)x) - x\big)/t \in \overline{\mathcal{R}(G)}$ and:

$$\{x, A_0x\} \subseteq \overline{\mathcal{R}(G)}. \tag{4.19}$$

Further, $\frac{d}{dt}S(t)x = AS(t)x - B_1x - 2tB_2x$, $t \in [0,\tau)$,

$$\begin{aligned}\frac{d}{dt}S^1_+(t)x + 2B_2x &= \lim_{h\to 0}\frac{A[S(t+h)x - S(0)x] - A[S(t)x - S(0)x]}{h}\\ &= \lim_{h\to 0} A\frac{S(t+h)x - S(t)x}{h}, \quad t \in [0,\sigma)\end{aligned}$$

and

$$\lim_{h\to 0}\frac{S(t+h)x - S(t)x}{h} = AS(t)x - B_1x - 2tB_2x, \quad t \in [0,\tau).$$

The closedness of A gives $AS(t)x - B_1x - 2tB_2x \in D(A)$, $t \in [0,\sigma)$ and $A[AS(t)x - B_1x - 2tB_2x] = \frac{d}{dt}S^1_+(t)x + 2B_2x$, $t \in [0,\sigma)$. Put $t = 0$ in the previous equality to obtain $A(AB_0 + B_1)x = -x - 2B_2x$. Hence,

$$\begin{aligned}A_0x &= \lim_{t\to 0+}\frac{\frac{d}{dt}(S^1_+(t)x) - x}{t} = \lim_{t\to 0+}\frac{A[AS(t)x - B_1x - 2tB_2x] - 2B_2x - x}{t}\\ &= \lim_{t\to 0+}\frac{A[AS(t)x - B_1x - 2tB_2x] + A(AB_0 + B_1)x}{t}\\ &= \lim_{t\to 0+} A\frac{A[S(t)x - S(0)x] - 2tB_2x}{t}.\end{aligned}$$

On the other hand,

$$\lim_{t\to 0+}\frac{A[S(t)x - S(0)x] - 2tB_2x}{t} = \lim_{t\to 0+}\frac{S^1_+(t)x - S^1_+(0)x}{t} = \Big(\frac{d}{dt}S^1_+(t)x\Big)_{t=0} = x.$$

Therefore, $x \in D(A)$, $A_0x = Ax$, $A_0 \subseteq A$ and (4.19) enables one to see that $A_0 \subseteq A_1$ and that $\overline{A_0} \subseteq A_1$. Furthermore, Theorem 4.1(b3) shows that A_1 is the generator of a once integrated semigroup $\big(\frac{d}{dt}S^2_+(t)\big)_{t\in[0,\tau)} \subseteq L(\overline{\mathcal{R}(G)})$ in the sense of Definition 2.3. Accordingly, $\big(\frac{d}{dt}S^2_+(t)\big)_{t\in[0,\tau)} \subseteq L\big(\overline{\mathcal{R}(G)}\big)$ is a local once integrated semigroup in the sense of ($\triangle$) and it can be easily proved that the c.i.g of $\big(\frac{d}{dt}S^2_+(t)\big)_{t\in[0,\tau)}$ is A_1. But, the c.i.g of $\big(S^1_+(t)\big)_{t\in[0,\tau)}$ is an extension of the c.i.g of $\big(\frac{d}{dt}S^2_+(t)\big)_{t\in[0,\tau)}$. Hence, $A_1 \subseteq \overline{A_0}$ and $A_1 = \overline{A_0}$. Further on, it is straightforward to see that $\frac{d}{dt}S^1_+(t)x = AS(t)Ax - B_1Ax - 2tB_2Ax - (2B_2 - C_1)x$, $t \in [0,\tau)$, $x \in D(A)$. Due to [**40**, Lemma 4.3(b)], we obtain that $x = \big(\frac{d}{dt}S^1_+(t)x\big)_{t=0}$, $x \in D(A)$ and an immediate consequence of this equality and (4.18) is $\lim_{n\to\infty} G(\rho_n)x = x$, $x \in D(A)$. By Theorem 4.1(b3), we have $D(A) \subseteq \overline{\mathcal{R}(G)} \subseteq \bigcap_{i=0}^{2} D(Y_i)$ and $Y_2x = -x$, $x \in D(A)$. Suppose $x \in D(A)$. By Proposition 3.2(iv), $Ax \in D(Y_1)$, $C_1x + AY_1x = Y_1Ax$ and, because of that, $2B_2x + Y_1Ax = C_1x + Y_2x = C_1x - x$. Now an application of Proposition 3.2(i) shows that $Y_1Ax = -(2B_2x - C_1x) - x \in D(A)$ and that $AY_1Ax = -2B_2Ax - Ax$. In other words, $Ax \in \bigcap_{i=0}^{2} D(Y_i)$ and $Y_2Ax = -Ax$.

Let us prove (c). First of all, suppose $\lambda \in \mathbb{C}$ and $\lambda - A$ is surjective. Assume $x = (\lambda - A)y$, for some $y \in D(A)$. We obtain $E = \bigcap_{i=0}^{2} D(Y_i) \ni x$ and $Y_2x = Y_2(\lambda y - Ay) = -\lambda y + Ay = -x$. Proceeding as in the proof of (b3) of Theorem 4.1, one gets that

$$\begin{aligned} A\int_0^t S_+^2(s)x\,ds &= S(t)x + B_0x + tB_1x + t^2B_2x + tAB_0x + \frac{t^2}{2}A(AB_0x + B_1x) \\ &= S_+^2(t)x - \frac{t^2}{2}x, \quad x \in E,\ t \in [0,\tau). \end{aligned}$$

This implies $\int_0^t S_+^1(s)x\,ds \in D(A)$, $t \in [0,\tau)$ and $A\int_0^t S_+^1(s)x\,ds = S_+^1(t)x - tx$, $x \in E$, $t \in [0,\tau)$. Assume $x \in D(A)$. Due to Proposition 3.2(i), we get $S_+^1(t)x = (S(t) - S(0))Ax - t(2B_2x - C_1x) \in D(A)$, $t \in [0,\tau)$ and $AS_+^1(t)x = A(S(t) - S(0))Ax - 2tB_2Ax = S_+^1(t)Ax$, $t \in [0,\tau)$. Suppose now that A is densely defined. Since $D(A) \subseteq \overline{\mathcal{R}(G)}$, we automatically obtain that $\overline{\mathcal{R}(G)} = E$ and that A is the c.i.g of $(S_+^1(t))_{t\in[0,\tau)}$. Due to [**40**, Proposition 4.5], $(S_+^1(t))_{t\in[0,\tau)}$ is a local once integrated semigroup in the sense of Definition 2.3. To prove (i), suppose $x \in E$, $\varphi, \psi \in \mathcal{D}_{(-\tau,\tau)}$ and $\operatorname{supp}\varphi + \operatorname{supp}\psi \subseteq (-\tau,\tau)$. Note that:

$$\begin{aligned} G(\varphi)G(\psi)x &= \int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0)}(s)S(t)S(s)x\,ds\,dt \\ &= -\int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s)S(t)\int_0^s S(v)x\,dv\,ds\,dt. \end{aligned}$$

Repeating literally the arguments given in the proof of Proposition 3.3, one obtains (3.7) and the last equality implies:

$$\begin{aligned} G(\varphi)G(\psi)x = -\int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s)\Big[-\sum_{j=0}^{n_0-1}\int_0^s (t+s-r)^j \int_0^r C_jS(v)x\,dv\,dr \\ + \sum_{j=1}^{n_0} j\int_0^s (t+s-r)^{j-1}\int_0^r B_jS(v)x\,dv\,dr - \sum_{j=0}^{n_0}\int_0^s r^jS(t+s-r)B_jx\,dr\Big]ds\,dt. \end{aligned}$$

Noticing that $\int_{-\infty}^{\infty}\varphi^{(n)}(t)t^j\,dt = 0$, $n \in \mathbb{N}$, $j \in \mathbb{N}_0$, $n > j$, one can deduce that:

$$I_1 := \int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s) \sum_{j=0}^{n_0-1}\int_0^s (t+s-r)^j \int_0^r C_jS(v)x\,dv\,dr\,ds\,dt = 0.$$

As a matter of fact,

$$I_1 = \int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s)\times$$

$$\times \sum_{j=0}^{n_0-1} \int_0^s \sum_{\substack{(k_1,k_2,k_3)\in\mathbb{N}_0^3 \\ k_1+k_2+k_3=j}} \frac{j!}{k_1!k_2!k_3!} t^{k_1} s^{k_2} (-r)^{k_3} \int_0^r C_j S(v)x\,dv\,dr\,ds\,dt.$$

Suppose $j \in \{0,\ldots,n_0-1\}$, $(k_1,k_2,k_3)\in\mathbb{N}_0^3$ and $k_1+k_2+k_3=j$. One gets:

$$\int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s) \int_0^s \frac{j!}{k_1!k_2!k_3!} t^{k_1} s^{k_2} (-r)^{k_3} \int_0^r C_j S(v)x\,dv\,dr\,ds\,dt$$
$$= \int_{-\infty}^{\infty} \varphi^{(n_0)}(t) t^{k_1}\,dt \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s) \int_0^s \frac{j!}{k_1!k_2!k_3!} s^{k_2} (-r)^{k_3} \int_0^r C_j S(v)x\,dv\,dr\,ds = 0.$$

Hence, $I_1 = 0$. Analogically,

$$\int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s) \sum_{j=1}^{n_0} j \int_0^s (t+s-r)^{j-1} \int_0^r B_j S(v)x\,dv\,dr\,ds\,dt = 0$$

and we obtain:

$$G(\varphi)G(\psi)x = \int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s) \sum_{j=0}^{n_0} \int_0^s r^j S(t+s-r) B_j x\,dr\,ds\,dt$$

$$= \int_{-\infty}^{\infty} \varphi^{(n_0)}(t) \int_{-\infty}^{\infty} \psi^{(n_0+1)}(s) \sum_{j=0}^{n_0} \int_t^{t+s} (t+s-r)^j S(r) B_j x\,dr\,ds\,dt. \tag{4.20}$$

Put, for every $t \in (-\tau,\tau)$ and $j \in \{1,\ldots,n_0+1\}$:

$$g_{j,t}(s) := \int_t^{t+s} (t+s-r)^{j-1} S(r) B_j x\,dr, \quad s \in (-\tau-t, \tau-t).$$

It is straightforward to check that $\frac{d}{ds} g_{j,t}(s) = (j-1)\int_t^{t+s}(t+s-r)^{j-2} S(r)B_j x\,dr$, $j>1$, $s\in(-\tau-t,\tau-t)$ and that $\frac{d}{ds} g_{1,t}(s) = S(t+s)B_1 x$, $s \in (-\tau-t,\tau-t)$. The partial integration and (4.20) imply:

$$G(\varphi)G(\psi)x = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi^{(n_0)}(t)\psi^{(n_0)}(s) \sum_{j=1}^{n_0} \int_t^{t+s} j(t+s-r)^{j-1} S(r) B_j x\,dr\,ds\,dt$$
$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi^{(n_0)}(t)\psi^{(n_0)}(s) S(t+s) B_0 x\,ds\,dt.$$

Apply again the partial integration in order to see that:

$$G(\varphi)G(\psi)x = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi^{(n_0)}(t)\psi^{(n_0)}(s) S(t+s) B_0 x\,ds\,dt$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi^{(n_0)}(t)\psi^{(n_0-1)}(s)S(t+s)B_1x\,ds\,dt$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi^{(n_0)}(t)\psi^{(n_0-1)}(s)\sum_{j=2}^{n_0}j(j-1)\int_{t}^{t+s}(t+s-r)^{j-2}S(r)B_jx\,dr\,ds\,dt.$$

Continuing this procedure, we finally obtain (4.12).

To prove (ii), suppose φ, $\psi\in\mathcal{D}_{(-\tau,\tau)}$ and $\operatorname{supp}\varphi+\operatorname{supp}\psi\subseteq(-\tau,\tau)$. Evidently,

$$G(\varphi*\psi)x=(-1)^{n_0}\int_{-\infty}^{\infty}(\varphi*\psi)^{(n_0)}(t)S(t)x\,dt=(-1)^{n_0}\int_{-\infty}^{\infty}(\varphi^{(n_0)}*\psi)(t)S(t)x\,dt$$

$$\tag{4.21}=(-1)^{n_0}\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty}\varphi^{(n_0)}(t-s)\psi(s)\,ds\Big]S(t)x\,dt$$

$$=(-1)^{n_0}\int_{-\infty}^{\infty}\varphi^{(n_0)}(t)\int_{-\infty}^{\infty}\psi(s)S(t+s)x\,ds\,dt,\quad x\in E.$$

Due to (4.21), we have:

$$\tag{4.22} G(\psi)x=\lim_{n\to\infty}G(\psi*\rho_n)x=\lim_{n\to\infty}(-1)^{n_0}\int_{-\infty}^{\infty}\rho_n^{(n_0)}(t)\int_{-\infty}^{\infty}\psi(s)S(t+s)x\,ds\,dt,$$

$x\in E$. Combining $(DG)_1$, (4.12) and (4.22) gives:

$$\tag{4.23}(-1)^{n_0}G(\psi)x=\sum_{i=0}^{n_0}(-1)^{i+1}i!G(\psi^{(n_0-i)})B_ix,\quad x\in E.$$

Suppose now $x\in\bigcap_{i=0}^{n_0}D(Y_i)$. A consequence of the definition of Y_{n_0} and (1.2) is $n_0!G(\psi)B_{n_0}x+AG(\psi)Y_{n_0-1}x=n_0!G(\psi)B_{n_0}x-G(\psi')Y_{n_0-1}x=G(\psi)Y_{n_0}x$. If $n_0\geqslant 2$, then we obtain $n_0!G(\psi)B_{n_0}x-G(\psi')(AY_{n_0-2}x+(n_0-1)!B_{n_0-1}x)=G(\psi)Y_{n_0}x$ and $n_0!G(\psi)B_{n_0}x-(n_0-1)!G(\psi')B_{n_0-1}x+G(\psi'')Y_{n_0-2}x=G(\psi)Y_{n_0}x$. By the definition of Y_i and (1.2), one concludes inductively:

$$\tag{4.24}\sum_{i=0}^{n_0}(-1)^{n_0+i}i!G(\psi^{(n_0-i)})B_ix=G(\psi)Y_{n_0}x.$$

This equality and (4.23) imply $G(\psi)(Y_{n_0}x+x)=0$; a simple consequence is $G(\eta)(Y_{n_0}x+x)=0$, $\eta\in\mathcal{D}$ and the proof of (ii) finishes an application of $(DG)_2$.

To prove (iii), one can argue as in the proof of (b5) of Theorem 4.1. We sketch the proof for the sake of completeness. Fix an $x\in D(A^{n_0-1})$. Since $S:(-\tau,\tau)\to L(E,[D(A)])$ is continuous, one can argue as in the proof of Proposition 3.2 in order to see that the mapping $t\mapsto S(t)x$, $t\in(-\tau,\tau)$ is n_0-times continuously differentiable and that there exists a function $M:(-\tau,\tau)\to(0,\infty)$ satisfying $\big\|\frac{d^{n_0}}{dt^{n_0}}S(t)x\big\|\leqslant M(t)\|x\|_{n_0-1}$, $t\in(-\tau,\tau)$. Furthermore, (3.3) holds for every $l \in \{0, \dots, n_0 - 1\}$ and one obtains inductively $Y_k x = -\big(\frac{d^k}{dt^k} S(t)x\big)_{t=0}$, $k \in \{0, \dots, n\}$. Denote $u(t;x) = \frac{d^{n_0}}{dt^{n_0}} S(t)x$, $t \in (-\tau, \tau)$; then the partial integration shows $G(\varphi)x = \int_{-\infty}^{\infty} \varphi(t)\, u(t;x)x\, dt$, $\varphi \in \mathcal{D}_{(-\tau,\tau)}$. The previous equality and (ii) imply $\lim\limits_{n\to\infty} G(\rho_n)x = u(0;x) = -Y_{n_0}x = x \in \overline{\mathcal{R}(G)}$. Therefore, $D(A^{n_0-1}) \subseteq \overline{\mathcal{R}(G)}$. Further on, Proposition 3.2(iv) implies

$$C_{n_0-1}x + \frac{1}{(n_0-1)!} AY_{n_0-1}x = \frac{1}{(n_0-1)!} Y_{n_0-1}(Ax), \quad \text{i.e.,}$$
$$C_{n_0-1}x + \frac{1}{(n_0-1)!}[-x - n_0! B_{n_0}x] = \frac{1}{(n_0-1)!} Y_{n_0-1}(Ax).$$

Due to Proposition 3.2(i), $Y_{n_0-1}(Ax) \in D(A)$ and a simple computation gives $Y_{n_0}Ax = -Ax$ which finishes the proof of (iii).

Further on, let us observe that (iii) implies $D(A^n) \subseteq D(A^{n_0-1}) \subseteq \overline{\mathcal{R}(G)} \subseteq \overline{D_\infty(A)} \subseteq \overline{D(A^{n+1})}$, for every $n \in \mathbb{N}$ such that $n \geqslant n_0 - 1$. Hence, A is stationary dense and $n(A) \leqslant n_0 - 1$.

To prove (v), suppose $\lambda \in \rho(A)$. We will prove that A generates a local (n_0-1)-times integrated group on $(-\tau, \tau)$. Repeating literally the arguments given in the proof of Theorem 4.1, one gets $A\int_0^t u(s;x)\, ds = u(t;x) - x$, $t \in (-\tau,\tau)$, $x \in D(A^{n_0-1})$ and $Au(t;x) = u(t;Ax)$, $t \in (-\tau,\tau)$, $x \in D(A^{n_0})$. Put $S^{n_0-1}(t)x = u(t; R(\lambda : A)^{n_0-1}x)$, $t \in [0,\tau)$, $x \in E$. Clearly, the mapping $t \mapsto S^{n_0-1}(t)x$, $t \in [0,\tau)$ is continuous for every $x \in E$ and an induction argument shows that, for every $k \in \mathbb{N}_0$, there exists an appropriate constant $M(k,\lambda) \in (0,\infty)$ which fulfills $\|A^k R(\lambda : A)^k x\| \leqslant M(k,\lambda)\|x\|$, $x \in E$. This implies

$$\begin{aligned}\|S^{n_0-1}(t)x\| &= \|u(t; R(\lambda : A)^{n_0-1}x)\| \leqslant M(t)\|R(\lambda : A)^{n_0-1}x\|_{n_0-1}\\ &\leqslant M(t)\sum_{i=0}^{n_0-1} M(i,\lambda)\|R(\lambda : A)\|^{n_0-1-i}\|x\|,\end{aligned}$$

$x \in E$ and $S^{n_0-1}(t) \in L(E)$, $t \in [0,\tau)$. In order to simplify the notation, denote $C = R(\lambda : A)^{n_0-1}$. We have $A\int_0^t S^{n_0-1}(s)x\, ds = A\int_0^t u(s;Cx)\, ds = u(t;Cx) - Cx = S^{n_0-1}(t)x - Cx$, $t \in [0,\tau)$, $x \in E$. Since $Au(t;x) = u(t;Ax)$, $t \in (-\tau,\tau)$, $x \in D(A^{n_0})$, one easily obtains $S^{n_0-1}(t)A \subseteq AS^{n_0-1}(t)$, $S^{n_0-1}(t)R(\lambda : A) = R(\lambda : A)S^{n_0-1}(t)$ and, by induction, $S^{n_0-1}(t)C = CS^{n_0-1}(t)$, $t \in [0,\tau)$. Now it is straightforward to prove that the abstract Cauchy problem:

$$\begin{cases} v \in C\big([0,\tau) : [D(A)]\big) \cap C^1([0,\tau) : E),\\ v'(t) = Av(t) + Cx,\ t \in [0,\tau),\\ v(0) = 0,\end{cases}$$

has a unique solution for every $x \in E$, given by $v(t) = \int_0^t S^{n_0-1}(s)x\, ds$, $t \in [0,\tau)$, $x \in E$. By [**44**, Theorem 4.4], A generates a local (n_0-1)-times integrated semigroup on $[0,\tau)$. Since $-A$ generates a (DG) $\check{G}$, we also obtain that $-A$ generates a local (n_0-1)-times integrated semigroup on $[0,\tau)$ and Lemma 2.2(v) implies that A generates a local (n_0-1)-times integrated group on $(-\tau,\tau)$. Thus, (v) is a consequence of Corollary 2.1(c).

To prove (vi), notice that the assumption $\overline{\mathcal{R}(G)} = E$ and $\mathcal{R}(G) \subseteq D_\infty(A)$ imply that $D_\infty(A)$ is dense in E. The converse statement is obvious since $\overline{D_\infty(A)} \subseteq \overline{D(A^{n_0-1})} \subseteq \overline{\mathcal{R}(G)}$ (cf. the proofs of (iii) and (iv)). In the case $\rho(A) \neq \emptyset$, the denseness of $\overline{D_\infty(A)}$ in E is equivalent to the denseness of $D(A)$ in E (see, for example, [**27**]) and the proof of (vi) completes a routine argument.

It remains to be proved (vii). Suppose $G(\varphi)x = 0$, $\varphi \in \mathcal{D}_0$. This implies $(-1)^{n_0}\int_{-\infty}^{\infty}\varphi^{n_0}(t)S(t)x\,dt = 0$, $\varphi \in \mathcal{D}_{[0,\tau)}$ and the existence of bounded linear operators $D_0, \dots, D_{n_0-1} \in L(E)$ satisfying $S(t)x = \sum_{j=0}^{n_0-1} t^j D_j x$, $t \in [0,\tau)$. Hence,

$$A\sum_{j=0}^{n_0-1}\frac{t^{j+1}}{j+1}D_jx = \sum_{j=0}^{n_0-1}t^jD_jx + \sum_{j=0}^{n_0}t^jB_jx, \quad t \in [0,\tau). \tag{4.25}$$

Substitute $t = 0$ in (4.25) to obtain $D_0 = -B_0$. Differentiating (4.25), it is straightforward to see that: $x \in \bigcap_{i=0}^{n_0} D(Y_i)$, $\bigcup_{i=0}^{n_0-1}\{D_ix\} \subseteq D(A)$, $D_ix = \frac{(-1)}{i!}Y_ix$, $i = 1, \dots, n_0-1$ and $A(D_{n_0-1}x) = n_0B_{n_0}x$. This implies $\big(n_0B_{n_0} + \frac{1}{(n_0-1)!}AY_{n_0-1}\big)x = 0$, i.e., $x \in N(Y_{n_0})$. Due to (ii), $x = 0$ and $\bigcap_{\varphi\in\mathcal{D}_0} N(G(\varphi)) = \{0\}$. The second equality in (vii) follows by passing to $-A$ and $\check{G}$. $\square$

Example 4.1. Let $E := L^\infty(\mathbb{R})$ and let $A := \frac{d}{dt}$ with maximal domain. Then A is not densely defined and generates a once integrated group $(S_1(t))_{t\in\mathbb{R}}$ given by $(S_1(t)f)(s) := \int_0^t f(r+s)\,dr$, $s \in \mathbb{R}$, $t \in \mathbb{R}$ (cf. also [**12**, Example 4.1]). Put $S_2(t)f := \int_0^t S_1(s)f\,ds$, $t \geqslant 0$, $f \in E$, $S_2(t)f := \int_0^{-t} S_1(-s)f\,ds$, $t < 0$, $f \in E$ and $G(\varphi)f := \int_{-\infty}^{\infty}\varphi''(t)S_2(t)f\,dt$, $\varphi \in \mathcal{D}$, $f \in E$. Then $(S_2(t))_{t\in\mathbb{R}}$ is a twice integrated group generated by A, the mapping $S_2 : \mathbb{R} \to L(E, [D(A)])$ is continuous and G is a non-dense (DG)vgenerated by A (cf. Theorem 4.2 with $n_0 = 2$). We would like to point out that there exists $f \in D(A)$ such that $Af \notin \overline{\mathcal{R}(G)}$. Suppose contrarily that $R(A) \subseteq \overline{\mathcal{R}(G)}$. By Theorem 6.2, $D(A) \subseteq \overline{\mathcal{R}(G)}$ and we obtain $(\lambda - A)f \in \overline{\mathcal{R}(G)}$, $\lambda \in \mathbb{C}$, $f \in D(A)$. Since $\mathbb{C} \smallsetminus i\mathbb{R} \subseteq \rho(A)$, one yields $E = \overline{\mathcal{R}(G)}$ and the contradiction is obvious. Hence, Theorem 4.2 implies that $(S_1(t))_{t\geqslant 0}$ is a once integrated semigroup generated by A in the sense of Definition 2.3 and that the c.i.g of $(S_1(t))_{t\in\mathbb{R}}$ is $A_{\overline{\mathcal{R}(G)}}$ $(\neq A)$. Furthermore, $\overline{\mathcal{R}(G)} \subsetneq \bigcap_{i=0}^{2} D(Y_i) = E$.

Proposition 4.1. *Suppose G_1 and G_2 are distribution groups generated by A and $\rho(A) \neq \emptyset$. Then $G_1 = G_2$.*

Proof. Suppose $x \in E$, $\lambda \in \rho(A)$ and $\varphi \in \mathcal{D}_{(-\tau,\tau)}$, for some $\tau \in (0,\infty)$. We will prove that $G_1(\varphi)x = G_2(\varphi)x$. Clearly, $G_i \in \mathcal{D}'\big(L(E,[D(A)])\big)$, $i = 1,2$ and an application of [**32**, Theorem 2.1.1] gives that there exist $n_1 \in \mathbb{N}$, $n_2 \in \mathbb{N}$ and continuous mappings $S_i : (-\tau,\tau) \to L\big(E,[D(A)]\big)$, $i = 1,2$ so that $G_i(\psi)x = (-1)^{n_i}\int_{-\infty}^{\infty}\psi^{(n_i)}(t)S_i(t)x\,dt$, $\psi \in \mathcal{D}_{(-\tau,\tau)}$, $x \in E$, $i = 1,2$. The proof of Theorem 4.1 shows that there are bounded linear operators

$$B_0, \dots, B_{n_1}, \overline{B_0}, \dots, \overline{B_{n_2}}, C_0, \dots, C_{n_1-1}, \overline{C_0}, \dots, \overline{C_{n_2-1}}$$

such that $(S_1(t))_{t\in(-\tau,\tau)}$, resp., $(S_2(t))_{t\in(-\tau,\tau)}$ is a $[B_0,\dots,B_{n_1},C_0,\dots,C_{n_1-1}]$-group, resp., $[\overline{B_0},\dots,\overline{B_{n_2}},\overline{C_0},\dots,\overline{C_{n_2-1}}]$-group with a subgenerator A. Without loss of generality, we may assume $n_1=n_2$. Further on, the proof of Theorem 4.2 implies that $\big(\frac{d^{n_1}}{dt^{n_1}}S_i(t)R(\lambda:A)^{n_1-1}\big)_{t\in[0,\tau)}$, $i=1,2$ are local $R(\lambda:A)^{n_1-1}$-semigroups generated by A. Hence, there exist $x_0,\dots,x_{n_1-1}\in E$ which satisfies $S_1(t)R(\lambda:A)^{n_1-1}x-S_2(t)R(\lambda:A)^{n_1-1}x=\sum_{i=0}^{n_1-1}t^ix_i$, $t\in[0,\tau)$. An immediate consequence is:

$$\begin{aligned}R(\lambda:A)^{n_1-1}G_1(\varphi)x&-R(\lambda:A)^{n_1-1}G_2(\varphi)x\\&=G_1(\varphi)R(\lambda:A)^{n_1-1}x-G_2(\varphi)R(\lambda:A)^{n_1-1}x\\&=(-1)^{n_1}\int_{-\infty}^{\infty}\varphi^{(n_1)}(t)\sum_{i=0}^{n_1-1}t^ix_i\,dt=0,\end{aligned}$$

which clearly implies $G_1(\varphi)x=G_2(\varphi)x$. □

REMARK 4.1. (i) Suppose A generates a (DG) G and $\rho(A)\neq\emptyset$. Then there exist $a>0$ and $b>0$ such that $E(a,b)\subseteq\rho(\pm A)$ and that the next representation formula holds for G:

$$G(\varphi)x=\frac{1}{2\pi i}\int_{\Gamma}\int_{-\infty}^{\infty}\varphi(t)[e^{\lambda t}R(\lambda:A)x+e^{-\lambda t}R(\lambda:-A)x]dtd\lambda,$$

$x\in E$, $\varphi\in\mathcal{D}$, where we assume that the curve $\Gamma=\partial E(a,b)$ is oriented upwards.

(ii) Suppose $G\in\mathcal{D}'(L(E))$ is regular, A is a closed linear operator so that (1.2) holds and there are no non-trivial solutions of the abstract Cauchy problem:

$$(ACP_1):\begin{cases}u\in C\big(\mathbb{R}:[D(A)]\big)\cap C^1(\mathbb{R}:E),\\u'(t)=Au(t),\ t\in\mathbb{R},\\u(0)=x,\end{cases}$$

when $x=0$ (cf. Theorem 4.1). Then $G(\varphi*\psi)x=G(\varphi)\,G(\psi)x$, $x\in\overline{\mathcal{R}(G)}$, $\varphi,\psi\in\mathcal{D}$. To show this, let us point out that $G(\varphi*\psi)x=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(t)\,\psi(s)\,u(t+s;x)\,ds\,dt$ and $G(\psi)G(\varphi)x=\int_{-\infty}^{\infty}\varphi(t)G(\psi)u(t;x)\,dt$, $x\in\mathcal{R}(G)$, $\varphi,\psi\in\mathcal{D}$. Since $G(\cdot)\in L(E)$, the consideration is over if we prove that $G(\psi)u(t;x)=\int_{-\infty}^{\infty}\psi(s)u(t+s;x)\,ds$, $\psi\in\mathcal{D}$, $x\in\mathcal{R}(G)$, $t\in\mathbb{R}$. Put, for fixed $\psi\in\mathcal{D}$ and $x\in\mathcal{R}(G)$, $f(t):=G(\psi)u(t;x)-\int_{-\infty}^{\infty}\psi(s)u(t+s;x)\,ds$, $t\in\mathbb{R}$. Then

$$\begin{aligned}A\int_0^t f(s)\,ds&=G(\psi)[u(t;x)-x]-\int_{-\infty}^{\infty}\psi(s)A\int_s^{t+s}u(r;x)\,dr\,ds\\&=G(\psi)[u(t;x)-x]-\int_{-\infty}^{\infty}\psi(s)[u(t+s;x)-u(s;x)]\,ds=f(t),\quad t\in\mathbb{R}.\end{aligned}$$

So, the function $u(t)=\int_0^t f(s)ds$, $t\in\mathbb{R}$ solves (ACP_1) and $u(0)=0$. This proves $f\equiv 0$.

(iii) Suppose $G \in \mathcal{D}'(L(E))$ is regular, (1.2) holds for A and G, $\tau \in (0,\infty)$ and $\rho(A) \neq \emptyset$. Set $G_1 := G_{|\overline{\mathcal{R}(G)}}$. Then G_1 is a dense (DG) in $\overline{\mathcal{R}(G)}$ generated by A_1. To this end, we employ the same terminology as in the proof of Theorem 4.1(b5); without loss of generality, we can assume that $0 \in \rho(A)$ so that $A_\lambda = A$, $u_\lambda = u$ and $G_\lambda = G$. Suppose (ρ_n) is a regularizing sequence. Choose an arbitrary $\tau \in (0,\infty)$ and notice that

$$\begin{aligned} C^2G(\varphi*\psi)x = CG(\varphi*\psi)Cx &= C\int_{-\infty}^{\infty}(\varphi*\psi)(t)\,u(t;Cx)\,dt \\ &= C\int_{-\infty}^{\infty}(\varphi*\psi)(t)\,T(t)x\,dt = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(t)\,\psi(s)\,T(t+s)x\,ds\,dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(t)\,\psi(s)\,T(t)\,T(s)x\,ds\,dt = G(\varphi)CG(\psi)Cx = C^2G(\varphi)\,G(\psi)x, \end{aligned}$$

for every $x \in \overline{\mathcal{R}(G)}$ and $\varphi, \psi \in \mathcal{D}_{[0,\tau)}$ with $\operatorname{supp}\varphi + \operatorname{supp}\psi \subseteq [0,\tau)$. The injectiveness of C combining with the argumentation used in the proof of Theorem 4.1(b1) enables one to deduce that $G(\varphi*\psi)x = G(\varphi)G(\psi)x$, $\varphi, \psi \in \mathcal{D}$, $x \in \overline{\mathcal{R}(G)}$ and that $G_1 \in \mathcal{D}'(L(\overline{\mathcal{R}(G)}))$ satisfies $(DG)_1$. The assumption $G_1(\varphi)x = 0$, $\varphi \in \mathcal{D}$ implies $G_1(\varphi)Cx = \int_{-\infty}^{\infty}\varphi(t)u(t;Cx)\,dt = \int_{-\infty}^{\infty}\varphi(t)T(t)x\,dt = 0$, for every $\varphi \in \mathcal{D}_{[0,\tau)}$ and $Cx = T(0)x = \lim_{n\to\infty} G_1(\rho_n)Cx = 0$. So, $x = 0$ and G_1 is a (DG) in $\overline{\mathcal{R}(G)}$. It can be easily seen that G_1 is generated by A_1.

(iv) Suppose G and A are as in (ii) and let $\lambda \in \rho(A)$. Then the $[B_0,\ldots,B_{n_0}, C_0,\ldots,C_{n_0-1}]$-group $(S(t))_{t\in(-\tau,\tau)}$, constructed in Theorem 4.1(a), satisfies (4.1), (4.2) as well as $Y_{n_0}x = -x$, $x \in \overline{\mathcal{R}(G)}$. Namely, (ii) shows that $G_1 = G_{|\overline{\mathcal{R}(G)}}$ is a (DG) in $\overline{\mathcal{R}(G)}$ generated by A_1. The proof of Theorem 4.1(b5) implies that A_1 generates a local n_0-times integrated group $(S_{n_0}(t))_{t\in(-\tau,\tau)}$ in $L(\overline{\mathcal{R}(G)})$ and it is not difficult to show that $G_1(\varphi)x = (-1)^{n_0}\int_0^\infty \varphi^{(n_0)}(t)S_{n_0}(t)x\,dt + \int_{-\infty}^0 \varphi^{(n_0)}(t)S_{n_0}(t)x\,dt$, $\varphi \in \mathcal{D}_{(-\tau,\tau)}$, $x \in \overline{\mathcal{R}(G)}$. Hence, $\int_0^\infty \varphi^{(n_0)}(t)S_{n_0}(t)x\,dt = \int_0^\infty \varphi^{(n_0)}(t)S(t)x\,dt$, $\varphi \in \mathcal{D}_{[0,\tau)}$, $x \in \overline{\mathcal{R}(G)}$ and an application of [**17**, Theorem 8.1.1] gives the existence of operators $D_i \in L(\overline{\mathcal{R}(G)}, E)$, $i = 0,\ldots,n_0-1$ satisfying $S(t)x = S_{n_0}(t)x + \sum_{i=0}^{n_0-1} t^iD_ix$, $t \in [0,\tau)$, $x \in \overline{\mathcal{R}(G)}$. Since $A\int_0^t S(s)x\,ds = S(t)x + \sum_{i=0}^{n_0} t^iB_ix$, $t \in [0,\tau)$, $x \in E$ one obtains

$$A\sum_{i=0}^{n_0-1}\frac{t^{i+1}}{i+1}D_ix = \sum_{i=0}^{n_0-1} t^i(D_ix + B_ix) + t^{n_0}B_{n_0}x + \frac{t^{n_0}}{n_0!}x, \quad t \in [0,\tau),\ x \in \overline{\mathcal{R}(G)}.$$

This implies $\overline{\mathcal{R}(G)} \subseteq D(Y_{n_0})$, $D_ix = \frac{(-1)}{i!}Y_ix$, $i = 1,\ldots,n_0-1$, (4.1), (4.2) and $Y_{n_0}x = -x$, $x \in \overline{\mathcal{R}(G)}$.

(iv) Suppose G is a (DG) and $\varphi \in \mathcal{D}$. Then $G(\varphi) = G(\varphi_+) + G(\varphi_-)$ if and only if $\{G(\varphi_+),\, G(\varphi_-)\} \subseteq L(E)$ if and only if $G(\varphi_+) \in L(E)$. Namely, the assumption

$G(\varphi) = G(\varphi_+) + G(\varphi_-)$ immediately implies $D(G(\varphi_+)) = D(G(\varphi_-)) = E$ and the Closed Graph Theorem gives $G(\varphi_+) \in L(E)$ and $G(\varphi_-) \in L(E)$. Clearly, $\{G(\varphi_+), G(\varphi_-)\} \subseteq L(E)$ implies $G(\varphi_+) \in L(E)$. Suppose now $G(\varphi_+) \in L(E)$. We will show that $G(\varphi_-) \in L(E)$ and that $G(\varphi_-) = G(\varphi) - G(\varphi_+)$. Fix an $x \in E$ and notice that $G(\varphi * \psi)x = G(\psi)G(\varphi)x$, $\psi \in \mathcal{D}$ implies $G(\varphi_+ * \psi)x + G(\varphi_- * \psi)x = G(\psi)G(\varphi)x$, $\psi \in \mathcal{D}$. Since $x \in D(G(\varphi_+))$, we obtain $G(\varphi_- * \psi)x = G(\psi)[G(\varphi)x - G(\varphi_+)x]$. So, $x \in D(G(\varphi_-))$ and $G(\varphi_-)x = G(\varphi)x - G(\varphi_+)x$.

Proposition 4.2. *Suppose G is a (DG) and $G(\varphi_+) \in L(E)$, $\varphi \in \mathcal{D}$. Put $G_+(\varphi) := G(\varphi_+)$ and $G_-(\varphi) := G((\check{\varphi})_-)$, $\varphi \in \mathcal{D}$. Then $\pm A$ are generators of distribution semigroups $G_\pm$.*

Proof. Owing to the previous remark, we have $G_\pm(\varphi) \in L(E)$, $\varphi \in \mathcal{D}$ and $G(\varphi) = G_+(\varphi) + G_-(\check{\varphi})$, $\varphi \in \mathcal{D}$. Evidently, $\operatorname{supp} G_+ \cup \operatorname{supp} G_- \subseteq [0,\infty)$, $G_\pm \in \mathcal{D}'_0(L(E))$ and Theorem 4.2(vii) implies $\bigcap_{\varphi \in \mathcal{D}_0} N(G_\pm(\varphi)) = \{0\}$. Since $(\varphi *_0 \psi)_+ = \varphi_+ * \psi_+$, $\varphi, \psi \in \mathcal{D}$, Proposition 2.2 yields that G_+ is a pre-(DSG). Analogously, G_- is a pre-(DSG) and one obtains that G_+, resp., G_- is a (DSG). Designate by A_+, resp., A_-, the generator of G_+, resp., G_-. Then it is straightforward to verify that $A_\pm$ are extensions of $\pm A$. The proof is completed if one shows:

$$A_+ \subseteq A \ \text{ and } \ A_- \subseteq -A.$$

We will first prove that $A_+ = -A_-$. To see this, suppose $x \in E$ and φ, $\psi \in \mathcal{D}$. Then one obtains: $G(\varphi_+ * \psi)x = G(\psi)G(\varphi_+)x$, $G_+(\varphi_+ * \psi)x + G_-\big((\varphi_+ * \psi)\check{}\big)x = (G_+(\psi) + G_-(\check{\psi}))\,G_+(\varphi)x$ and:

$$G_+(\varphi_+ * \psi)x + G_-\big((\varphi_+ * \psi)\check{}\big)x = G_+(\varphi *_0 \psi)x + G_-(\check{\psi})\,G_+(\varphi)x. \tag{4.26}$$

Further, notice that $(\psi * \varphi_+ - \varphi * \psi_- - \varphi *_0 \psi)(t) = 0$, $t \geqslant 0$. The last equality and (4.26) give $G_+(\psi * \varphi_+)x = G_+(\varphi * \psi_-)x + G_+(\varphi *_0 \psi)x$ and:

$$G_+(\varphi * (\check{\psi})_-)x + G_-\big((\varphi_+ * \check{\psi})\check{}\big)x = G_-(\psi)\,G_+(\varphi)x. \tag{4.27}$$

Suppose now $(x,y) \in D(A_+)$, $a > 0$, $\psi \in \mathcal{D}_{(a,\infty)}$ and (ρ_n) is a regularizing sequence satisfying $\operatorname{supp}\rho_n \subseteq [0,\frac{1}{n}]$, $n \in \mathbb{N}$. Since $G_+(-\varphi')x = G_+(\varphi)y$, $\varphi \in \mathcal{D}_0$, (4.27) enables one to establish the following equalities:

$$\begin{aligned} G_+(\rho_n * (\check{\psi})_-)y + G_-\big((\rho_n * \check{\psi})\check{}\big)y &= G_-(\psi)\,G_+(\rho_n)y \\ &= G_-(\psi)\,G_+(-\rho_n')x = G_+(-\rho_n' * (\check{\psi})_-)x + G_-\big((-\rho_n' * \check{\psi})\check{}\big)x. \end{aligned} \tag{4.28}$$

Clearly, $\operatorname{supp}(\rho_n * (\check{\psi})_-) \cup \operatorname{supp}(-\rho_n' * (\check{\psi})_-) \subseteq [0,\frac{1}{n}] + (-\infty,-a) \subseteq (-\infty,0]$, $n \geqslant \frac{1}{a}$ and an application of (4.28) gives:

$$G_-\big((\rho_n * \check{\psi})\check{}\big)y = G_-\big((-\rho_n * (\check{\psi})')\check{}\big)x. \tag{4.29}$$

Letting $n \to \infty$ in (4.29), one concludes that $G_-(\psi)y = -G_-\big(((\check{\psi})')\check{}\big)x = G_-(\psi')x$ and, as a matter of routine, one can see that the previous equalities remain true for every $\psi \in \mathcal{D}_0$. In conclusion, one gets $(x,-y) \in A_-$ and $A_+ \subseteq -A_-$; analogously,

$A_+ \supset -A_-$ and this finally gives $A_+ = -A_-$. Taking into account [**28**, Lemma 3.6], one can deduce the following:

$$\begin{aligned} G(\varphi)A_+x &= G_+(\varphi)A_+x + G_-(\check{\varphi})A_+x = G_+(-\varphi')x - \varphi(0)x - A_-G_-(\check{\varphi})x \\ &= G_+(-\varphi')x - \varphi(0)x - (G_-(-\check{\varphi}')x - \check{\varphi}(0)x) \\ &= G_+(-\varphi')x + G_-(-\check{\varphi'})x = G(-\varphi')x, \quad \varphi \in \mathcal{D}. \end{aligned}$$

Hence, $(x, A_+x) \in A$, $A_+ \subseteq A$ and $A_+ = A$. □

REMARK 4.2. Suppose G is a (DG) generated by A and $\rho(A) \neq \emptyset$. Due to Lemma 2.2 and Theorem 4.2, we have that A, resp., $-A$, is the generator of a (DSG) G_1, resp., G_2. Obviously, $G(\varphi) = G_1(\varphi)+G_2(\check{\varphi})$, $\varphi \in \mathcal{D}$ and $G_1(\varphi)\,G_2(\psi) = G_2(\psi)\,G_1(\varphi)$, $\varphi, \psi \in \mathcal{D}$. Let $x \in E$ and $\varphi \in \mathcal{D}$ be fixed. We will prove that $G_+(\varphi) = G(\varphi_+) = G_1(\varphi)$. To this end, it is enough to show $G(\psi * \varphi_+)x = G(\psi)\,G_1(\varphi)x$, $\psi \in \mathcal{D}$, i.e.,

$$G_1(\psi * \varphi_+)x + G_2\big((\psi * \varphi_+)\check{}\,\big)x = G_1(\varphi)\,G_2(\check{\psi})x + G_1(\varphi)\,G_1(\psi), \quad \psi \in \mathcal{D}. \tag{4.30}$$

Notice that the proof of [**33**, Theorem 6] (see [**33**, (9), p. 61]) enables one to see that $G_1(\varphi * \psi_-)x + G_2\big((\varphi_+ * \psi)\check{}\,\big)x = G_1(\varphi)\,G_2(\check{\psi})x$, $\psi \in \mathcal{D}$. As in the proof of Proposition 4.2, one has $(\psi * \varphi_+ - \varphi * \psi_- - \varphi *_0 \psi)(t) = 0$, $t \geqslant 0$, $\psi \in \mathcal{D}$, which gives $G_1(\psi * \varphi_+)x = G_1(\varphi * \psi_-)x + G_1(\varphi *_0 \psi)x = G_1(\varphi * \psi_-)x + G_1(\varphi)\,G_1(\psi)x$, $\psi \in \mathcal{D}$. Hence,

$$\begin{aligned} G_1(\psi * \varphi_+)x + G_2\big((\psi * \varphi_+)\check{}\,\big)x &= G_1(\varphi * \psi_-)x + G_1(\varphi)\,G_1(\psi)x + G_2\big((\psi * \varphi_+)\check{}\,\big)x \\ &= G_1(\varphi * \psi_-)x + G_1(\varphi)\,G_1(\psi)x + G_1(\varphi)\,G_2(\check{\psi})x - G_1(\varphi * \psi_-)x \\ &= G_1(\varphi)\,G_2(\check{\psi})x + G_1(\varphi)\,G_1(\psi)x, \quad \psi \in \mathcal{D} \end{aligned}$$

and this proves (4.30). Accordingly, A is the generator of $G_+ = G_1$ and the previous remark implies that $G(\varphi_-) = G_2(\check{\varphi}) \in L(E)$, $\varphi \in \mathcal{D}$ and that $G((\check{\cdot})_-)$ is a (DSG) generated by $-A$.

THEOREM 4.3. *Suppose $B_0, \dots, B_n, C_0, \dots, C_{n-1} \in L(E)$ and A is a subgenerator of a $[B_0, \dots, B_n, C_0, \dots, C_{n-1}]$-group $(S(t))_{t\in\mathbb{R}}$. Set*

$$G(\varphi)x := (-1)^n \int_{-\infty}^{\infty} \varphi^{(n)}(t)S(t)x\,dt, \quad \varphi \in \mathcal{D},\ x \in E.$$

Then:

(a) *(1.2) holds and (4.12) holds for every $\varphi \in \mathcal{D}$ and $\psi \in \mathcal{D}$.*
(b) *$\mathcal{N}(G) \subseteq N(Y_n)$ and, in particular, the injectiveness of Y_n implies $(DG)_2$ for G.*
(c) *For every $\varphi \in \mathcal{D}$ and $\psi \in \mathcal{D}$, $N(Y_n) \subseteq N(G(\varphi)G(\psi))$; especially, if G is regular, then $N(Y_n) = \mathcal{N}(G)$.*
(d) *Assume $B_0 = \cdots = B_{n-1} = 0$ and $B_n = -\frac{1}{n!}I$. Then G is a (DG) generated by A.*

PROOF. To prove (a), notice that $G \in \mathcal{D}'(L(E))$. Suppose $x \in E$ and $\varphi \in \mathcal{D}$. Then $AG(\varphi)x = G(-\varphi')x$; indeed,

$$G(\varphi)x = (-1)^n \int_{-\infty}^{\infty} \varphi^{(n)}(t)S(t)x\,dt = (-1)^{n+1} \int_{-\infty}^{\infty} \varphi^{(n+1)}(t) \int_0^t S(s)x\,ds\,dt \in D(A)$$

and

$$\begin{aligned} AG(\varphi)x &= (-1)^{n+1} \int_{-\infty}^{\infty} \varphi^{(n+1)}(t)\Big[S(t)x + \sum_{j=0}^{n} t^j B_j x\Big]dt \\ &= (-1)^{n+1} \int_{-\infty}^{\infty} \varphi^{(n+1)}(t)S(t)x\,dt = G(-\varphi')x. \end{aligned}$$

Further,

$$\begin{aligned} G(\varphi)Ax &= (-1)^{n+1} \int_{-\infty}^{\infty} \varphi^{(n+1)}(t) \int_0^t S(s)Ax\,ds\,dt \\ &= (-1)^{n+1} \int_{-\infty}^{\infty} \varphi^{(n+1)}(t) \int_0^t \Big[AS(s)x - \sum_{j=0}^{n-1} s^j C_j x\Big]ds\,dt \\ &= (-1)^{n+1} \int_{-\infty}^{\infty} \varphi^{(n+1)}(t)A \int_0^t S(s)x\,ds\,dt = AG(\varphi)x, \quad x \in D(A). \end{aligned}$$

Hence, $G(\varphi)A \subseteq AG(\varphi)$ and (1.2) holds. The proofs of (4.12) and (b) are contained in those of Theorem 4.2. Let $\varphi \in \mathcal{D}$, $\psi \in \mathcal{D}$ and let $x \in N(Y_n)$ be fixed. Arguing as in the proof of Theorem 4.2, one gets the validity of (4.24). Hence,

$$\begin{aligned} 0 &= \sum_{i=0}^{n} (-1)^{i+1} i! G\big((\varphi^{(n)} * \psi)^{(n-i)}\big)B_i x = \sum_{i=0}^{n} (-1)^{i+1} i! G\big(\varphi^{(n)} * \psi^{(n-i)}\big)B_i x \\ &= \sum_{i=0}^{n} (-1)^{i+1} i! (-1)^n \int_{-\infty}^{\infty} \varphi^{(n)}(t) \int_{-\infty}^{\infty} \psi^{(n-i)}(s)S(t+s)B_i x\,ds\,dt. \end{aligned}$$

Owing to (4.12), $G(\varphi)G(\psi)x = 0$ and $N(Y_n) \subseteq N(G(\varphi)G(\psi))$. Let (ρ_k) be a regularizing sequence and G be regular. Then $G(\psi)x = \lim_{k\to\infty} G(\rho_k)G(\psi)x = 0$ and $x \in \mathcal{N}(G)$. This proves $N(Y_n) \subseteq \mathcal{N}(G)$, and due to (b), we have $\mathcal{N}(G) \subseteq N(Y_n)$. The proof of (c) is completed.

To prove (d), notice that the proof of Theorem 4.2 implies (4.21) for G. Since $B_0 = \cdots = B_{n-1} = 0$ and $B_n = -\frac{1}{n!}I$, we immediately obtain $(DG)_1$ from (4.12). Clearly, $Y_n = n!B_n = -I$ and $(DG)_2$ follows from an application of (b). Hence, G is a (DG). Put now $\overline{S}(t) := S(t)$, $t \geqslant 0$ and $\overline{S}(t) := (-1)^n S(t)$, $t < 0$. It is straightforward to verify that $(\overline{S}(t))_{t\in\mathbb{R}}$ is an n-times integrated group generated by A. Furthermore, it is evident that

$$G(\varphi)x = (-1)^n \int_0^\infty \varphi^{(n)}(t)\overline{S}(t)x\,dt + \int_{-\infty}^0 \varphi^{(n)}(t)\overline{S}(t)x\,dt, \quad \varphi \in \mathcal{D},\ x \in E.$$

By the proof of Theorem 2.1, we have that G is generated by A. □

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