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ON THE SOLID HULL OF THE HARDY–LORENTZ SPACE

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ABSTRACT. The solid hulls of the Hardy–Lorentz spaces $H^{p,q}$, $0 < p < 1$, $0 < q \leqslant \infty$ and $H_0^{p,\infty}$, $0 < p < 1$, as well as of the mixed norm space $H_0^{p,\infty,\alpha}$, $0 < p \leqslant 1$, $0 < \alpha < \infty$, are determined.

Introduction

In [**JP1**] the solid hull of the Hardy space H^p, $0 < p < 1$, is determined. In this article we determine the solid hulls of the Hardy–Lorentz spaces $H^{p,q}$, $0 < p < 1$, $0 < q \leqslant \infty$ and $H_0^{p,\infty}$, $0 < p < 1$, as well as of the mixed norm space $H_0^{p,\infty,\alpha}$, $0 < p \leqslant 1$, $0 < \alpha < \infty$. Since $H^{p,p} = H^p$ our results generalize [**JP1**, Theorem 1].

Recall, the Hardy space H^p, $0 < p \leqslant \infty$, is the space of all functions f holomorphic in the unit disk U, ($f \in H(U)$), for which $\|f\|_p = \lim_{r\to 1} M_p(r,f) < \infty$, where, as usual,

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt\right)^{1/p}, \quad 0 < p < \infty,$$
$$M_\infty(r,f) = \sup_{0\leqslant t<2\pi} |f(re^{it})|.$$

Now we introduce a generalization and refinement of the spaces H^p; the Hardy–Lorentz spaces $H^{p,q}$, $0 < p < \infty$, $0 < q \leqslant \infty$.

Let σ denotes normalized Lebesgue measure on $T = \partial U$ and let $L^0(\sigma)$ be the space of complex-valued Lebesgue measurable functions on T. For $f \in L^0(\sigma)$ and $s \geqslant 0$ we write

$$\lambda_f(s) = \sigma\big(\{\xi \in T : |f(\xi)| > s\}\big)$$

for the distribution function and

$$f^\star(s) = \inf\big(\{t \geqslant 0 : \lambda_f(t) \leqslant s\}\big)$$

for the decreasing rearrangement of $|f|$ each taken with respect to σ.

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The Lorentz functional $\|\cdot\|_{p,q}$ is defined at $f \in L^0(\sigma)$ by

$$\|f\|_{p,q} = \left(\int_0^1 \big(f^\star(s)s^{1/p}\big)^q \frac{ds}{s}\right)^{1/q} \quad \text{for } 0 < q < \infty,$$
$$\|f\|_{p,\infty} = \sup\{f^\star(s)s^{1/p} : s \geqslant 0\}.$$

The corresponding Lorentz space is $L^{p,q}(\sigma) = \{f \in L^0(\sigma) : \|f\|_{p,q} < \infty\}$. The space $L^{p,q}(\sigma)$ is separable if and only if $q \neq \infty$. The class of functions $f \in L^0(\sigma)$ satisfying $\lim_{s\to 0}(f^\star(s)s^{1/p}) = 0$ is a separable closed subspace of $L^{p,\infty}(\sigma)$, which is denoted by $L_0^{p,\infty}(\sigma)$.

The Nevanlinna class N is the subclass of functions $f \in H(U)$ for which

$$\sup_{0<r<1} \int_T \log^+ |f(r\xi)|\, d\sigma(\xi) < \infty.$$

Functions in N are known to have non-tangential limits σ-a.e. on T. Consequently every $f \in N$ determines a boundary value function which we also denote by f. Thus

$$f(\xi) = \lim_{r\to 1} f(r\xi) \quad \sigma\text{-a.e.} \quad \xi \in T.$$

The Smirnov class N^+ is the subclass of N consisting of those functions f for which

$$\lim_{r\to 1} \int_T \log^+ |f(r\xi)|\, d\sigma(\xi) = \int_T \log^+ |f(\xi)|\, d\sigma(\xi).$$

We define the Hardy–Lorentz space $H^{p,q}$, $0 < p < \infty$, $0 < q \leqslant \infty$, to be the space of functions $f \in N^+$ with boundary value function in $L^{p,q}(\sigma)$ and we put $\|f\|_{H^{p,q}} = \|f\|_{p,q}$. The functions in $H^{p,\infty}$ with a boundary value function in $L_0^{p,\infty}(\sigma)$ form a closed subspace of $H^{p,\infty}$, which is denoted by $H_0^{p,\infty}$. The cases of major interest are of course $p = q$ and $q = \infty$; indeed $H^{p,p}$ is nothing but H^p, and $H^{p,\infty}$ is the weak-H^p.

The mixed norm space $H^{p,q,\alpha}$, $0 < p \leqslant \infty$, $0 < q$, $\alpha < \infty$, consists of all $f \in H(U)$ for which

$$\|f\|_{H^{p,q,\alpha}} = \|f\|_{p,q,\alpha} = \left(\int_0^1 (1-r)^{q\alpha-1} M_p(r,f)^q dr\right)^{1/q} < \infty.$$

$H^{p,q,\alpha}$ can also be defined when $q = \infty$, in which case it is sometimes known as the weighted Hardy space $H^{p,\infty,\alpha}$, and consists of all $f \in H(U)$ for which

$$\|f\|_{p,\infty,\alpha} = \sup_{0<r<1} (1-r)^\alpha M_p(r,f) < \infty.$$

The functions in $H^{p,\infty,\alpha}$ $0 < p \leqslant \infty$ for which $\lim_{r\to 1}(1-r)^\alpha M_p(r,f) = 0$ form a closed subspace which is denoted by $H_0^{p,\infty,\alpha}$.

Throughout this paper, we identify the holomorphic function $f(z) = \sum_{k=0}^\infty \hat{f}(k)z^k$ with its sequence of Taylor coefficients $\{\hat{f}(k)\}_{k=0}^\infty$.

If $f(z) = \sum_{k=0}^\infty \hat{f}(k)z^k$ belongs to $H^{p,q}$, then

$$\hat{f}(k) = O\big((k+1)^{(1/p)-1}\big), \quad \text{if } 0 < p < 1 \text{ and } 0 < q \leqslant \infty. \tag{1}$$

(See [**Al**] and [**Co**].)

In this paper we find the strongest condition that the moduli of an $H^{p,q}$, $0 < p < 1$, $0 < q \leqslant \infty$, satisfy. Our result shows that the estimate (1) is optimal only if $q = \infty$.

To state our results in a form of theorems we need to introduce some more notations

A sequence space X is solid if $\{b_n\} \in X$ whenever $\{a_n\} \in X$ and $|b_n| \leqslant |a_n|$. More generally, we define $S(X)$, the solid hull of X. Explicitly,

$$S(X) = \big\{\{\lambda_n\} : \text{there exists } \{a_n\} \in X \text{ such that } |\lambda_n| \leqslant |a_n|\big\}.$$

A complex sequence $\{a_n\}$ is of class $l(p,q)$, $0 < p, q \leqslant \infty$, if

$$\|\{a_n\}\|_{p,q}^q = \|\{a_n\}\|_{l(p,q)}^q = \sum_{n=0}^{\infty} \bigg(\sum_{k \in I_n} |a_k|^p\bigg)^{q/p} < \infty,$$

where $I_0 = \{0\}$, $I_n = \{k \in N : 2^{n-1} \leqslant k < 2^n\}$, $n = 1, 2, \ldots$ In the case where p or q is infinite, replace the corresponding sum by a supremum. Note that $l(p,p) = l^p$.

For $t \in R$ we write D^t for the sequence $\{(n+1)^t\}$, for all $n \geqslant 0$. If $\lambda = \{\lambda_n\}$ is a sequence and X a sequence space, we write $\lambda X = \big\{\{\lambda_n x_n\} : \{x_n\} \in X\big\}$; thus, for example, $\{a_n\} \in D^t l^\infty$ if and only if $|a_n| = O(n^t)$.

We are now ready to state our first result.

Theorem 1. *If $0 < p < 1$ and $0 < q \leqslant \infty$, then $S(H^{p,q}) = D^{(1/p)-1} l(\infty, q)$.*

In particular, $S(H^p) = D^{(1/p)-1} l(\infty, p)$, $0 < p < 1$. This was proved in [**JP1**]. Also, $S(H^{p,\infty}) = D^{(1/p)-1} l^\infty$ means that the estimate (1) valid for the Taylor coefficients of an $H^{p,\infty}$, $0 < p < 1$, function is sharp.

Our second result is as follows:

Theorem 2. *If $0 < p < 1$, then $S(H_0^{p,\infty}) = D^{(1/p)-1} c_0$, where c_0 is the space of all null sequences.*

Our method of proving Theorem 1 and Theorem 2 depend upon nested embedding [**Le**, Theorem 4.1] for Hardy–Lorentz spaces. Thus, the strategy is to trap $H^{p,q}$ between a pair of mixed norm spaces and then deduce the results for $H^{p,q}$ from the corresponding results for the mixed norm spaces. Our Theorem 1 will follow from the following two theorems:

Theorem L. [**Le**] *Let $0 < p_0 < p < s \leqslant \infty$, $0 < q \leqslant t \leqslant \infty$ and $\beta > (1/p_0) - (1/p)$. Then*

$$D^{-\beta} H^{p_0, q, \beta + (1/p) - (1/p_0)} \subset H^{p,q} \subset H^{s, q, (1/p) - (1/s)}, \tag{2}$$

$$D^{-\beta} H_0^{p_0, \infty, \beta + (1/p) - (1/p_0)} \subset H_0^{p,\infty} \subset H_0^{s, \infty, (1/p) - (1/s)}. \tag{3}$$

Theorem JP 1. [**JP1**] *If $0 < p \leqslant 1$, $0 < q \leqslant \infty$ and $0 < \alpha < \infty$, then $S(H^{p,q,\alpha}) = D^{\alpha + (1/p) - 1} l(\infty, q)$.*

To prove Theorem 2 we first determine the solid hull of the space $H_0^{p,\infty,\alpha}$, $0 < p \leqslant 1$, $0 < \alpha < \infty$. More precisely, we prove

Theorem 3. *If $0 < p \leqslant 1$ and $0 < \alpha < \infty$, then $S(H_0^{p,\infty,\alpha}) = D^{\alpha + (1/p) - 1} c_0$.*

Given two vector spaces X, Y of sequences we denote by (X,Y) the space of multipliers from X to Y. More precisely,

$$(X,Y) = \big\{\lambda = \{\lambda_n\} : \{\lambda_n a_n\} \in Y, \text{ for every } \{a_n\} \in X\big\}.$$

As an application of our results we calculate multipliers $(H^{p,q}, l(u,v))$, $0<p<1$, $0 < q \leqslant \infty$, $(H_0^{p,\infty}, l(u,v))$, $0 < p < 1$, and $(H^{p,\infty}, X)$, $0 < p < 1$, where X is a solid space. These results extend some of the results obtained by Lengfield [**Le**, Section 5].

1. The solid hull of the Hardy–Lorentz space $H^{p,q}$, $0 < p < 1$, $0 < q \leqslant \infty$

Proof of Theorem 1. Let $0 < p < 1$. Choose p_0 and s so that $p_0 < p < s \leqslant 1$ and a real number β so that $\beta + (1/p) - (1/p_0) > 0$. As an easy consequence of Theorem JP we have

$$S\big(D^{-\beta} H^{p_0,q,\beta+(1/p)-(1/p_0)}\big) = D^{(1/p)-1} l(\infty, q).$$

Also, by Theorem JP,

$$S\big(H^{s,q,(1/p)-(1/s)}\big) = D^{(1/p)-1} l(\infty, q),$$

and consequently $S(H^{p,q}) = D^{(1/p)-1} l(\infty, q)$, by Theorem L. $\square$

2. The solid hull of mixed norm space $H_0^{p,\infty,\alpha}$, $0 < p \leqslant 1$, $0 < \alpha < \infty$

If $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$ and $g(z) = \sum_{k=0}^{\infty} \hat{g}(k) z^k$ are holomorphic functions in U, then the function $f \star g$ is defined by $(f \star g)(z) = \sum_{k=0}^{\infty} \hat{f}(k)\hat{g}(k) z^k$.

The main tool for proving Theorem 3 are polynomials W_n, $n \geqslant 0$, constructed in [**JP1**]and [**JP3**]. Recall the construction and some of their properties.

Let $\omega : R \to R$ be a nonincreasing function of class C^∞ such that $\omega(t) = 1$, for $t \leqslant 1$, and $\omega(t) = 0$, for $t \geqslant 2$. We define polynomials $W_n = W_n^\omega$, $n \geqslant 0$, in the following way:

$$W_0(z) = \sum_{k=0}^{\infty} \omega(k) z^k \quad \text{and} \quad W_n(z) = \sum_{k=2^{n-1}}^{2^{n+1}} \varphi\Big(\frac{k}{2^{n-1}}\Big) z^k, \text{ for } n \geqslant 1,$$

where $\varphi(t) = \omega(t/2) - \omega(t)$, $t \in R$.

The coefficients $\hat{W}_n(k)$ of these polynomials have the following properties:

$$\operatorname{supp}\{\hat{W}_n\} \subset [2^{n-1}, 2^{n+1}]; \tag{4}$$

$$0 \leqslant \hat{W}_n(k) \leqslant 1, \quad \text{for all } k, \tag{5}$$

$$\sum_{n=0}^{\infty} \hat{W}_n(k) = 1, \quad \text{for all } k, \tag{6}$$

$$\hat{W}_n(k) + \hat{W}_{n+1}(k) = 1, \quad \text{for } 2^n \leqslant k \leqslant 2^{n+1},\ n \geqslant 0. \tag{7}$$

Property (5) implies that

$$f(z) = \sum_{n=0}^{\infty} (W_n \star f)(z), \quad f \in H(U),$$

the series being uniformly convergent on compact subsets of U.

If $0 < p < 1$, then there exists a constant $C > 0$ depending only on p such that

$$\|W_n\|_p^p \leqslant C_p 2^{-n(1-p)}, \quad n \geqslant 0. \tag{8}$$

Proof of Theorem 3. Let $f \in H_0^{p,\infty,\alpha}$, $0 < p < 1$, $0 < \alpha < \infty$. By using the familiar inequality

$$M_p(r, f) \geqslant C(1-r)^{(1/p)-1} M_1(r^2, f), \quad 0 < p \leqslant 1,$$

(see [**Du**, Theorem 5.9]), we obtain

$$\sup_{k \in I_n} |\hat{f}(k)| r^{2k} \leqslant M_1(r^2, f) \leqslant C M_p(r, f)(1-r)^{1-(1/p)}, \quad 0 < r < 1.$$

Now we take $r_n = 1 - 2^{-n}$ and let $n \to \infty$, to get $\{\hat{f}(k)\} \in D^{\alpha+(1/p)-1} c_0$. Thus $H_0^{p,\infty,\alpha} \subset D^{\alpha+(1/p)-1} c_0$.

To show that $D^{\alpha+(1/p)-1} c_0$ is the solid hull of $H_0^{p,\infty,\alpha}$, it is enough to prove that if $\{a_n\} \in D^{\alpha+(1/p)-1} c_0$, then there exists $\{b_n\} \in H_0^{p,\infty,\alpha}$ such that $|b_n| \geqslant |a_n|$, for all n.

Let $\{a_n\} \in D^{\alpha+(1/p)-1} c_0$. Define

$$g(z) = \sum_{j=0}^{\infty} B_j \big(W_j(z) + W_{j+1}(z) \big) = \sum_{k=0}^{\infty} c_k z^k,$$

where $B_j = \sup_{2^j \leqslant k < 2^{j+1}} |a_k|$. Using (4) and (8) we find that

$$M_p^p(r, g) \leqslant \sum_{j=0}^{\infty} B_j^p \big(M_p^p(r, W_j) + M_p^p(r, W_{j+1}) \big) \leqslant C \bigg(B_0^p + \sum_{j=1}^{\infty} B_j^p r^{p 2^{j-1}} 2^{-j(1-p)} \bigg)$$

Set $B_j^p 2^{-j(\alpha p + 1 - p)} = \lambda_j$. Then

$$M_p^p(r, g) \leqslant C \bigg(\lambda_0 + \sum_{j=1}^{\infty} \lambda_j r^{p 2^{j-1}} 2^{j \alpha p} \bigg),$$

where $\lambda_j \to 0$, as $j \to \infty$. From this it easily follows that $(1-r)^{\alpha p} M_p^p(r, g) \to 0$, as $r \to 1$. Thus $g \in H_0^{p,\infty,\alpha}$.

To prove that $|c_k| \geqslant |a_k|$, $k = 1, 2, \ldots$, choose n so that $2^n \leqslant k < 2^{n+1}$. It follows from (7)

$$\begin{aligned} c_k &= \sum_{j=0}^{\infty} B_j \big(\hat{W}_j(k) + \hat{W}_{j+1}(k) \big) \geqslant B_n \big(\hat{W}_n(k) + \hat{W}_{n+1}(k) \big) \\ &= B_n = \sup_{2^n \leqslant j < 2^{n+1}} |a_j| \geqslant |a_k|. \end{aligned}$$

Now the function $h(z) = \sum_{n=0}^{\infty} b_n z^n$, where $b_0 = a_0$ and $b_n = c_n$, for $n \geqslant 1$, belongs to $H_0^{p,\infty,\alpha}$ and $|b_n| \geqslant |a_n|$ for all $n \geqslant 0$. This finishes the proof of Theorem 3. □

3. The solid hull of the space $H_0^{p,\infty}$, $0 < p < 1$

PROOF OF THEOREM 2. Let $0 < p < 1$. Choose p_0 and s so that $p_0 < p < s \leqslant 1$ and $\beta \in R$ so that $\beta + (1/p) - (1/p_0) > 0$. Then

$$S\big(D^{-\beta} H_0^{p_0,\infty,\beta+(1/p)-(1/p_0)}\big) = D^{(1/p)-1} c_0,$$
$$S\big(H_0^{s,\infty,(1/p)-(1/s)}\big) = D^{(1/p)-1} c_0,$$

by Theorem 3. By Theorem L we have $S(H_0^{p,\infty}) = D^{(1/p)-1} c_0$. □

4. Applications to multipliers

As it was noticed in the introduction, another objective of this paper is to extend some of the results given in [**Le**, Section 5].

The next lemma due to Kellog (see [**K**]) (who states it for exponents no smaller than 1, but it then follows for all exponents, since $\{\lambda_n\} \in (l(a,b), l(c,d)))$ if and only if $\{\lambda_n^{(1/t)})\} \in (l(at,bt), l(ct,dt))$.

LEMMA 1. *If $0 < a,b,c,d \leqslant \infty$, then $(l(a,b), l(c,d)) = l(a \ominus c, b \ominus d)$, where $a \ominus c = \infty$ if $a \leqslant c$, $b \ominus d = \infty$, if $b \leqslant d$, and*

$$\frac{1}{a \ominus c} = \frac{1}{c} - \frac{1}{a}, \quad \textit{for } 0 < c < a,$$
$$\frac{1}{b \ominus d} = \frac{1}{d} - \frac{1}{b}, \quad \textit{for } 0 < d < b.$$

In particular, $(l^\infty, l(u,v)) = l(u,v)$. Also, it is known that $(c_0, l(u,v)) = l(u,v)$.

In [**AS**] it is proved that if X is any solid space and A any vector space of sequences, then $(A, X) = (S(A), X)$.

Since $l(u,v)$ are solid spaces, we have $(H^{p,q}, l(u,v)) = (S(H^{p,q}), l(u,v))$ and $(H_0^{p,\infty}, l(u,v)) = (S(H_0^{p,\infty}), l(u,v))$. Using this, Lemma 1, Theorem 1 and Theorem 2 we get

THEOREM 4. *Let $0 < p < 1$ and $0 < q \leqslant \infty$. Then*

$$(H^{p,q}, l(u,v)) = D^{1-(1/p)} l(u, q \ominus v).$$

THEOREM 5. *Let $0 < p < 1$. Then*

$$(H_0^{p,\infty}, l(u,v)) = D^{1-(1/p)} l(u,v).$$

In particular, $(H^{p,\infty}, l(u,v)) = D^{1-(1/p)} l(u,v)$. In fact more is true.

THEOREM 6. *Let $0 < p < 1$ and let X be a solid space. Then*

$$(H^{p,\infty}, X) = D^{1-(1/p)} X.$$

PROOF. Since X is a solid space, we have $(l^\infty, X) = X$. Hence, using Theorem 1 we get

$$\begin{aligned}(H^{p,\infty}, X) &= (S(H^{p,\infty}), X) = (D^{(1/p)-1} l^\infty, X)\\ &= D^{1-(1/p)} (l^\infty, X) = D^{1-(1/p)} X.\end{aligned}$$
□

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