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ON THE DIFFERENTIABILITY OF A DISTANCE FUNCTION

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ABSTRACT. Let M be a simply connected complete Kähler manifold and N a closed complete totally geodesic complex submanifold of M such that every minimal geodesic in N is minimal in M. Let U_ν be the unit normal bundle of N in M. We prove that if a distance function ρ is differentiable at $v \in U_\nu$, then ρ is also differentiable at $-v$.

1. Introduction

Let N be a closed submanifold of a complete Riemannian manifold M and $\pi : U_\nu \mapsto N$ the unit normal bundle of N in M. For $v \in T_pM$, $p \in M$, throughout this paper, let $\gamma_v(t)$ denote always the geodesic curve such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. Define a function $\rho : U_\nu \mapsto \mathbb{R}$ by

$$\rho(v) := \sup\{t > 0 \mid d(N, \gamma_v(t)) = t\} \quad \text{for } v \in U_\nu,$$

where $d(N, \gamma_v(t))$ denotes the distance between N and $\gamma_v(t)$. For each positive integer $k \in \mathbb{N}$, define a function $\lambda_k : U_\nu \mapsto \mathbb{R}$ by

$$\lambda_k(v) := \sup\{t > 0 \mid \gamma_v|_{[0,t]} \text{ has no } k\text{-th focal point of } N\}$$

for $v \in U_\nu$ [2]. The followings are well known: ρ is continuous [10] and λ_1 is smooth where λ_1 is finite [2]. Itoh and Tanaka [2] proved that the function ρ on U_ν is locally Lipschitz, where ρ is finite. So, by Rademacher's theorem ([1], [6]), the function $\min(\rho, r)$ is differentiable almost everywhere for each $r > 0$. Generally, it is well known that ρ is differentiable at $v \in U_\nu$ if $\gamma_v(\rho(v))$ is a normal cut point, i.e., there exist exactly two N-segments through $\gamma_v(\rho(v))$ such that $\gamma_v(\rho(v))$ is not a focal point along all of these two N-segments. Furthermore, in the case $\dim M = 2$, Tanaka [8] proved that a point $v \in U_\nu$ with $\rho(v) < \infty$ is a differentiable point of the function ρ if and only if $\gamma_v(\rho(v))$ is a 1-st focal point of N along γ_v or there exist at most two N-segments through $\gamma_v(\rho(v))$. Here, a curve $\gamma : [0, r] \mapsto M$ is called

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an N-segment if γ is a geodesic curve such that $\gamma'(0) \in U_\nu$ and $d(N, \gamma(t)) = t$ for $t \in [0, r]$. This fact is obviously very nice but didn't have any information about the n-dimensional manifold M with $n \geqslant 3$. So, we plan to consider the manifold M such that it has some good conditions. Then we have

Main Theorem. *Let M be a simply connected complete Kähler manifold and N a closed complete totally geodesic complex submanifold of M such that every minimal geodesic in N is minimal in M. Let U_ν be the unit normal bundle of N in M. If ρ is differentiable at $v \in U_\nu$, then ρ is also differentiable at $-v$.*

2. Proof of the Main Theorem

Now, we need the following theorem

Ambrose Theorem. *Let M and $\widetilde{M}$ be m-dimensional complete Riemannian manifolds and $I : T_pM \mapsto T_{\widetilde{p}}\widetilde{M}$ a linear isometry. Suppose that M is simply connected and for any once broken geodesic $\gamma : [0, l] \mapsto M$ in M*

$$I_t\big(R(u, v)w\big) = \widetilde{R}\big(I_t(u), I_t(v)\big)I_t(w)$$

for any $u, v, w \in T_{\gamma(t)}M$, $0 \leqslant t \leqslant l$, where R and $\widetilde{R}$ denote the curvature tensors of M and $\widetilde{M}$, respectively. For any minimal geodesic $\gamma : [0, l] \mapsto M$ with $\gamma(0) = p$, define a geodesic $\widetilde{\gamma}$ by $\widetilde{\gamma}(t) := \gamma_{I(\gamma'(0))}(t)$ and define a map $\Phi : M \mapsto \widetilde{M}$ by $\Phi(\gamma(t)) := \widetilde{\gamma}(t)$. Then Φ is well defined and a C^∞ Riemannian covering. In particular, if $\widetilde{M}$ is also simply connected, then M and $\widetilde{M}$ are isometric [7].

In our case, since M is complete Kähler, let g and I denote the corresponding Kähler metric and the corresponding complex structure, respectively. Let ∇ and R be the Levi–Civita connection and the curvature tensor of the metric g, respectively. For each $p \in M$, we know that $I|_{T_pM} : T_pM \mapsto T_pM$ is a linear isometry, where $I|_{T_pM}$ means the restriction of the complex structure I to the tangent space T_pM. We see $\nabla I = 0$. Furthermore [5], $R(I, I) = R(\ ,\ )$ and $I \circ R = R \circ I$. For any minimal geodesic $\gamma : [0, l] \mapsto M$ with $\gamma(0) = p$, define a map $\Phi_p : M \mapsto M$ by

$$\Phi_p(\gamma(t)) := \gamma_{I(\gamma'(0))}(t) \quad \text{for } t \in [0, l].$$

Then, by Ambrose Theorem, Φ_p is an isometry for each $p \in M$.

Proposition 1. *$\Phi_p(N) = N$ for each $p \in N$.*

Proof. Firstly, we claim $\Phi_p(N) \supset N$. For any $q \in N$, there exists a minimal geodesic curve $\gamma : [0, 1] \mapsto N$ such that $\gamma(0) = p$ and $\gamma(1) = q$. By the hypothesis, γ is also a minimal geodesic curve in M. Since Φ_p is isometric and N is complex, $\Phi_p^k \circ \gamma$ is minimal in N for each $k \in \{1, 2, 3, 4\}$. Hence,

$$q = (\Phi_p^4 \circ \gamma)(1) = \Phi_p\big(\Phi_p^3(\gamma(1))\big) \in \Phi_p(N).$$

Secondly, we claim $\Phi_p(N) \subset N$. For any $q \in \Phi_p(N)$, by definition, there exists a point $\widetilde{q} \in N$ such that $\Phi_p(\widetilde{q}) = q$. Choose a minimal geodesic curve $\gamma : [0, 1] \mapsto N$ such that $\gamma(0) = p$ and $\gamma(1) = \widetilde{q}$. Then, γ is also minimal in M. As the above, $\Phi_p \circ \gamma$ is minimal in N. Thus, $(\Phi_p \circ \gamma)(1) = q \in N$. This completes the proof. $\square$

PROOF OF THE MAIN THEOREM. Since (M, I) is a complex manifold, there exists an atlas $\{(z_\alpha, U_\alpha) \mid \alpha \in A\}$ of M, being a subfamily of the maximal atlas of M, such that

(i) $\{U_\alpha \mid \alpha \in A\}$ is a locally finite open covering of M,

(ii) there exists a partition of unity $\{\varphi_\alpha : M \mapsto \mathbb{R} \mid \alpha \in A\}$ such that $\operatorname{supp}\varphi_\alpha \subset U_\alpha$ for all $\alpha \in A$.

Let $\pi : TM \mapsto M$ be the natural projection map, given by $\pi(p, v) = p$ for $(p, v) \in TM$. Conveniently, identify the tangent space TM with the holomorphic tangent space $T'M$ [5]. Given a chart $z_\alpha : U_\alpha \mapsto \mathbb{C}^m$, $\alpha \in A$, we can naturally have the corresponding chart $dz_\alpha : T'U_\alpha \mapsto \mathbb{C}^m \times \mathbb{C}^m$ by

$$dz_\alpha(v) = (z_\alpha^1, z_\alpha^2, \ldots, z_\alpha^m; \xi_\alpha^1, \xi_\alpha^2, \ldots, \xi_\alpha^m), \text{ where } v = \sum_{k=1}^m \xi_\alpha^k \frac{\partial}{\partial z_\alpha^k} \in T'_pU_\alpha \text{ with } p \in U_\alpha.$$

For $v', w' \in T'_v(TM)$ with $v \in TU_\alpha (\equiv T'U_\alpha)$ and $\alpha \in A$ let their coordinate representations be $(v'_{\alpha 1}, \ldots, v'_{\alpha m}; \eta_{\alpha 1}, \ldots, \eta_{\alpha m})$ and $(w'_{\alpha 1}, \ldots, w'_{\alpha m}; \eta'_{\alpha 1}, \cdots, \eta'_{\alpha m})$. Then we put

$$h(v', w') := \sum_{\substack{\alpha \in A \\ v \in TU_\alpha \\ i \in \{1, \ldots, m\}}} \varphi_\alpha(p)\big(v'_{\alpha i}\overline{w'_{\alpha i}} + \eta_{\alpha i}\overline{\eta'_{\alpha i}}\big),$$

where $p = \pi(v)$. This defines a Hermitian metric on the complex manifold TM. Let G be the Riemannian metric on TM which is naturally induced from the Hermitian metric h.

Assume that ρ is differentiable at $v \in U_\nu \cap T_pM$. By definition, the differential $d\Phi_p$ of the map Φ_p has the following properties

$$(d\Phi_p)_p(v) = Iv \quad \text{and} \quad (d\Phi_p)_p(Iv) = I(Iv) = -v.$$

We know that ρ is differentiable at $v \in U_\nu$ if and only if for any unit speed smooth curve $c : (-\epsilon, \epsilon) \mapsto U_\nu$ with $c(0) = v$ and $\epsilon > 0$ the following limit exists:

$$\lim_{t \to 0} \frac{\rho(c(t)) - \rho(c(0))}{t}.$$

Take any unit speed smooth curve $\widetilde{c} : (-\epsilon, \epsilon) \mapsto U_\nu$ with $\widetilde{c}(0) = Iv$ and sufficiently small $\epsilon > 0$. Let $p_t := \pi(-I\widetilde{c}(t))$ for each $t \in (-\epsilon, \epsilon)$. By Proposition 1,

$$d(N, \gamma_{\widetilde{c}(t)}(s)) = d(\Phi_{p_t}(N), \Phi_{p_t}(\gamma_{-I\widetilde{c}(t)}(s))) = d(N, \gamma_{-I\widetilde{c}(t)}(s))$$

for all $s \in [0, l_t]$ with $l_t := \sup\{r > 0 \mid \gamma_{-I\widetilde{c}(t)}|_{[0,r]} \text{ is minimal}\}$ so that

$$\rho(\widetilde{c}(t)) = \sup\{s > 0 \mid d(N, \gamma_{-I\widetilde{c}(t)}(s)) = s\} = \rho(-I\widetilde{c}(t))$$

for $t \in (-\epsilon, \epsilon)$. Note that $-I\widetilde{c}(t)$ is a unit speed smooth curve in U_ν with the property $-I\widetilde{c}(0) = v$. Thus, by the hypothesis, the following limit

$$\lim_{t \to 0} \frac{\rho(\widetilde{c}(t)) - \rho(\widetilde{c}(0))}{t} = \lim_{t \to 0} \frac{\rho(-I\widetilde{c}(t)) - \rho(-I\widetilde{c}(0))}{t}$$

exists. Hence, ρ is differentiable at Iv. Furthermore, from this result, ρ is also differentiable at $I(Iv) = -v$. Therefore, we complete the proof. $\square$

REMARKS. 1. In particular, if ρ is differentiable at $v \in U_\nu$, then ρ is also differentiable at $w \in \{v, Iv, I^2v = -v, I^3v = -Iv\}$.

2. Let $\langle\Phi_p\rangle$ be the group generated by the element Φ_p. Then $\langle\Phi_p\rangle$ is a cyclic group of order 4. Let $G := \bigcup_{p\in M}\langle\Phi_p\rangle$. Then $G \subset \mathrm{iso}(M)$, where $\mathrm{iso}(M)$ denotes the group of all isometries of M

3. For each $p \in M$, let $N = \{p\}$ as a 0-dimensional complex submanifold of M. Then $U_\nu = U_pM$, where U_pM denotes the unit tangent vector space of M at p. If ρ is differentiable at $v \in U_\nu$, then ρ is also differentiable at $w \in \{v, Iv, -v, -Iv\}$.

4. Consider the complex projective space $\mathbb{P}^n$ with the Fubini–Study metric [3]. Let $\mathbb{P}^k := \{(z_0 : \cdots : z_k : 0 : \cdots : 0) \mid z_i \in \mathbb{C}, 0 \leqslant i \leqslant k\} \subset \mathbb{P}^n$ for $k = 1, \ldots, n-1$. Then $\mathbb{P}^n$ is a simply connected complete Kähler amnifold and $\mathbb{P}^k$ is a closed complete totally geodesic complex submanifold of $\mathbb{P}^n$ such that every minimal geodesic in $\mathbb{P}^k$ is minimal in $\mathbb{P}^n$ [4]. Let U_ν be the unit normal bundle of $\mathbb{P}^k$ in $\mathbb{P}^n$. If ρ is differentiable at $v \in U_\nu$, then ρ is also differentiable at $w \in \{v, Iv, -v, -Iv\}$.

5. Let (M, g) be a simply connected complete Riemannian manifold with a hyperkähler structure (g, I, J, K) and N a closed complete totally geodesic trianalytic submanifold of M such that every minimal geodesic in N is minimal in M ([3], [9]). If ρ is differentiable at $v \in U_\nu$, then ρ is also differentiable at $w \in \{R^iv \mid i \in \{1, 2, 3, 4\},\ R \in S^2\}$, where $S^2 := \{aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\}$.

Now, we consider

QUESTION 1. Let M be a simply connected complete Kähler manifold and N a closed complete totally geodesic complex submanifold of M. Then, is it true that every minimal geodesic in N is also minimal in M?

The author believes that it may be true, but can not prove it.

QUESTION 2. Let (M, g, I) be a 2-dimensional simply connected complete Kähler manifold and N a 1-dimensional closed complex submanifold of M. Let U_ν be the unit normal bundle of N in M. Then, at which $v \in U_\nu$ is $\rho : U_\nu \mapsto \mathbb{R}$ differentiable?

Note that if $v \in T_pN$ with $g(v, v) = 1$ and $u \in T_pM \cap U_\nu$ for $p \in N$, then we easily get

$$T_pM = \mathbb{R}\langle v, Iv, u, Iu\rangle \quad \text{and} \quad T_pM \cap U_\nu = \{au + bIu \mid a^2 + b^2 = 1\}.$$

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