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CONSTRUCTIONS OF $(2,n)$-VARIETIES OF GROUPOIDS FOR $n = 7,8,9$

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Dedicated to Prof. Dr. Kazimierz Glazek

Abstract. Given positive integer $n > 2$, an algebra is said to be a $(2,n)$-algebra if any of its subalgebras generated by two distinct elements has n elements. A variety is called a $(2,n)$-variety if every algebra in that variety is a $(2,n)$-algebra. There are known only $(2,3)$-, $(2,4)$- and $(2,5)$-varieties of groupoids, and there is no $(2,6)$-variety. We present here $(2,n)$-varieties of groupoids for $n = 7,8,9$.

1. Introduction

The notion of variety of algebras having the property (k,n) was given in [**4**] and equationally defined classes of cancellative groupoids having the property $(2,4)$ and $(2,5)$ were considered there. This notion was qeneralized in [**1**], where it was shown that the condition of the cancellativity is superfluous, that is, any variety of groupoids with the property $(2,n)$ is a variety of quasigroups.

Let k and n be two positive integers and $k \leqslant n$. An algebra $\mathbf{A}$ is said to have the property (k,n) if every subalgebra of $\mathbf{A}$ generated by k distinct elements has exactly n elements. We also say that $\mathbf{A}$ is a (k,n)-algebra. A class $\mathcal{K}$ of algebras is said to be a (k,n)-class if every algebra in $\mathcal{K}$ is a (k,n)-algebra. A variety is called a (k,n)-variety if it is a (k,n)-class of algebras.

Trivially, the variety of Steiner quasigroups ($xx = x, xy = yx, x\cdot xy = y$) is a $(2,3)$-variety. It is the unique variety of groupoids with the stated property, and the same holds for the $(2,4)$-variety ($x{\cdot}xy = yx, xy{\cdot}yx = x$) given by Padmanabhan in [**4**]. He has also constructed two $(2,5)$-varieties. One of them is commutative ($xy = yx, x(y{\cdot}xy) = y, x(x{\cdot}xy) = y{\cdot}xy$), while the other one ($x{\cdot}xy = y, xy{\cdot}y = yx$) consists of anticommutative quasigroups. These two varieties together with the variety whose defining identities ($x{\cdot}xy = yx, xy{\cdot}y = x$) are dual to the identities of the preceeding variety are the only $(2,5)$-varieties of groupoids. The non-existence

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of a $(2,6)$-variety can be deduced from the correspondence between the (k,n)-varieties and Steiner systems $S(k,n,v)$ [**1**].

Here we present $(2,n)$-varieties of groupoids for $n = 7, 8$ and 9. Their construction is given in Sections 2, 3 and 4 respectively. It is an open problem the existence of $(2,n)$-varieties for $n \geqslant 10$, as well as the answer of the question whether the set of integers $\{n \mid \text{There exists a } (2,n)\text{-variety of groupoids}\}$ is finite.

2. A construction of (2,7)-variety of groupoids

We use the fact that any member of a $(2,n)$-variety of groupoids is a quasigroup, i.e., the choosing of the defining identities of the $(2,7)$-variety $\mathcal{V}_7$ (as well as the varieties $\mathcal{V}_8$ and $\mathcal{V}_9$ in the next sections) is made in a manner that enables a variety of quasigroups to be obtained.

THEOREM 2.1. *Let $\mathcal{V}_7$ be a variety of groupoids, defined by the identities:*

$$(1)\ xy = yx, \quad (2)\ x(x\cdot xy) = y, \quad (3)\ xy\cdot(y\cdot xy) = y(x\cdot xy)$$

Then $\mathcal{V}_7$ is a $(2,7)$-variety of quasigroups.

PROOF. Let $(G,\cdot)$ be arbitrary groupoid in $\mathcal{V}_7$ and $a, b \in G$. Since $ab = ac \implies b = a(a\cdot ab) = a(a\cdot ac) = c$, $x = a\cdot ab$ is the unique solution of the equation $ax = b$. By commutativity $ax = xa$ we have that $(G,\cdot)$ is a quasigroup.

Next we show that the following identities also hold in $(G,\cdot)$. (The commutativity will not be pointed out when used.)

$$
\begin{array}{llll}
(4) & xx = x, & (8) & (x\cdot xy)(y\cdot xy) = xy,\\
(5) & x\cdot x(y\cdot xy) = y(x\cdot xy), & (9) & (x\cdot xy)\cdot x(y\cdot xy) = y\cdot xy,\\
(6) & x\cdot y(x\cdot xy) = y\cdot xy, & (10) & (x\cdot xy)\cdot y(x\cdot xy) = x,\\
(7) & xy\cdot x(y\cdot xy) = x, & (11) & x(y\cdot xy)\cdot y(x\cdot xy) = xy.
\end{array}
$$

Namely, we have the following transformations:

$$
\begin{aligned}
xx &\overset{(2)}{=} x(xx\cdot(xx\cdot(xx\cdot x))) \overset{(3)}{=} x(xx\cdot x(x\cdot xx)) \overset{(2)}{=} x(xx\cdot x) \overset{(2)}{=} x;\\
x\cdot x(y\cdot xy) &\overset{(3)}{=} x(xy\cdot(x\cdot xy)) \overset{(3)}{=} (x\cdot xy)\cdot x(x\cdot xy) \overset{(2)}{=} (x\cdot xy)y;\\
x\cdot y(x\cdot xy) &\overset{(5)}{=} x(x\cdot x(y\cdot xy)) \overset{(2)}{=} y\cdot xy;\\
xy\cdot x(y\cdot xy) &\overset{(3)}{=} xy\cdot(xy\cdot(x\cdot xy)) \overset{(2)}{=} x;\\
(x\cdot xy)(y\cdot xy) &\overset{(2)}{=} (x\cdot xy)(x(x\cdot xy)\cdot xy) \overset{(7)}{=} xy;\\
(x\cdot xy)\cdot x(y\cdot xy) &\overset{(3)}{=} (x\cdot xy)(xy\cdot(x\cdot xy)) \overset{(3)}{=} xy\cdot x(x\cdot xy) \overset{(2)}{=} xy\cdot y;\\
(x\cdot xy)\cdot y(x\cdot xy) &\overset{(5)}{=} (x\cdot xy)\cdot x(x(y\cdot xy)) \overset{(3)}{=} (x\cdot xy)\cdot x(xy\cdot(x\cdot xy)) \overset{(7)}{=} x;\\
x(y\cdot xy)\cdot y(x\cdot xy) &\overset{(3)}{=} (xy\cdot(x\cdot xy))(xy\cdot(y\cdot xy)) \overset{(9)}{=}\\
&\overset{(9)}{=} (xy\cdot(x\cdot xy))(xy\cdot((x\cdot xy)\cdot x(y\cdot xy))) \overset{(3)}{=}\\
&\overset{(3)}{=} (xy\cdot(x\cdot xy))(xy\cdot((x\cdot xy)(xy\cdot(x\cdot xy)))) \overset{(7)}{=} xy.
\end{aligned}
$$

Therefore, the multiplication table of any subquasigroup of a quasigroup in $\mathcal{V}_7$, generated by the elements x and y $(x \neq y)$, is the following one:

	x	y	xy	$x{\cdot}xy$	$y{\cdot}xy$	$x(y{\cdot}xy)$	$y(x{\cdot}xy)$
x	x	xy	$x{\cdot}xy$	y	$x(y{\cdot}xy)$	$y(x{\cdot}xy)$	$y{\cdot}xy$
y	xy	y	$y{\cdot}xy$	$y(x{\cdot}xy)$	x	$x{\cdot}xy$	$x(y{\cdot}xy)$
xy	$x{\cdot}xy$	$y{\cdot}xy$	xy	$x(y{\cdot}xy)$	$y(x{\cdot}xy)$	x	y
$x{\cdot}xy$	y	$y(x{\cdot}xy)$	$x(y{\cdot}xy)$	$x{\cdot}xy$	xy	$y{\cdot}xy$	x
$y{\cdot}xy$	$x(y{\cdot}xy)$	x	$y(x{\cdot}xy)$	xy	$y{\cdot}xy$	y	$x{\cdot}xy$
$x(y{\cdot}xy)$	$y(x{\cdot}xy)$	$x{\cdot}xy$	x	$y{\cdot}xy$	y	$x(y{\cdot}xy)$	xy
$y(x{\cdot}xy)$	$y{\cdot}xy$	$x(y{\cdot}xy)$	y	x	$x{\cdot}xy$	xy	$y(x{\cdot}xy)$

In order to complete the proof, it suffices to show that the elements x, y, xy, $x{\cdot}xy$, $y{\cdot}xy$, $x(y{\cdot}xy)$, $y(x{\cdot}xy)$ are distinct:

$$\begin{aligned}
x = xy &\implies xx = xy \implies x = y\\
x = x{\cdot}xy &\implies xx = x{\cdot}xy \implies x = xy\\
x = y{\cdot}xy &\implies xy = (y{\cdot}xy)y \implies xy = x\\
x = x(y{\cdot}xy) &\implies xx = x(y{\cdot}xy) \implies x = y{\cdot}xy\\
x = y(x{\cdot}xy) &\implies y(y{\cdot}xy) = y(x{\cdot}xy) \implies y{\cdot}xy = x{\cdot}xy \implies x = y\\
xy = x{\cdot}xy &\implies y = xy\\
xy = x(y{\cdot}xy) &\implies y = y{\cdot}xy\\
x{\cdot}xy = y{\cdot}xy &\implies x = y\\
x{\cdot}xy = x(y{\cdot}xy) &\implies xy = y{\cdot}xy\\
x{\cdot}xy = y(x{\cdot}xy) &\implies x(x{\cdot}xy) = x(y(x{\cdot}xy)) \implies y = y{\cdot}xy\\
x(y{\cdot}xy) = y(x{\cdot}xy) &\implies x(y{\cdot}xy) = xy{\cdot}(y{\cdot}xy) \implies x = xy.
\end{aligned}$$

□

3. A construction of (2,8)-variety of groupoids

THEOREM 3.1. *Let $\mathcal{V}_8$ be the variety of groupoids, defined by the identities:*

$$(1)\ x{\cdot}xy = xy{\cdot}y, \quad (2)\ x{\cdot}yx = xy{\cdot}x, \quad (3)\ x(y{\cdot}yx) = y.$$

Then $\mathcal{V}_8$ is a $(2, 8)$-variety of quasigroups.

PROOF. First we show that the following identities are satisfied by any $\mathcal{V}_8$-groupoid:

(4)	$x(x{\cdot}xy) = yx,$	(16)	$(x{\cdot}xy){\cdot}xy = yx,$
(5)	$xx = x,$	(17)	$xy{\cdot}(yx{\cdot}y) = yx,$
(6)	$xy{\cdot}yx = x,$	(18)	$(y{\cdot}yx){\cdot}xy = x,$
(7)	$(xy{\cdot}x)x = y,$	(19)	$(xy{\cdot}x){\cdot}xy = y{\cdot}yx,$
(8)	$x(xy{\cdot}x) = yx{\cdot}y,$	(20)	$(yx{\cdot}y){\cdot}xy = xy{\cdot}x,$

(9) $x(yx{\cdot}y) = y{\cdot}yx,$ (21) $(x{\cdot}xy)(y{\cdot}yx) = xy{\cdot}x,$

(10) $(x{\cdot}xy)x = yx{\cdot}y,$ (22) $(x{\cdot}xy)(xy{\cdot}x) = x,$

(11) $(y{\cdot}yx)x = x{\cdot}xy,$ (23) $(x{\cdot}xy)(yx{\cdot}y) = xy,$

(12) $(yx{\cdot}y)x = xy,$ (24) $(y{\cdot}yx)(x{\cdot}xy) = yx{\cdot}y,$

(13) $xy{\cdot}(x{\cdot}xy) = y{\cdot}yx,$ (25) $(yx{\cdot}y)(x{\cdot}xy) = y,$

(14) $xy{\cdot}(y{\cdot}yx) = yx{\cdot}y,$ (26) $(xy{\cdot}x)(yx{\cdot}y) = x{\cdot}xy,$

(15) $xy{\cdot}(xy{\cdot}x) = y.$

Namely, we have:

$$x(x{\cdot}xy) \overset{(3)}{=} y(x{\cdot}xy){\cdot}(x{\cdot}xy) \overset{(1)}{=} y{\cdot}y(x{\cdot}xy) \overset{(3)}{=} yx;$$

$$xx \overset{(4)}{=} x(x{\cdot}xx) \overset{(3)}{=} x;$$

$$xy{\cdot}yx \overset{(4)}{=} xy{\cdot}(x{\cdot}(x{\cdot}xy)) \overset{(3)}{=} x;$$

$$(xy{\cdot}x)x \overset{(2)}{=} (x{\cdot}yx)x \overset{(2)}{=} x(yx{\cdot}x) \overset{(1)}{=} x(y{\cdot}yx) \overset{(3)}{=} y;$$

$$\begin{aligned} x(xy{\cdot}x) &\overset{(2)}{=} x(x{\cdot}yx) \overset{(1)}{=} (x{\cdot}yx){\cdot}yx \overset{(4)}{=} yx{\cdot}(yx{\cdot}(yx{\cdot}(x{\cdot}yx))) \\ &\overset{(2)}{=} yx{\cdot}(yx{\cdot}((yx{\cdot}x){\cdot}yx)) \overset{(1)}{=} yx{\cdot}(yx{\cdot}((y{\cdot}yx){\cdot}yx)) \\ &\overset{(2)}{=} yx{\cdot}((yx{\cdot}(y{\cdot}yx)){\cdot}yx) \overset{(2)}{=} yx{\cdot}(((yx{\cdot}y){\cdot}yx){\cdot}yx) \overset{(7)}{=} yx{\cdot}y; \end{aligned}$$

$$x(yx{\cdot}y) \overset{(8)}{=} x{\cdot}x(xy{\cdot}x) \overset{(2)}{=} x{\cdot}x(x{\cdot}yx) \overset{(4)}{=} yx{\cdot}x \overset{(1)}{=} y{\cdot}yx;$$

$$(x{\cdot}xy)x \overset{(2)}{=} x(xy{\cdot}x) \overset{(8)}{=} yx{\cdot}y;$$

$$(y{\cdot}yx)x \overset{(4)}{=} x(x{\cdot}x(y{\cdot}yx)) \overset{(3)}{=} x{\cdot}xy;$$

$$(yx{\cdot}y)x \overset{(4)}{=} x(x{\cdot}x(yx{\cdot}y)) \overset{(9)}{=} x{\cdot}x(y{\cdot}yx) \overset{(3)}{=} xy;$$

$$xy{\cdot}(x{\cdot}xy) \overset{(1)}{=} xy{\cdot}(xy{\cdot}y) \overset{(1)}{=} (xy{\cdot}y)y \overset{(1)}{=} (x{\cdot}xy)y \overset{(11)}{=} y{\cdot}yx;$$

$$xy{\cdot}(y{\cdot}yx) \overset{(4)}{=} (y(y{\cdot}yx))(y{\cdot}yx) \overset{(1)}{=} y{\cdot}y(y{\cdot}yx) \overset{(4)}{=} yx{\cdot}y;$$

$$xy{\cdot}(xy{\cdot}x) \overset{(1)}{=} (xy{\cdot}x)x \overset{(7)}{=} y;$$

$$(x{\cdot}xy){\cdot}xy \overset{(1)}{=} x(x{\cdot}xy) \overset{(4)}{=} yx;$$

$$xy{\cdot}(yx{\cdot}y) \overset{(2)}{=} xy{\cdot}(y{\cdot}xy) \overset{(2)}{=} (xy{\cdot}y){\cdot}xy \overset{(1)}{=} (x{\cdot}xy){\cdot}xy \overset{(16)}{=} yx;$$

$$(y{\cdot}yx){\cdot}xy \overset{(13)}{=} (xy{\cdot}(x{\cdot}xy)){\cdot}xy \overset{(2)}{=} ((xy{\cdot}x){\cdot}xy){\cdot}xy \overset{(7)}{=} x;$$

$$(xy{\cdot}x){\cdot}xy \overset{(2)}{=} xy{\cdot}(x{\cdot}xy) \overset{(13)}{=} y{\cdot}yx;$$

$$(yx{\cdot}y){\cdot}xy \overset{(14)}{=} (xy{\cdot}(y{\cdot}yx)){\cdot}xy \overset{(2)}{=} xy{\cdot}((y{\cdot}yx){\cdot}xy) \overset{(18)}{=} xy{\cdot}x;$$

$$(x{\cdot}xy)(y{\cdot}yx) \overset{(11)}{=} ((y{\cdot}yx)x)(y{\cdot}yx) \overset{(2)}{=} (y{\cdot}yx){\cdot}x(y{\cdot}yx) \overset{(3)}{=} (y{\cdot}yx)y \overset{(10)}{=} xy{\cdot}x;$$

$(x{\cdot}xy)(xy{\cdot}x) \overset{(6)}{=} x;$

$(x{\cdot}xy)(yx{\cdot}y) \overset{(19)}{=} ((yx{\cdot}y){\cdot}yx)(yx{\cdot}y) \overset{(19)}{=} y(y{\cdot}yx) \overset{(4)}{=} xy;$

$(y{\cdot}yx)(x{\cdot}xy) \overset{(13)}{=} (y{\cdot}yx)(yx{\cdot}(y{\cdot}yx)) \overset{(2)}{=} (y{\cdot}yx)((yx{\cdot}y){\cdot}yx) \overset{(17)}{=} yx{\cdot}y;$

$(yx{\cdot}y)(x{\cdot}xy) \overset{(10)}{=} ((x{\cdot}xy)x)(x{\cdot}xy) \overset{(19)}{=} xy{\cdot}(xy{\cdot}x) \overset{(15)}{=} y;$

$(xy{\cdot}x)(yx{\cdot}y) \overset{(8)}{=} (xy{\cdot}x)(x(xy{\cdot}x)) \overset{(2)}{=} (xy{\cdot}x)((x{\cdot}xy)x) \overset{(17)}{=} x{\cdot}xy.$

In each groupoid in $\mathcal{V}_8$, the equations $ax = b$ and $ya = b$ have solutions $x = b{\cdot}ba$ and $y = ab{\cdot}a$. Moreover, the cancellation laws hold:

$$ac = ad \implies c = (ac{\cdot}a)a = (ad{\cdot}a)a = d,$$
$$ca = da \implies c = a(c{\cdot}ca) = a(ca{\cdot}a) = a(da{\cdot}a) = a(d{\cdot}da) = d.$$

Hence, $\mathcal{V}_8$ is a variety of quasigroups.

The multiplication table of the subquasigroup of a quasigroup in $\mathcal{V}_8$, generated by the set $\{x, y\}$, is given as follows:

	x	y	xy	yx	$x{\cdot}xy$	$y{\cdot}yx$	$xy{\cdot}x$	$yx{\cdot}y$
x	x	xy	$x{\cdot}xy$	$xy{\cdot}x$	yx	y	$yx{\cdot}y$	$y{\cdot}yx$
y	yx	y	$yx{\cdot}y$	$y{\cdot}yx$	x	xy	$x{\cdot}xy$	$xy{\cdot}x$
xy	$xy{\cdot}x$	$x{\cdot}xy$	xy	x	$y{\cdot}yx$	$yx{\cdot}y$	y	yx
yx	$y{\cdot}yx$	$yx{\cdot}y$	y	yx	$xy{\cdot}x$	$x{\cdot}xy$	xy	x
$x{\cdot}xy$	$yx{\cdot}y$	$y{\cdot}yx$	yx	y	$x{\cdot}xy$	$xy{\cdot}x$	x	xy
$y{\cdot}yx$	$x{\cdot}xy$	$xy{\cdot}x$	x	xy	$yx{\cdot}y$	$y{\cdot}yx$	yx	y
$xy{\cdot}x$	y	yx	$y{\cdot}yx$	$yx{\cdot}y$	xy	x	$xy{\cdot}x$	$x{\cdot}xy$
$yx{\cdot}y$	xy	x	$xy{\cdot}x$	$x{\cdot}xy$	y	yx	$y{\cdot}yx$	$yx{\cdot}y$

All of the elements in the multiplication table are distinct:

$$x = xy \implies xx = xy \implies x = y;$$
$$x = x{\cdot}xy \implies xx = x{\cdot}xy \implies x = xy;$$
$$x = y{\cdot}yx \implies xx = x(y{\cdot}yx) \implies x = y;$$
$$x = xy{\cdot}x \implies xx = xy{\cdot}x \implies x = xy;$$
$$x = yx{\cdot}y \implies xy = (yx{\cdot}y)y \implies xy = x;$$
$$xy = yx \implies xy{\cdot}yx = yx{\cdot}yx \implies x = yx;$$
$$xy = x{\cdot}xy \implies y = xy;$$
$$xy = y{\cdot}yx \implies x{\cdot}xy = x(y{\cdot}yx) \implies x{\cdot}xy = y;$$
$$xy = xy{\cdot}x \implies xy{\cdot}xy = xy{\cdot}x \implies xy = x;$$
$$xy = yx{\cdot}y \implies xy{\cdot}y = (yx{\cdot}y)y \implies x{\cdot}xy = x;$$
$$x{\cdot}xy = y{\cdot}yx \implies x(x{\cdot}xy) = x(y{\cdot}yx) \implies yx = y;$$
$$x{\cdot}xy = xy{\cdot}x \implies x{\cdot}xy = x{\cdot}yx \implies xy = yx;$$
$$x{\cdot}xy = yx{\cdot}y \implies x{\cdot}xy = y{\cdot}xy \implies x = y;$$
$$xy{\cdot}x = yx{\cdot}y \implies (xy{\cdot}x)x = (yx{\cdot}y)x \implies y = xy.$$

□

4. A construction of (2,9)-variety of groupoids

THEOREM 4.1. *Let $\mathcal{V}_9$ be the variety of groupoids defined by the identities*

$$(1)\ \ x{\cdot}xy = yx, \quad (2)\ \ xy{\cdot}(y{\cdot}xy) = x.$$

Then $\mathcal{V}_9$ is a $(2,9)$-variety of quasigroups.

PROOF. One can check that the identities (3)–(30), given below, are satisfied by any groupoid in $\mathcal{V}_9$. We emphasis the identities that can be applied to the left-hand side of each equality in order to obtain its right-hand side.

Identity	Left-hand side = Right-hand side	Applyed identities
(3)	$xy{\cdot}x = x{\cdot}yx$	$(1),(1)$
(4)	$xx = x$	$(1),(1),(3),(2)$
(5)	$(x{\cdot}yx){\cdot}xy = xy{\cdot}yx$	$(3),(3),(1)$
(6)	$xy{\cdot}(x{\cdot}yx) = yx$	$(3),(1),(1)$
(7)	$(xy{\cdot}y){\cdot}xy = x$	$(3),(2)$
(8)	$(x{\cdot}yx){\cdot}yx = y{\cdot}xy$	$(1),(2),(3)$
(9)	$(x{\cdot}yx)y = x$	$(2),(3),(7)$
(10)	$yx{\cdot}x = xy{\cdot}y$	$(1),(2),(3),(2)$
(11)	$(x{\cdot}yx)x = yx{\cdot}xy$	$(3),(10),(1)$
(12)	$x(xy{\cdot}y) = yx{\cdot}xy$	$(10),(3),(11)$
(13)	$xy{\cdot}(yx{\cdot}xy) = y$	$(1),(1),(2)$
(14)	$(xy{\cdot}yx){\cdot}yx = xy{\cdot}y$	$(1),(13),(10)$
(15)	$(xy{\cdot}y)x = y{\cdot}xy$	$(10),(10),(8)$
(16)	$(xy{\cdot}y)(x{\cdot}yx) = xy{\cdot}yx$	$(15),(10),(1),(12)$
(17)	$x(yx{\cdot}xy) = y{\cdot}xy$	$(12),(1),(15)$
(18)	$x(y{\cdot}xy) = xy{\cdot}yx$	$(2),(10),(8),(5)$
(19)	$x(xy{\cdot}yx) = y$	$(18),(1),(9)$
(20)	$(xy{\cdot}yx)x = xy$	$(1),(19)$
(21)	$(x{\cdot}yx)(xy{\cdot}yx) = yx$	$(5),(1),(6)$
(22)	$(xy{\cdot}yx)(x{\cdot}yx) = y{\cdot}xy$	$(1),(21),(8)$
(23)	$(xy{\cdot}yx)y = yx{\cdot}xy$	$(1),(17),(18)$
(24)	$(xy{\cdot}yx)(yx{\cdot}xy) = x{\cdot}yx$	$(23),(1),(17)$
(25)	$(x{\cdot}yx)(xy{\cdot}y) = xy$	$(10),(1),(3),(2)$
(26)	$(xy{\cdot}yx)(y{\cdot}xy) = yx$	$(11),(3),(2)$
(27)	$(x{\cdot}yx)(yx{\cdot}xy) = xy{\cdot}y$	$(1),(26),(14),(10)$
(28)	$(xy{\cdot}y)(xy{\cdot}yx) = xy$	$(16),(1),(25)$
(29)	$(xy{\cdot}yx)(xy{\cdot}y) = x$	$(1),(28),(7)$
(30)	$(x{\cdot}yx)(y{\cdot}xy) = y$	$(8),(1),(3),(7)$

Next, we show that every groupoid in $\mathcal{V}_9$ is a quasigroup.

The equations $ax = b$ and $ya = b$ have solutions $x = ab{\cdot}ba$ and $y = b{\cdot}ab$ respectively, and they are unique. Namely, if $ac = b$ and $da = b$, we have that $c = ca{\cdot}(a{\cdot}ca) = (a{\cdot}ac)(ac{\cdot}a) = ab{\cdot}ba$ and $d = da{\cdot}(a{\cdot}da) = b{\cdot}ab$.

By the above identities, we have that a subquasigroup generated by two distinct elements x and y is represented by the following table, and one can check that all of the elements are distinct.

	x	y	xy	yx	$x\cdot yx$	$y\cdot xy$	$xy\cdot yx$	$yx\cdot xy$	$xy\cdot y$
x	x	xy	yx	$x\cdot yx$	$xy\cdot y$	$xy\cdot yx$	y	$y\cdot xy$	$yx\cdot xy$
y	yx	y	$y\cdot xy$	xy	$yx\cdot xy$	$xy\cdot y$	$x\cdot yx$	x	$xy\cdot yx$
xy	$x\cdot yx$	$xy\cdot y$	xy	$xy\cdot yx$	yx	x	$yx\cdot xy$	y	$y\cdot xy$
yx	$xy\cdot y$	$y\cdot xy$	$yx\cdot xy$	yx	y	xy	x	$xy\cdot yx$	$x\cdot yx$
$x\cdot yx$	$yx\cdot xy$	x	$xy\cdot yx$	$y\cdot xy$	$x\cdot yx$	y	yx	$xy\cdot y$	xy
$y\cdot xy$	y	$xy\cdot yx$	$x\cdot yx$	$yx\cdot xy$	x	$y\cdot xy$	$xy\cdot y$	xy	yx
$xy\cdot yx$	xy	$yx\cdot xy$	y	$xy\cdot y$	$y\cdot xy$	yx	$xy\cdot yx$	$x\cdot yx$	x
$yx\cdot xy$	$xy\cdot yx$	yx	$xy\cdot y$	x	xy	$x\cdot yx$	$y\cdot xy$	$yx\cdot xy$	y
$xy\cdot y$	$y\cdot xy$	$x\cdot yx$	x	y	$xy\cdot yx$	$yx\cdot xy$	xy	yx	$xy\cdot y$

□

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