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CANONICAL BIASSOCIATIVE GROUPOIDS

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Abstract. In the paper *Free biassociative groupoids*, the variety of biassociative groupoids (i.e., groupoids satisfying the condition: every subgroupoid generated by at most two elements is a subsemigroup) is considered and free objects are constructed using a chain of partial biassociative groupoids that satisfy certain properties. The obtained free objects in this variety are not canonical. By a *canonical groupoid* in a variety $\mathcal{V}$ of groupoids we mean a free groupoid $(R, *)$ in $\mathcal{V}$ with a free basis B such that the carrier R is a subset of the absolutely free groupoid $(T_B, \cdot)$ with the free basis B and $(tu \in R \Rightarrow t, u \in R \;\&\; t*u = tu)$. In the present paper, a canonical description of free objects in the variety of biassociative groupoids is obtained.

1. Preliminaries

Let $\boldsymbol{G} = (G, \cdot)$ be a groupoid and $a, b \in G$. We denote by $\langle a, b\rangle$ the subgroupoid of $\boldsymbol{G}$ generated by a, b and by $\langle a\rangle$ the subgroupoid generated by a. Clearly, $\langle a\rangle \subseteq \langle a, b\rangle$ and if $b \in \langle a\rangle$, then $\langle a, b\rangle = \langle a\rangle$; specially, $\langle a, a\rangle = \langle a\rangle$. The subgroupoids $\langle a, b\rangle$ and $\langle b, a\rangle$ are equal.

Let $a_1, a_2, \ldots, a_n$ be a finite sequence of elements in a groupoid $\boldsymbol{G}$. We denote by $a_1a_2\cdots a_n$ the product of the sequence $a_1, a_2, \ldots, a_n$ in $\boldsymbol{G}$ defined as follows:

i) if $n = 3$, then $a_1a_2a_3 \overset{\text{def}}{=} a_1(a_2a_3)$ and

ii) if $n \geqslant 3$, then $a_1a_2\cdots a_n \overset{\text{def}}{=} a_1(a_2\cdots a_n)$.

We call $a_1a_2\cdots a_n$ the *main product* of the sequence $a_1, a_2, \ldots, a_n$. If $n = 1$ and $n = 2$, then a_1 and a_1a_2 will also be called the main products of the sequences a_1 and a_1, a_2 respectively. If $c = a_1a_2\cdots a_n$, then we say that c is *presented* as a main product of the sequence $a_1, a_2, \ldots, a_n$.

Let $\boldsymbol{G}$ be a groupoid and $A \subseteq G$. If $\boldsymbol{Q}$ is the subgroupoid of $\boldsymbol{G}$ generated by A, i.e., $\boldsymbol{Q} = \langle A\rangle$, then $Q = \bigcup\{A_k : k \geqslant 0\}$, where $A_0 = A$, $A_{k+1} = A_k \cup A_kA_k$.

If $x \in Q$, then a *hierarchy* of x in $\boldsymbol{Q}$ is the nonnegative integer $\chi_{\boldsymbol{Q}}(x)$, defined by $\chi_{\boldsymbol{Q}}(x) = \min\{k \in \mathbb{N}_0 : x \in A_k\}$, where $\mathbb{N}_0$ is the set of nonnegative integers.

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In the sequel B will be an arbitrary nonempty set whose elements are called variables. By T_B we will denote the set of all groupoid terms over B in the signature $\cdot$. The terms are denoted by $t, u, v, \ldots, x, y, \cdots$ $\boldsymbol{T}_B = (T_B, \cdot)$ is the absolutely free groupoid with the free basis B, where the operation is defined by $(u, v) \mapsto uv$. The groupoid $\boldsymbol{T}_B$ is injective, i.e., if $x, y, v, w \in T_B$, then $xy = vw \Rightarrow x = v, y = w$; in other words the operation $\cdot$ is an injective mapping.

Note that $\boldsymbol{T}_B = \bigcup\{B_k : k \geqslant 0\}$, where $B_0 = B$, $B_{k+1} = B_k \cup B_k B_k$. The hierarchy $\chi : T_B \to \mathbb{N}_0$, defined by $\chi(t) = \min\{k \in \mathbb{N}_0 : t \in B_k\}$, for any $t \in T_B$, has the property:

$$\chi(tu) = 1 + \max\{\chi(t), \chi(u)\},$$

for all $t, u \in T_B$.

For any term $v \in T_B$ we define the *length* $|v|$ of v and the *set of subterms* $P(v)$ of v in the following way:

$$|b| = 1,\ |tu| = |t| + |u|\,;\ P(b) = \{b\},\ P(tu) = \{tu\} \cup P(t) \cup P(u),$$

for any $b \in B$ and $t, u \in T_B$.

2. Main biproducts

Let $t, u \in T_B$ and $\langle t, u\rangle$ be the subgroupoid of $\boldsymbol{T}_B$ generated by t, u:

$$\langle t, u\rangle = \{t, u, tt, tu, ut, uu, t(tt), t(tu), t(ut), t(uu), (tt)t, (tu)t, \cdots\}.$$

Each element x of $\langle t, u\rangle$ is a product of a finite sequence of elements $x_1, \ldots, x_n$ ($n \geqslant 1$), where each x_i is either t or u, i.e., $\{x_1, x_2, \ldots, x_n\} \subseteq \{t, u\}$. Any such product is constructed by the two generators t, u and therefore we call it a *binary product* or shortly *biproduct*.

Thus, if a term $x \in T_B$ is an element of $\langle t, u\rangle$, then we say that *x has a representation as a biproduct* (or shortly, *x is a biproduct*) *with the generating pair* $\{t, u\}$ and denote it by $x_{\langle t,u\rangle}$. (In this case we also say that x is the *carrier* of the biproduct $x_{\langle t,u\rangle}$.)

If $u = t$ or $u \in \langle t\rangle$, then $\langle t, u\rangle = \langle t\rangle$. In that case if $x \in \langle t\rangle$, we say again that x is a biproduct with the generator t and denote it by $x_{\langle t\rangle}$. Specially, $t \in \langle t\rangle$ and t has a representation as a biproduct with the generator t: $t_{\langle t\rangle} = t$. We say that $t_{\langle t\rangle}$ is a *trivial biproduct* of t. Since $t \in \langle t, u\rangle$ we have $t_{\langle t,u\rangle} = t$ and we say also that $t_{\langle t,u\rangle}$ is the trivial biproduct of t in $\langle t, u\rangle$.

If $t \notin \langle u\rangle$ and $u \notin \langle t\rangle$, then no two elements of the subgroupoid $\langle t, u\rangle$ are equal, since the groupoid $\boldsymbol{T}_B$ is injective. Therefore:

PROPOSITION 2.1. *If t, u, x are terms of T_B and x is such that $x \in \langle t, u\rangle$, $t \notin \langle u\rangle$ and $u \notin \langle t\rangle$, then x has a unique representation as a biproduct with the generating pair $\{t, u\}$.*

Note that a term of T_B may have representations as biproducts with different pairs of generators.

EXAMPLE 2.1. Let a, b be two distinct variables and x the term $((ab)b)(ab)$.

1) $x \in \langle x\rangle$, and thus $x_{\langle x\rangle} = x$ is the biproduct of x with the generator x.

2) Put $t = (ab)b$ and $u = ab$. Then $x \in \langle t, u\rangle$ and $x_{\langle t,u\rangle} = tu$ is the biproduct of x with the generating pair $\{t, u\}$.

3) If $u = ab$ and $v = b$, then $x \in \langle u, v\rangle$ and $x_{\langle u,v\rangle} = (uv)u$ is the biproduct of x with the generating pair $\{u, v\}$.

4) $x \in \langle a, b\rangle$ and thus $x_{\langle a,b\rangle} = ((ab)b)(ab)$ is the biproduct of x with the generating pair $\{a, b\}$.

(Note that there is no biproduct of x other than those enumerated above.)

A biproduct $x_{\langle t,u\rangle}$ of a term x is said to be *maximal* in $\boldsymbol{T}_B$ if and only if for any biproduct $x_{\langle \alpha,\beta\rangle}$ of x, the hierarchy $\chi_{\langle \alpha,\beta\rangle}(x)$ does not exceed the hierarchy $\chi_{\langle t,u\rangle}(x)$, i.e., $\chi_{\langle \alpha,\beta\rangle}(x) \leqslant \chi_{\langle t,u\rangle}(x)$.

Proposition 2.2. *Any term x of T_B has a finite number of representations as a biproduct in $\boldsymbol{T}_B$, i.e., $x \in T_B$ is the carrier of a finite number of biproducts in $\boldsymbol{T}_B$. Any term x of T_B is the carrier of maximal biproducts in $\boldsymbol{T}_B$.*

Proof. The length $|x|$ of any $x \in T_B$ is finite, and thus the set $P(x)$ of subterms of x is finite. As the generators of any biproduct of x are subterms of x, and the set of subterms $P(x)$ of x is finite, it follows that x has a finite number of biproducts. The set of nonnegative integers that are hierarchies of x (with respect to the pair of generators of all biproducts of x, including the pairs $\{t, t\} = \{t\}$) is finite, and thus it has the largest element. Therefore, there is the largest hierarchy of x, i.e., a maximal biproduct of x. □

Note that a given term x of T_B may have more than one maximal biproducts.

Example 2.2. Let $x = ((ab)b)(b^2(ab))$ (where a, b are variables). Put $t = ab$ and $u = b$. Then $x_{\langle t,u\rangle} = (tu)(u^2t)$ and $\chi_{\langle t,u\rangle}(x) = 3$. If we take $\{a, b\}$ as the generating pair, then $x_{\langle a,b\rangle} = ((ab)b)(b^2(ab))$ is a biproduct of x and $\chi_{\langle a,b\rangle}(x) = 3$. For all other biproducts $x_{\langle \alpha,\beta\rangle}$ one obtains that $\chi_{\langle \alpha,\beta\rangle}(x) \leqslant 3$. Thus, $x_{\langle t,u\rangle}$ and $x_{\langle a,b\rangle}$ are maximal biproducts of x.

Let $x = x_1x_2\cdots x_m$ be the main product of $x_1, x_2, \ldots, x_m$ in $\boldsymbol{T}_B$. If

$$\{x_1, x_2, \ldots, x_m\} \subseteq \{t, u\},$$

for some terms t, u of T_B, then we call $x_1x_2\cdots x_m$ the *main biproduct* of x in $\boldsymbol{T}_B$ with the generating pair $\{t, u\}$ and denote it by $x_{t,u}$. (If $u = t$, i.e., the generating "pair" is $\{t, t\}$, we write x_t instead of $x_{t,t}$.)

Below we will state some properties about main biproducts.

(1) Note that any term x of T_B has at least one main biproduct – the trivial one, x_x. If $x \in T_B \setminus B$, then $x = \alpha\beta$ for some $\alpha, \beta \in T_B$, and $x_{\alpha,\beta} = \alpha\beta$ is another main biproduct of x in $\boldsymbol{T}_B$.

(2) The hierarchy of a main biproduct $x_1x_2\cdots x_m$, with a generating pair $\{t, u\}$, equals $m - 1$. Therefore, if two main biproducts $x_1x_2\cdots x_m$ and $y_1y_2\cdots y_{m+k}$ are maximal biproducts of x in $\boldsymbol{T}_B$, then they have to satisfy $k = 0$ (or the hierarchies would differ) and $x_i = y_i$, for $1 \leqslant i \leqslant m$.

Proposition 2.3. *If $x \in T_B$ has two nontrivial main biproducts $x_{t,u}$ and $x_{v,w}$ in $\boldsymbol{T}_B$, then one generator of the one generating pair coincides with a generator of the other generating pair.*

PROOF. Let $x_{t,u} = x_1x_2 \cdots x_m$ and $x_{v,w} = y_1y_2 \cdots y_n$ be two main biproducts of x in $\boldsymbol{T}_B$. Then $x_1x_2 \cdots x_m = y_1y_2 \cdots y_n$ implies $x_1 = y_1$. Since $x_\nu \in \{t,u\}$ and $y_\lambda \in \{v,w\}$ it follows that x_1 is either t or u, and y_1 is either v or w. If, for example, $x_1 = t$ and $y_1 = v$, then $v = t$ (and in that case $x_{t,u} = x_{t,w}$). □

Using the property (2) stated above, we obtain the following:

THEOREM 2.1. *If $x = x_1x_2 \cdots x_m$ and $x = x'_1x'_2 \cdots x'_n$ are main biproducts of x in $\boldsymbol{T}_B$ with the same generating pair $\{t,u\}$, then $m = n$ and $x_i = x'_i$, for $i = 1,2,\ldots,m$. Specially, any maximal biproduct of $x \in \boldsymbol{T}_B$, that is a main biproduct, is uniquely determined.*

3. A construction of canonical biassociative groupoids

A groupoid $\boldsymbol{G} = (G,\cdot)$ is said to be *biassociative* [**1**] if and only if for any $a,b \in G$ the subgroupoid S of $\boldsymbol{G}$ generated by $\{a,b\}$, i.e., $S = \langle a,b\rangle$, is a subsemigroup of $\boldsymbol{G}$. The class of all biassociative groupoids will be denoted by **Bass**. This class is hereditary and closed under the formation of homomorphic images and direct products, i.e., **Bass** is a variety of groupoids.

Assuming that B is a nonempty set and $\boldsymbol{T}_B = (T_B,\cdot)$ the absolutely free groupoid with the free basis B, we are looking for a *canonical groupoid* in **Bass**, i.e., a groupoid $\boldsymbol{R} = (R,*)$ with the following properties:

i) $B \subset R \subset T_B$; ii) $tu \in R \Rightarrow t,u \in R$; iii) $tu \in R \Rightarrow t*u = tu$

iv) $\boldsymbol{R}$ is a free groupoid in **Bass** with the free basis B.

A "candidate" for the carrier R of the desired groupoid $\boldsymbol{R}$ is the set defined by:

(3.1) $R = \{x \in T_B :$ every biproduct of any subterm of x is a main biproduct$\}$.

The following properties of R are obvious corollaries of (3.1).

PROPOSITION 3.1. a) *R satisfies* i) *and* ii).

b) $x,y \in R \Rightarrow \{xy \notin R \Leftrightarrow xy$ *has a biproduct that is not a main biproduct in* $\boldsymbol{T}_B\}$.

c) $x,y \in T_B \Rightarrow \{xy \in R \Leftrightarrow x,y \in R \;\&$ *every biproduct of any subterm of xy in $\boldsymbol{T}_B$ is a main biproduct*$\}$.

LEMMA 3.1. *For any $x \in R$ there is a unique maximal biproduct of x in $\boldsymbol{T}_B$ that is a main biproduct.*

PROOF. *Existence.* By Proposition 2.2, any $x \in T_B$ has maximal biproducts in $\boldsymbol{T}_B$ and thus any $x \in R$ has maximal biproducts in $\boldsymbol{T}_B$. By the definition of R, every biproduct of any subterm of x is a main biproduct and therefore the maximal biproducts of x are main biproducts, too.

Uniqueness. Let $x \in R$ and $x_{\langle t,u\rangle}$, $x_{\langle v,w\rangle}$ be maximal biproducts of x in $\boldsymbol{T}_B$. Since $x \in R$, both maximal biproducts $x_{\langle t,u\rangle}$, $x_{\langle v,w\rangle}$ are main biproducts and we will denote them by $x_{t,u}$, $x_{v,w}$. Let $x = x_1x_2 \cdots x_m$ and $x = x'_1x'_2 \cdots x'_m x'_{m+1} \cdots x'_{m+k}$, $k \geqslant 0$, be the representations of x as main biproducts in $\langle t,u\rangle$ and $\langle v,w\rangle$, respectively. By the property (2) we have that

$$m-1 = \chi_{\langle t,u\rangle}(x_1x_2\cdots x_m) = \chi_{\langle v,w\rangle}(x'_1x'_2\cdots x'_{m+k}) = m+k-1,$$

which implies that $k = 0$ and that $x_i = x'_i$, for $1 \leqslant i \leqslant m$. Therefore, the maximal biproducts $x_{t,u}$ and $x_{v,w}$ are in fact the same biproduct. $\square$

Bellow, for $x \in R$, we will denote by $x = x_1x_2\cdots x_m$ the maximal main biproduct of x in $\boldsymbol{T}_B$ (if it is not stated otherwise).

LEMMA 3.2. *Let $x \in R$, let the maximal biproduct of x be generated by $\{t,u\}$, and let another biproduct of x be generated by $\{v,w\}$. Then $v,w \in \langle t,u\rangle$.*

PROOF. Let $x = x_1\cdots x_m$ be the maximal biproduct of x generated by $\{t,u\}$ and let $x = x'_1\cdots x'_n$ be another biproduct of x generated by $\{v,w\}$. By Proposition 2.3 we may put $t = v$. Both biproducts are equal and since $x \in R$, they are main biproducts. By Lemma 3.1, $n < m$, i.e., $m = n + k$, $k \geqslant 1$, so

$$x'_1\cdots x'_n = x_1\cdots x_n x_{n+1}\cdots x_{n+k}.$$

Using this facts, we obtain that $x'_i = x_i = t$, for $i \in \{1,\dots,n-1\}$. Clearly, $x'_n = w$ and $x_i \in \{t,u\}$, for $i \in \{n,\dots,n+k\}$. Therefore, $v,w \in \langle t,u\rangle$. $\square$

PROPOSITION 3.2. *Let $x,y \in R$ and the maximal biproducts $x = x_1x_2\cdots x_m$, $y = y_1y_2\cdots y_n$ have generating pairs $\{t,u\}$, $\{v,w\}$, respectively. Then $xy \in R$ if and only if* (a) *or* (b)*, where*

(a) *$y \notin \langle t,u\rangle$, and for any biproduct of x with a generating pair $\{t_1,u_1\}$, if $t_1,u_1 \in \langle v,w\rangle$, then $t_1 = u_1 = x$*

(b) *$y \in \langle t,u\rangle$ and $t = u = x \in B$.*

PROOF. Let $xy \in R$. There are two possible cases for y: 1) $y \notin \langle t,u\rangle$ and 2) $y \in \langle t,u\rangle$.

Case 1). Since $\{v,w\}$ is the generating pair for the biproduct $y = y_1y_2\cdots y_n$, and $y \notin \langle t,u\rangle$, we should consider the cases when some of the biproducts of x has a generating pair $\{t_1,u_1\}$, such that $t_1,u_1 \in \langle v,w\rangle$. Let $x = z_1z_2\cdots z_k$ be such a biproduct of x. Then $z_i \in \{t_1,u_1\} \subseteq \langle v,w\rangle$. The product $xy = (z_1z_2\cdots z_k)(y_1y_2\cdots y_n)$ will be a main biproduct only if $k = 1$, i.e., $x = z_1$, and $z_1 = v$ or $z_1 = w$. Since $x = z_1$ is generated by $\{t_1,u_1\}$, it follows that $t_1 = u_1 = x$.

Case 2). In this case xy has a biproduct with a generating pair $\{t,u\}$. xy is a main biproduct, since $xy \in R$ and, therefore $m = 1$, i.e., $x = x_1$. The maximal biproduct of x is generated by $\{t,u\}$, so $t = u = x$. Moreover, $x \in B$, because if $x \notin B$ (for example $x = ab$, i.e., $t = u = ab$), then the biproduct of xy generated by $\{a,b\}$ can not be a main biproduct, that contradicts the assumption that $xy \in R$.

For the converse, let (a) or (b) hold. If (b) holds, then it is clear that $xy \in R$.

Let (a) holds and suppose $xy \notin R$. From 1) we obtain that $x \in \langle v,w\rangle$. Therefore, there is a biproduct of x with a generating pair $\{v,w\}$. By Lemma 3.2 it follows that $v,w \in \langle t,u\rangle$, that contradicts the assumption that $y \notin \langle t,u\rangle$. $\square$

Now we define an operation $*$ on R as follows. Let $x,y \in R$, $x = x_1x_2\cdots x_m$, $y = y_1y_2\cdots y_n$ and put

$$x * y = \begin{cases} xy, & \text{if } xy \in R \\ x_1x_2\cdots x_my_1y_2\cdots y_n, & \text{if } xy \notin R. \end{cases} \tag{3.2}$$

The operation $*$ is well-defined, i.e., $\boldsymbol{R} = (R, *)$ is a groupoid. Namely, let $x, y \in R$. If $xy \in R$, then $x * y$ is a uniquelly determined element of R. If $xy \notin R$, then $z = x_1x_2 \cdots x_my_1y_2 \cdots y_n$ is a term of T_B that is a main biproduct. Clearly, every biproduct of any subterm of $x_1x_2 \cdots x_my_1y_2 \cdots y_n$ is a main biproduct. Therefore, by (3.1), $z \in R$. Since $x_1x_2 \cdots x_my_1y_2 \cdots y_n$ as a maximal biproduct in $\boldsymbol{T}_B$ is unique (by Lemma 3.1), it follows that $x * y$ is uniquely determined element of R in the case $xy \notin R$. Thus, $\boldsymbol{R} = (R, *)$ is a groupoid.

By (3.2) it follows directly that:

1°. If $xy \in R$, then $x, y \in R$ & $x * y = xy$ (i.e., $\boldsymbol{R}$ satisfies ii) and iii)).

2°. $(\forall x, y \in R)\ |x * y| = |x| + |y|$.

The following three properties of $\boldsymbol{R}$ (3°–5°) show that the groupoid $\boldsymbol{R} = (R, *)$ is free in **Bass** with the free basis B.

3°. $\boldsymbol{R} \in$ **Bass**.

Proof of 3°. We have to show that every subgroupoid of $\boldsymbol{R}$ generated by two elements is a subsemigroup of $\boldsymbol{R}$.

For this purpose, let $t, u \in R$ and $\langle t, u\rangle_*$ be the subgroupoid of $\boldsymbol{R}$ generated by $\{t, u\}$. According to the definition of $*$, any $x \in \langle t, u\rangle_*$ is a maximal biproduct with the generating pair $\{t, u\}$. Therefore, if $x, y, z \in \langle t, u\rangle_*$, then $x = x_1x_2 \cdots x_m$, $y = y_1y_2 \cdots y_n$, $z = z_1z_2 \cdots z_p$ $(x_i, y_j, z_k \in \{t, u\})$ and by (3.2):

$$(x * y) * z = x_1x_2 \cdots x_my_1y_2 \cdots y_nz_1z_2 \cdots z_p = x * (y * z),$$

i.e., the subgroupoid $\langle t, u\rangle_*$ is a subsemigroup of $\boldsymbol{R}$. Hence, $\boldsymbol{R} \in$ **Bass**.

4°. The set of primes in $\boldsymbol{R}$ coincides with B and generates $\boldsymbol{R}$.

(An element a in a groupoid $\boldsymbol{G} = (G, \cdot)$ is said to be *prime* in $\boldsymbol{G}$ if and only if $a \neq xy$, for any $x, y \in G$.)

Proof of 4°. If $b \in B$, then by (3.2) $b \neq x * y$, for all $x, y \in R$. Hence, every $b \in B$ is prime in $\boldsymbol{R}$. To show that no element of $R \setminus B$ is prime in $\boldsymbol{R}$, let $x \in T_B \setminus B$ be a term belonging to R. Then by (3.1), every biproduct of any subterm of x is a main biproduct, and thus the maximal biproduct of x in $\boldsymbol{T}_B$ is a main biproduct. Therefore, $x = x_1x_2 \cdots x_m$, where $m \geqslant 2$ (since $x \in T_B \setminus B$). Thus, $x = x_1 * (x_2 \cdots x_m)$, i.e., x is not prime in $\boldsymbol{R}$.

Let $\boldsymbol{Q}$ be the subgroupoid of $\boldsymbol{R}$ generated by B, $\boldsymbol{Q} = \langle B\rangle_*$. We will show that $R = Q$. Clearly, $Q \subseteq R$. To show that $R \subseteq Q$, let $x \in R$. If $x \in B$, then $x \in \langle B\rangle_* = Q$, i.e., $(x \in R \ \&\ |\,x\,| = 1 \Rightarrow x \in Q)$.

Suppose that $(x \in R \ \&\ |\,x\,| \leqslant k \Rightarrow x \in Q)$ is true. If $x \in R$ is such that $|\,x\,| = k + 1$, then $x = x_1x_2$ in $\boldsymbol{T}_B$ and $|\,x_1\,|, |\,x_2\,| \leqslant k$. By the inductive hypothesis we have $x_1, x_2 \in Q$, and since $\boldsymbol{Q}$ is a groupoid, it follows that $x = x_1x_2 = x_1 * x_2 \in Q$. Thus, $R \subseteq Q$. Therefore, $\boldsymbol{R} = \boldsymbol{Q} = \langle B\rangle_*$.

5°. If $\boldsymbol{G} \in$ **Bass** and $\lambda : B \to G$ is a mapping, then there is a homomorphism $\psi : \boldsymbol{R} \to \boldsymbol{G}$ that extends λ, i.e., $\psi(b) = \lambda(b)$, for all $b \in B$.

Proof of 5°. Let $\varphi : \boldsymbol{T}_B \to \boldsymbol{G}$ be the homomorphism that extends λ. Denote by ψ the restriction of φ on R (i.e., $\psi = \varphi|_R$). It suffices to show that

$$(\forall x, y \in R)\ \varphi(x * y) = \varphi(x)\varphi(y).$$

Let $x, y \in R$. If $xy \in R$, then $\varphi(x * y) = \varphi(xy) = \varphi(x)\varphi(y)$. If $xy \notin R$ (i.e., $x = x_1x_2 \cdots x_m$, $y = y_1y_2 \cdots y_n$, where $x_i, y_j \in \{t, u\}$ and $m \geqslant 2$) then using the fact: $(x_i, y_j \in \{t, u\} \Rightarrow \varphi(x_i), \varphi(y_j) \in \{\varphi(t), \varphi(u)\})$ we have

$$\begin{aligned}\varphi(x * y) &= \varphi(x_1 \cdots x_m y_1 \cdots y_n) = \varphi(x_1) \cdots \varphi(x_m)\varphi(y_1) \cdots \varphi(y_n)\\ &= [\boldsymbol{G} \in \mathbf{Bass}] = \varphi(x_1 \cdots x_m)\varphi(y_1 \cdots y_n) = \varphi(x)\varphi(y).\end{aligned}$$

So, the conditions i)–iv) at the beginning of this section are fulfilled and thus we proved the following

THEOREM 3.1. *The groupoid* $\boldsymbol{R} = (R, *)$, *defined by* (3.1) *and* (3.2) *is a canonical biassociative groupoid with a free basis* B.

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