PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 81(95) (2007), 69–78 DOI 102298/PIM0795069L

ANALYTIC EQUIVALENCE OF PLANE CURVE SINGULARITIES $y^n + x^\alpha y + x^\beta A(x) = 0$

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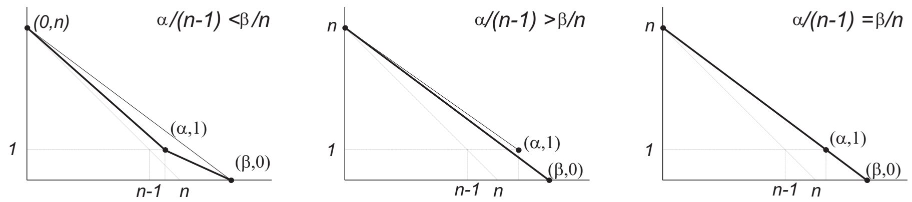
Abstract. There are not many examples of complete analytical classification of specific families of singularities, even in the case of plane algebraic curves. In 1989, Kang and Kim published a paper on analytical classification of plane curve singularities $y^n+a(x)y+b(x)=0$, or, equivalently, $y^n+x^\alpha y+x^\beta A(x)=0$ where $A(x)$ is a unit in $\mathbb{C}t\{x\}$, α and β are integers, $\alpha \geqslant n-1$ and $\beta \geqslant n$. The classification was not complete in the most difficult case $\frac{\alpha}{n-1} = \frac{\beta}{n}$. In the present paper, the classification is extended also in this case, the proofs are improved and some gaps are removed.

1. Introduction

In the theory of algebraic curves, the relationship between topological and analytical classifications of local germs is an old, interesting, and sometimes rather involved question [BK]. In the literature there are not many examples of curves and curve families with complete answer to this question. In 1989 Kang and Kim published a paper about topological and analytic classification of germs of plane curve singularities $f(x,y)=0$ defined by a square-free polynomial $f(x,y) = y^n + a(x)y + b(x)$ of multiplicity n [KK]. The problem is equivalent to classification of singularities defined by $y^n + x^\alpha y + x^\beta A(x)$, where $A(x)$ is a unit in $\mathbb{C}\{x\}$ (i.e., $A(0) = A_0 \neq 0$), α and β are integers, $\alpha \geqslant n-1$ and $\beta \geqslant n$. Recall that, if the singularity germs at $(0,0)$ defined by $f=0$ and $g=0$ are analytically equivalent [BK], we shall write $f \approx g$; else we shall write $f \not\approx g$. There are two cases depending on whether $\frac{\alpha}{n-1} \neq \frac{\beta}{n}$ (case A) or $\frac{\alpha}{n-1} = \frac{\beta}{n}$ (case B). The difference between cases A and B can be easily seen on Newton diagrams of these singularities [BK], [L] (see the figure).

If $\frac{\alpha}{n-1} < \frac{\beta}{n}$ the germ is reducible, if $\frac{\alpha}{n-1} > \frac{\beta}{n}$ it is irreducible. Kang and Kim found a list of singularities representing all analytic classes in the case A and partly in the case B. If $\frac{\alpha}{n-1} \neq \frac{\beta}{n}$, one can prove that $f \approx y^n + x^\alpha y + x^\beta$. However, in the case B there are two possibilities depending on the y-discriminant $D(f) = R(f, f')$ of f. This discriminant is of the form

2000 *Mathematics Subject Classification*: 14B05, 14H20, 32S15.



FIGURE

$$D(f) = cx^{(n-1)\beta}\Big[\Big(-\frac{1}{n}\Big)^n - \Big(\frac{A(x)}{n-1}\Big)^{n-1}\Big].$$

If $A(x) = A_0 + A_r x^r + \cdots$ $(A_0 \cdot A_r \neq 0)$, then

$$D(f) = cx^{(n-1)\beta}\Big[D_0 - \Big(\frac{A_0}{n-1}\Big)^{n-2} A_r x^r - \cdots\Big],$$

where

$$D_0 = \Big(\Big(-\frac{1}{n}\Big)^n - \Big(\frac{A_0}{n-1}\Big)^{n-1}\Big).$$

Since f is square-free, $D(x) \neq 0$. Now if $D_0 = 0$, then $D(f) = x^{(n-1)\beta+r}u(x)$ where $u(x)$ is a unit in $\mathbb{C}\{x\}$, and for each $r \in \mathbb{N}$ there is a single analytic class represented by the singularity $y^n - nx^\alpha y + (n-1)x^\beta + x^{\beta+r} = 0$ [KK]. If $D_0 \neq 0$, then $D(f) = x^{(n-1)\beta}u(x)$ where $u(x)$ is a unit in $\mathbb{C}\{x\}$. In the present paper we analyze this generic case B, and obtain results stated below.

Let $M(f)$ be the ideal in $\mathbb{C}\{x, y\}$ generated by f, $x\frac{\partial f}{\partial x}$, $y\frac{\partial f}{\partial x}$, $x\frac{\partial f}{\partial y}$ and $y\frac{\partial f}{\partial y}$, and let $\dim \mathbb{C}\{x, y\}/M(f)$ be the dimension of the algebra $\mathbb{C}\{x, y\}/M(f)$ as a vector space over $\mathbb{C}$.

THEOREM 1.1. *Let* $\frac{\alpha}{n-1} = \frac{\beta}{n}$. *If* $1 \leqslant r < \frac{\beta}{n}$, *then* $f \approx y^n + x^\alpha y + A_0 x^\beta + x^{\beta+r}$ *and* $\dim \mathbb{C}\{x, y\}/M(f) = \frac{\beta}{n}\big(n^2 - 2n + 2\big) - n + r + 4$. *For different* r *we have different analytic classes.*

THEOREM 1.2. *Let* $\frac{\alpha}{n-1} = \frac{\beta}{n}$, $n \geqslant 4$ *and let the discriminant of* f *be of the form* $D(f) = x^{(n-1)\beta}u(x)$, *where* $u(x)$ *is a unit in* $\mathbb{C}\{x\}$ *(generic case B). If* $1 \leqslant r < \frac{\beta}{n}$, *then* $f \not\approx y^n + x^\alpha y + cx^\beta$ *for all* $c \in \mathbb{C}$. *The same holds for* $n = 3$, *if* $1 \leqslant r < \frac{\beta}{n} - 1$.

If we omit the condition $r < \frac{\beta}{n}$ the situation becomes more complex, especially the calculation of the dimension d. Nevertheless we have the following.

THEOREM 1.3. *Let* $\frac{\alpha}{n-1} = \frac{\beta}{n}$ *and* $r \geqslant 1$. *Then* $f \approx y^n + x^\alpha y + A_0 x^\beta + x^{\beta+r}$. *Two such singularities (with the same* r*) are analytically isomorphic* $y^n + x^\alpha y + cx^\beta + x^{\beta+r} \approx y^n + x^\alpha y + dx^\beta + x^{\beta+r}$ *if and only if* $c^{n-1} = d^{n-1}$.

The question of analytical equivalence for different $r \geqslant \frac{\beta}{n}$ remains open.

2. Results and Proofs

For any series $f \in \mathbb{C}\{x,y\}$ let $M(f)$ denote the ideal $\left(f, y\frac{\partial f}{\partial y}, x\frac{\partial f}{\partial y}, y\frac{\partial f}{\partial x}, x\frac{\partial f}{\partial x}\right)$ in $\mathbb{C}\{x,y\}$. We shall use the following standard fact.

Theorem 2.1 (see [MY]). *$f \approx g$ if and only if $\mathbb{C}\{x,y\}/M(f)$ and $\mathbb{C}\{x,y\}/M(g)$ are isomorphic as $\mathbb{C}$-algebras.*

Throughout the paper we use the following notations. Let $f = y^n + x^\alpha y + x^\beta A(x) \in \mathbb{C}\{x,y\}$, where $A(0) = A_0 \neq 0$, α and β are integers, $\alpha \geqslant n-1$ and $\beta \geqslant n$. We shall consider the case B (see Introduction) where $\frac{\alpha}{n-1} = \frac{\beta}{n}$. Then $\alpha = k(n-1)$ and $\beta = kn$ for some $k \in \mathbb{N}$, so $f = y^n + x^{k(n-1)}y + A(x)x^{kn}$. Let $r \in \mathbb{N}_0$ be the multiplicity of $A(x) - A_0$ i.e., $A(x) = A_0 + A_r x^r + \cdots$ with $A_r \neq 0$.

Let us describe the algebra $\mathbb{C}\{x,y\}/M(f)$. First we describe the ideal $M(f) = \left(f, y\frac{\partial f}{\partial y}, x\frac{\partial f}{\partial y}, y\frac{\partial f}{\partial x}, x\frac{\partial f}{\partial x}\right)$. Note that $ky\frac{\partial f}{\partial y} + x\frac{\partial f}{\partial x} - knf = x^{kn+1}A'(x) \in M(f)$. We have to consider three essentially different cases: (i) $r = 0$; (ii) $1 \leqslant r < k$; and (iii) $r \geqslant k$.

Case (i) $r = 0$. Here $A(x) \equiv A_0 \neq 0$ and the y-discriminant of f is

$$D(f) = cx^{kn(n-1)}\left(\left(-\frac{1}{n}\right)^n - \left(\frac{A_0}{n-1}\right)^{n-1}\right).$$

Since it is $\neq 0$, we have

$$(A_0)^{n-1} \neq \left(-\frac{1}{n}\right)^n (n-1)^{n-1}. \tag{2.1}$$

The theorem of [KK] gives the criterion for analytic equivalence of two singularities of the considered type.

Theorem 2.2. *Let $n \geqslant 4$ and $c \in \mathbb{C}$ be such that $c^{n-1} \neq (-\frac{1}{n})^n (n-1)^{n-1}$. Then for $d \in \mathbb{C}$ we have $y^n + x^\alpha y + dx^\beta \approx y^n + x^\alpha y + cx^\beta$, if and only if $c^{n-1} = d^{n-1}$.*

Though this gives complete analytic classification in the case (i), in order to compare it with the two remaining types, we shall describe the algebra $\mathbb{C}\{x,y\}/M(f)$. One has

$$\begin{aligned} M(f) = \big(&ny^n + x^{k(n-1)}y,\ ny^{n-1}x + x^{k(n-1)+1}, \\ &(n-1)x^{k(n-1)}y + nx^{kn}A_0,\ (n-1)x^{k(n-1)-1}y^2 + nA_0x^{kn-1}y, f\big). \end{aligned}$$

Here f could be omitted, since

$$nf = (ny^n + x^{k(n-1)}y) + \left\{(n-1)x^{k(n-1)}y + nA_0x^{kn}\right\} \in \left(y\frac{\partial f}{\partial y}, x\frac{\partial f}{\partial y}, y\frac{\partial f}{\partial x}, x\frac{\partial f}{\partial x}\right).$$

We have $\frac{1}{k}y\frac{\partial f}{\partial x} \cdot y^{n-3} = (n-1)x^{kn-k-1}y^{n-1} + nA_0x^{kn-1}y^{n-2} \in M(f)$, so

$$x^{k(n-1)-1}y^{n-1} \equiv -\frac{nA_0}{n-1}x^{kn-1}y^{n-2} \bmod M(f). \tag{2.2}$$

Also, $\frac{1}{k}x\frac{\partial f}{\partial x} = (n-1)x^{k(n-1)}y + nA_0x^{kn} \in M(f)$, and therefore $x^{k(n-1)}y \equiv -\frac{nA_0}{n-1}x^{kn} \bmod M(f)$. Multiplying with $-\frac{nA_0}{n-1}x^{k-1}y^{n-3}$, $(-\frac{nA_0}{n-1})^2x^{2k-1}y^{n-4}$,..., $(-\frac{nA_0}{n-1})^{n-2}x^{(n-2)k-1}$ respectively, we obtain:

$$\begin{aligned}-\frac{nA_0}{n-1}x^{kn-1}y^{n-2} &\equiv \Big(-\frac{nA_0}{n-1}\Big)^2 x^{kn+k-1}y^{n-3}\\ &\equiv \cdots \equiv \Big(-\frac{nA_0}{n-1}\Big)^{n-1} x^{2kn-2k-1} \bmod M(f).\end{aligned} \tag{2.3}$$

Together with (2.2) this gives

$$x^{k(n-1)-1}y^{n-1} \equiv \Big(-\frac{nA_0}{n-1}\Big)^{n-1} x^{2kn-2k-1} \bmod M(f). \tag{2.4}$$

On the other hand, $\dfrac{1}{k}\,x\,\dfrac{\partial f}{\partial y}\cdot x^{k(n-1)-2} = ny^{n-1}x^{k(n-1)-1} + x^{2kn-2k-1} \in M(f)$, so we have

$$x^{k(n-1)-1}y^{n-1} \equiv -\frac{1}{n}x^{2kn-2k-1} \bmod M(f). \tag{2.5}$$

From (2.4) and (2.5) we have

$$\Big(-\frac{nA_0}{n-1}\Big)^{n-1} x^{2kn-2k-1} \equiv -\frac{1}{n}x^{2kn-2k-1} \bmod M(f)$$

and

$$\Big[\Big(-\frac{nA_0}{n-1}\Big)^{n-1} + \frac{1}{n}\Big]x^{2kn-2k-1} \equiv 0 \bmod M(f). \tag{2.6}$$

From (2.1) it follows that $x^{2kn-2k-1} \in M(f)$. Therefore the monomials $x^{kn-1}y^{n-2}$, $x^{kn+k-1}y^{n-3}$, $x^{kn+2k-1}y^{n-4}$,..., $x^{2kn-3k-1}y$ are in $M(f)$, and so is $x^{k(n-1)-1}y^{n-1}$ (see (2.3) and (2.4)).

From the definition of $M(f)$ it follows that $y^n \equiv -\frac{1}{n}x^{k(n-1)}y \bmod M(f)$, therefore $y^{2n-2} \equiv -\frac{1}{n}x^{k(n-1)}y^{n-1} \bmod M(f)$, and

$$y^{2n-2} \in M(f). \tag{2.7}$$

Multiplying $y^n \equiv -\frac{1}{n}x^{k(n-1)}y \bmod M(f)$ with $x^{k(n-2)-1}$, $x^{k(n-3)-1}y$,..., $x^{k-1}y^{n-3}$ respectively, we obtain that monomials $x^{k(n-2)-1}y^n$, $x^{k(n-3)-1}y^{n+1}$,..., $x^{k-1}y^{2n-3}$ are congruent with $-\frac{1}{n}x^{2kn-3k-1}y$, $-\frac{1}{n}x^{2kn-4k-1}y^2$,..., $-\frac{1}{n}x^{kn-1}y^{n-2}$ respectively, so:

$$x^{k(n-2)-1}y^n, x^{k(n-3)-1}y^{n+1}, \dots, x^{k-1}y^{2n-3} \in M(f). \tag{2.8}$$

Now we describe the algebra $\mathbb{C}\{x,y\}/M(f)$. It is finitely generated as a vector space over $\mathbb{C}$, for instance by the set of monomials $\big\{x^\alpha y^\beta \mid \alpha < 2kn-2k-1; \beta < 2n-2\big\}$ (for simplicity we identify monomials with their congruence classes $\bmod M(f)$). It is easy to see that all linear dependence relations in $\mathbb{C}\{x,y\}/M(f)$ between these monomials are finite linear combinations of the expressions obtained by multiplying $x\frac{\partial f}{\partial y}$, $y\frac{\partial f}{\partial y}$, $x\frac{\partial f}{\partial x}$, $y\frac{\partial f}{\partial x}$ with different x^py^q $(p,q \in \mathbb{N}_0)$. In such way we obtain linear combinations of monomials $x^\alpha y^\beta$ and $x^{\alpha+lk}y^{\beta-l}$ for some α and β, i.e., monomials $x^\alpha y^\beta$ and $x^{\alpha'}y^{\beta'}$ for which $\alpha + k\beta = \alpha' + k\beta'$, that lie in the set

$I_c := \{x^\alpha y^\beta \mid \alpha + k\beta = c\}$. So the algebra $\mathbb{C}\{x,y\}/M(f)$ splits as a vector space over $\mathbb{C}$ into direct sums of vector spaces V_c generated by I_c, $0 \leqslant c < 2kn - 2k - 1$, because $I_c = \{0\}$ for $c \geqslant 2kn-2k-1$ (this follows from (2.3), (2.4), (2.7) and (2.8):

$$\mathbb{C}\{x,y\}/M(f) = \bigoplus_{0\leqslant c\leqslant 2kn-2k-1} V_c.$$

LEMMA 2.1. *The number $2kn-2k-1$ is the least number a such that $x^a \in M(f)$.*

PROOF. Suppose that $x^a \in M(f)$ for some $a < 2kn-2k-1$; then $x^{2kn-2k-2} \in M(f)$. But this means that $x^{2kn-2k-2}$ is, in $\mathbb{C}\{x,y\}$, equal to a linear combination of the expressions of the type $x^\gamma y^\delta \frac{\partial f}{\partial y}$, $x^\alpha y^\beta \frac{\partial f}{\partial x}$, where $\gamma^2+\delta^2 \neq 0$, $\alpha^2+\beta^2 \neq 0$, $\gamma + k\delta = kn - k - 2$ and $\alpha + k\beta = kn - 2k - 1$. The linear combination must contain $x^{2kn-2k-2} + nx^{kn-k-2}y^{n-1}$ or $nx^{2kn-2k-2} + (n-1)x^{2kn-3k-2}y$. But the expression $x^{2kn-2k-2} + nx^{kn-k-2}y^{n-1}$ could not appear with any nonzero coefficient, because $x^{kn-k-2}y^{n-1}$ could not be eliminated since it does not appear in any other expression above. So the linear combination must contain $c[nx^{2kn-2k-2}+(n-1)x^{2kn-3k-2}y]$ with $c \neq 0$, and $x^{2kn-3k-2}y$ must be found in at least one of the two remaining terms which contain $x^{2kn-3k-2}y$: $x^{2kn-3k-2}y + nx^{kn-2k-2}y^n$ or $nx^{2kn-3k-2}y + (n-1)x^{2kn-4k-2}y^2$. Again $x^{kn-2k-2}y^n$ is not found in any other expression, so $nx^{2kn-3k-2}y + (n-1)x^{2kn-4k-2}y^2$ must be contained in a linear combination with a nonzero coefficient etc. We continue by induction, and finally we get that the term $x^{kn+k-2}y^{n-3}$ which is contained in $nx^{kn+2k-2}y^{n-4} + (n-1)x^{kn+k-2}y^{n-3}$ could be eliminated only by the expression $nx^{kn+k-2}y^{n-3} + (n-1)x^{kn-2}y^{n-2}$ taken with the appropriate nonzero coefficient. But then we get $x^{kn-2}y^{n-2}$ with a nonzero coefficient, and it is not found in any other of the above expressions. So it can not be eliminated, and we have a contradiction. □

This fact will help us to distinguish analytic types of the expressions in the case (i) from those found in the sequel.

Case (ii) $\boldsymbol{1 \leqslant r < k}$. Let $1 \leqslant r < k$. We have

$$\begin{aligned} M(f) \ni ky\frac{\partial f}{\partial y} + x\frac{\partial f}{\partial x} - knf = x^{kn+1}A'(x) = rA_r x^{kn+r} + (r+1)A_{r+1}x^{kn+r+1} + \cdots \\ = x^{kn+r}u(x), \end{aligned}$$

where $u(x)$ is a unit, so $x^{kn+r} \in M(f)$ and in the generating set of $M(f)$ we can replace f with x^{kn+r}. We have $(ny^{n-1}x + x^{k(n-1)+1})x^{k+r-1} = ny^{n-1}x^{k+r} + x^{kn+r} \in M(f)$, and we get $y^{n-1}x^{k+r} \in M(f)$ because $x^{kn+r} \in M(f)$. Now $x\frac{\partial f}{\partial x} \cdot x^r = k(n-1)x^{k(n-1)+r}y + x^{kn+r}(\dots)$, so we get $x^{k(n-1)+r}y \in M(f)$. Since $r < k$, $x^{kn-1}y, x^{kn}y \in M(f)$, so we have

$$\begin{aligned} y\frac{\partial f}{\partial x} = k(n-1)x^{k(n-1)-1}y^2 + knx^{kn-1}yA(x) + x^{kn}yA'(x) \\ \equiv k(n-1)x^{k(n-1)-1}y^2 (\operatorname{mod} M(f)), \end{aligned}$$

and we get $x^{k(n-1)-1}y^2 \in M(f)$. If we add $x^{k(n-1)+r}y$ and $x^{k(n-1)-1}y^2$ to the generating set of the ideal $M(f)$, we can omit $y\frac{\partial f}{\partial x}$. Also,

$$yf = y^{n+1} + x^{k(n-1)}y^2 + A(x)yx^{kn} \in M(f),$$

so $y^{n+1} \in M(f)$. We also have $y\frac{\partial f}{\partial y} \cdot x^r = ny^n x^r + x^{k(n-1)+r}y \in M(f)$, therefore $y^n x^r \in M(f)$. Finally,

$$xf'_x = k(n-1)x^{k(n-1)}y + knx^{kn}A(x) + x^{kn+1}A'(x) \equiv k(n-1)x^{k(n-1)}y + knx^{kn}A_0,$$

because $x^{kn+r} \in M(f)$, therefore we can take $k(n-1)x^{k(n-1)}y + knx^{kn}A_0$ instead of xf'_x in the generating set of $M(f)$. Finally, we have

$$\begin{aligned} M(f) = \big(&y^{n+1}, y^n x^r,\, y^{n-1}x^{k+r},\, x^{k(n-1)-1}y^2,\, x^{k(n-1)+r}y, x^{kn+1},\\ &ny^n + x^{k(n-1)}y, ny^{n-1}x + x^{k(n-1)+1}, (n-1)x^{k(n-1)}y + nx^{kn}A_0\big). \end{aligned}$$

The ideal $M(f)$ depends only on A_0 and r, and so does the algebra $\mathbb{C}\{x,y\}/M(f)$. Finally we have: $f \approx y^n + x^\alpha y + x^\beta\,(A_0 + x^r)$.

Now we can prove the following theorem.

Theorem 2.3. *If $1 \leqslant r < k$, then $f \approx y^n + x^\alpha y + A_0 x^\beta + x^{\beta+r}$. For different r we get different analytic classes, because*

$$\dim \mathbb{C}\{x,y\}/M(f) = k\left(n^2 - 2n + 2\right) - n + r + 4.$$

Proof. Algebra $\mathbb{C}\{x,y\}/M(f)$ is generated as a vector space over $\mathbb{C}$ by the following monomials:

$$\begin{array}{ccccccc}
1 & y & y^2 & \dots & y^{n-2} & y^{n-1} & y^n \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
x^r & x^r y & x^r y^2 & \dots & x^r y^{n-2} & x^r y^{n-1} & \\
\dots & \dots & \dots & \dots & \dots & \dots & \\
x^{k+r} & x^{k+r}y & x^{k+r}y^2 & \dots & x^{k+r}y^{n-2} & & \\
\dots & \dots & \dots & \dots & \dots & & \\
x^{k(n-1)-2} & x^{k(n-1)-2}y & x^{k(n-1)-2}y^2 & \dots & x^{k(n-1)-2}y^{n-2} & & \\
x^{k(n-1)-1} & x^{k(n-1)-1}y & & & & & \\
\dots & \dots & & & & & \\
x^{kn+r-k-1} & x^{kn+r-k-1}y & & & & & \\
x^{kn+r-k} & & & & & & \\
\dots & & & & & & \\
x^{kn+r-1} & & & & & &
\end{array}.$$

The total number of monomials in this generating set is

$$(n+1)r + kn + (kn - 2k - 1 - r)\cdot(n-1) + (r+1)2 + k = k(n^2 - 2n + 3) - n + 4r + 3.$$

On the basis of our previous considerations,

$$\begin{aligned} M(f) = \big(&x^{kn+r},\, x^{kn-k+r}y,\, x^{kn-k-1}y^2,\, x^{k+r}y^{n-1},\, x^r y^n, y^{n+1},\\ &ny^n + x^{k(n-1)}y,\, ny^{n-1}x + x^{k(n-1)+1},\, (n-1)x^{k(n-1)}y + nx^{kn}A_0\big). \end{aligned}$$

Multiplying xf'_y with monomials 1, $x,\dots,x^{k+r-2}$ respectively, we obtain $k+r-1$ linear relations between generating monomials

$$ny^{n-1}x + x^{k(n-1)+1} \equiv \dots \equiv ny^{n-1}x^{k+r-1} + x^{kn+r-1} \equiv 0 \bmod M(f).$$

Multiplying yf'_y with 1, $x,\dots, x^{r-1}$ we obtain another r linear relations

$$ny^n + x^{k(n-1)}y \equiv \dots \equiv ny^n x^{r-1} + x^{k(n-1)+r-1}y \equiv 0 \bmod M(f).$$

Finally, multiplying $(n-1)x^{k(n-1)}y+nx^{kn}A_0$ with $1,x,\dots, x^{r-1}$ we get the following r linear relations between the generating monomials

$$(n-1)x^{k(n-1)}y+nx^{kn}A_0 \equiv \dots \equiv (n-1)x^{k(n-1)+r-1}y+nx^{kn+r-1}A_0 \equiv 0 \bmod M(f).$$

It is easy to see that these $k+3r-1$ linear relations are independent, so

$$\begin{aligned}\dim \mathbb{C}\{x,y\}/M(f) &= k(n^2-2n+3) - n + 4r + 3 - (k+3r-1)\\ &= k(n^2-2n+2) - n + r + 4.\end{aligned}$$ □

On the basis of the description of algebra $\mathbb{C}\{x,y\}/M(f)$ given in this proof, and the description of the same algebra in the case (i), we can now prove the following theorem.

THEOREM 2.4. *Let $1 \leqslant r < k$ and $n \geqslant 4$. If the y-discriminant of f is of the form $D(f) = x^{(n-1)\beta}u(x)$, where $u(x)$ is unit in $\mathbb{C}\{x\}$, then $y^n + x^\alpha y + x^\beta A(x) \not\approx y^n + x^\alpha y + cx^\beta$ for every $c \in \mathbb{C}$. The same holds for $n = 3$, if we replace the condition $1 \leqslant r < k$ with the condition $1 \leqslant r < k-1$.*

PROOF. Consider the degrees of the finite number of (classes of) monomials which generate the algebra $\mathbb{C}\{x,y\}/M(f)$. They do not exceed the biggest of the following numbers: $n+r-1$, $k+r-1+n-1$, $k(n-1)+n-4$, $kn+r-k$, $kn+r-1$, which is, in the case $r<k$ and $n \geqslant 4$, less than $2kn-2k-2$ (also in the case $n=3$ and $r<k-1$). Therefore, every element of $\mathbb{C}\{x,y\}/M(f)$ is of nilpotent degree $< 2k(n-1)-1$. But in the case (i) where $A(x) \equiv c \neq 0$, we have an element x of nilpotent degree $= 2k(n-1)-1$. Therefore, $\mathbb{C}\{x,y\}/M(f)$ in the case (i) is not isomorphic to the same algebra in the case (ii). To complete the proof, we should prove that $y^n + x^\alpha y + x^\beta A(x) \not\approx y^n + x^\alpha y + cx^\beta$ for $c = 0$. If $c=0$, the ideal $M(f)$ is

$$M\big(y^n + x^{k(n-1)}y\big) = \big(y^n,\ x^{k(n-1)}y,\ x^{k(n-1)-1}y^2,\ ny^{n-1}x + x^{k(n-1)+1}\big).$$

Since $x^{k(n-1)-1}y^2 \in M(f)$, we have that $ny^{n-1}x^{k(n-1)-1} \in M(f)$. On the other hand,

$$\big(ny^{n-1}x + x^{k(n-1)+1}\big)x^{k(n-1)-2} = ny^{n-1}x^{k(n-1)-1} + x^{2k(n-1)-1} \in M(f),$$

so $x^{2k(n-1)-1} \in M(f)$. We shall prove that $2k(n-1)-1$ is the least number α such that $x^\alpha \in M(f)$. Let $x^\alpha \in M(f)$. In the same way as before, we conclude that x^α may be obtained as a linear combination of expressions we obtain multiplying $x\frac{\partial f}{\partial y}$, $y\frac{\partial f}{\partial y}$, $x\frac{\partial f}{\partial x}$, $y\frac{\partial f}{\partial x}$ with $x^p y^q$ for different $p, q \in \mathbb{N}_0$. But the single expression where x^α appears is $x^{\alpha-k(n-1)-1}\big(ny^{n-1}x + x^{k(n-1)+1}\big) = ny^{n-1}x^{\alpha-k(n-1)} + x^\alpha$, and only if $\alpha - k(n-1) - 1 \geqslant 0$, that is $\alpha \geqslant k(n-1)+1$. The expression $ny^{n-1}x^{\alpha-k(n-1)} + x^\alpha$

must appear with the coefficient 1, so the monomial $y^{n-1}x^{\alpha-k(n-1)}$ will appear with a nonzero coefficient in another expression in the linear combination. But the only way we can obtain $y^{n-1}x^{\alpha-k(n-1)}$ is $x^{k(n-1)}y\cdot x^{\alpha-2k(n-1)}y^{n-2}$ or $x^{k(n-1)-1}y^2\cdot x^{\alpha-2k(n-1)+1}y^{n-3}$, so it is necessary that $\alpha \geqslant 2k(n-1)-1$. Therefore, the algebra $\mathbb{C}\{x,y\}/M\left(y^n+x^{k(n-1)}y\right)$ contains an element of the nilpotent degree $2k(n-1)-1$, and it is not isomorphic to the algebra $\mathbb{C}\{x,y\}/M(f)$ in the case (ii). So we have proved that, under the given conditions $y^n+x^\alpha y+x^\beta A(x)\not\approx y^n+x^\alpha y+cx^\beta$ for every $c\in\mathbb{C}$. □

Case (iii) $r\geqslant k$. Let $r\geqslant k$. In the same way as in the case (ii) we prove that $x^{kn+r}\in M(f)$, $x^{kn-k+r}\in M(f)$, and that we can replace f with x^{kn+r} in the set of generators of $M(f)$.

Since $x^{kn-k+r}y\in M(f)$, we have that $x^{kn+r-1}y\in M(f)$, and therefore

$$\begin{aligned} y\frac{\partial f}{\partial x} &= k(n-1)x^{k(n-1)-1}y^2+knx^{kn-1}yA(x)+x^{kn}yA'(x)=\cdots\\ &\equiv k\left[(n-1)x^{k(n-1)-1}y^2+nx^{kn-1}yA_0\right] \operatorname{mod} M(f), \end{aligned}$$

so if we add $x^{kn-k+r}y$ and $(n-1)x^{k(n-1)-1}y^2+nx^{kn-1}yA_0$ to the generating set of $M(f)$, we can omit $y\frac{\partial f}{\partial x}$. Also,

$$\begin{aligned} x\frac{\partial f}{\partial x} &= k(n-1)x^{k(n-1)}y+knx^{kn}A(x)+x^{kn}yA'(x)=\cdots\\ &\equiv k\left[(n-1)x^{k(n-1)}y+nx^{kn}A_0\right] \operatorname{mod} M(f), \end{aligned}$$

so we may take $(n-1)x^{k(n-1)}y+nx^{kn}A_0$ instead of $x\frac{\partial f}{\partial x}$ in the generating set of the ideal $M(f)$. Finally,

$$\begin{aligned} M(f)=\big(&x^{k(n-1)+r}y,\ x^{kn+r},\ ny^n+x^{k(n-1)}y,\ ny^{n-1}x+x^{k(n-1)+1},\\ &(n-1)x^{k(n-1)}y+nx^{kn}A_0,\ (n-1)x^{k(n-1)-1}y^2+nx^{kn-1}yA_0\big). \end{aligned}$$

Notice that the ideal $M(f)$ depends only on A_0, so in this case we also have $f\approx y^n+x^\alpha y+x^\beta\left(A_0+x^r\right)=y^n+x^\alpha y+A_0x^\beta+x^{\beta+r}$. Together with the previous case, this gives the following theorem. Recall that if $p,q\in\mathbb{N}$, the (p,q)-quasidegree of the monomial $a_{\alpha\beta}x^\alpha y^\beta$ $(a_{\alpha\beta}\in\mathbb{C})$ is the number $\frac{\alpha}{p}+\frac{\beta}{q}$, and polynomial f is (p,q)-quasihomogeneous, if it is a sum of monomials of the same (p,q)-quasidegree.

THEOREM 2.5. *Let $r\geqslant 1$. We have $f\approx y^n+x^\alpha y+A_0x^\beta+x^{\beta+r}$. Also, $y^n+x^\alpha y+cx^\beta+x^{\beta+r}\approx y^n+x^\alpha y+dx^\beta+x^{\beta+r}$ if and only if $c^{n-1}=d^{n-1}$.*

PROOF. The second assertion remains to be proved. Let $c^{n-1}=d^{n-1}$. If $c=0$, then $d=0$, and there is nothing to be proved. Let $c\neq 0$. Then $\left(\frac{d}{c}\right)^n=\frac{d}{c}$, and we have

$$\begin{aligned} y^n+x^\alpha y+cx^\beta+x^{\beta+r} &= \frac{c}{d}\Big(\frac{d}{c}y^n+x^\alpha\cdot\frac{d}{c}y+dx^\beta+\frac{d}{c}x^{\beta+r}\Big)\\ &= \frac{c}{d}\left[\Big(\frac{d}{c}y\Big)^n+x^\alpha\Big(\frac{d}{c}y\Big)+dx^\beta+\frac{d}{c}x^{\beta+r}\right]. \end{aligned}$$

Let $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be given by $\Phi(x,y) = (x, \frac{d}{c}y)$. It maps the singularity $y^n + x^\alpha y + cx^\beta + x^{\beta+r} = 0$ into $y^n + x^\alpha y + dx^\beta + \frac{d}{c}x^{\beta+r} = 0$. So

$$y^n + x^\alpha y + cx^\beta + x^{\beta+r} \approx y^n + x^\alpha y + dx^\beta + \frac{d}{c}x^{\beta+r} \approx y^n + x^\alpha y + dx^\beta + x^{\beta+r}.$$

Now suppose that $y^n+x^\alpha y+\gamma x^\beta+x^{\beta+r} \approx y^n+x^\alpha y+\delta x^\beta+x^{\beta+r}$; it means that there is an isomorphism $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, such that $u \cdot (y^n + x^\alpha y + \gamma x^\beta + x^{\beta+r}) = (y^n + x^\alpha y + \delta x^\beta + x^{\beta+r}) \circ \Phi$, where $u \in \mathbb{C}\{x, y\}$ is a unit. Let $\Phi(x,y) = (L, H)$, where $H = H(x,y) = ay + bx + H_2 + \cdots$ and $L = L(x,y) = cy + dx + L_2 + \cdots$. H_n and L_n are homogeneous polynomials of the homogeneous degree n. Now we have

$$\begin{aligned}(2.9)\qquad &(ay+bx+H_2+\cdots)^n+(cy+dx+L_2+\cdots)^\alpha(ay+bx+H_2+\cdots)\\ &+\delta(cy+dx+L_2+\cdots)^\beta+(cy+dx+L_2+\cdots)^{\beta+r} = u\cdot(y^n+x^\alpha y+\gamma x^\beta+x^{\beta+r}).\end{aligned}$$

(i) Let $k = 1$, then taking terms of the degree n in (2.9), we get

$$(ay + bx)^n + (cy + dx)^{n-1}(ay + bx) + \delta(cy + dx)^n = u_0 \cdot (y^n + x^{n-1}y + \gamma x^n).$$

This implies that $y^n + x^{n-1}y + \gamma x^n \approx y^n + x^{n-1}y + \delta x^n$. By theorem 2.2 we have $\delta^{n-1} = \gamma^{n-1}$.

(ii) Let $k > 1$, then taking the terms of the homogeneous degree n in (2.9), we get $b^n x^n = 0$, so $b = 0$.

Let's first suppose $y \mid H$. Taking from (2.9) terms of $(k, 1)$-quasidegree n, i.e., the terms with $x^r y^s$, where $\frac{r}{k} + \frac{s}{1} = n$, we get

$$(2.10)\qquad (ay)^n + (dx)^\alpha(ay) + \delta(dx)^\beta = u_0 \cdot (y^n + x^\alpha y + \gamma x^\beta),$$

where $u_0 = u(0)$. This implies that $y^n + x^\alpha y + \gamma x^\beta \approx y^n + x^\alpha y + \delta x^\beta$ and again, by theorem 2.2, $\gamma^{n-1} = \delta^{n-1}$.

If $y \nmid H$, let H be x-regular of the order r ($r \geqslant 2$, because $b = 0$). Then H contains a term $b_r x^r$ with $b_r \neq 0$. First suppose $r < k$, then the monomial $(b_r x^r)^n \in H^n$ is the only term of $(k, 1)$-quasidegree rn/k in H^n, and also in (2.9): namely, $(dx)^\alpha(b_r x^r) \in L^\alpha H$ is the term of the minimal $(k, 1)$-quasidegree in $L^\alpha H$ - $\frac{\alpha+r}{k} = (n-1) + \frac{r}{k} > rn/k$; also L^{kn} and $u \cdot (y^n + x^\alpha y + \gamma x^\beta + x^{\beta+r})$ obviously have no term of $(k, 1)$-quasidegree less then n, and $n > rn/k$. So, if $r < k$, taking in (2.9) terms of $(k, 1)$-quasidegree rn/k we get $(b_r x^r)^n = 0$, that contradicts $b_r \neq 0$. Therefore $r \geqslant k$, and in (2.9) there are no terms of $(k, 1)$-quasidegree less than n. Taking in (2.9) terms of the minimal $(k, 1)$-quasidegree, we get (2.10) if $r > k$, and if $r = k$ we have $(ay + b_k x^k)^n + (dx)^\alpha(ay + b_k x^k) + \delta(dx)^\beta = u_0 \cdot (y^n + x^\alpha y + \gamma x^\beta)$. In both cases we obtain that $y^n + x^\alpha y + \gamma x^\beta \approx y^n + x^\alpha y + \delta x^\beta$, and therefore $\gamma^{n-1} = \delta^{n-1}$. □

In the case (iii) $r \geqslant k$ it is not easy to calculate the dimension of the algebra $\mathbb{C}\{x, y\}/M(f)$, since there is a multitude of cases depending on r. Therefore we can not distinguish the analytic classes for different r in the case (iii) as we could in (ii). Also, we can not distinguish these classes from those in (i) and (ii).

Let us also note that in all three cases (i)–(iii), the ideal $M(f)$ is generated by $(k, 1)$-quasihomogeneous polynomials. Therefore, all relations between monomials in the algebra $\mathbb{C}\{x,y\}/M(f)$ are generated by quasihomogeneous relations, and the algebra splits as a vector space into the direct sum of subspaces generated by monomials of the same $(k,1)$-quasidegree.

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