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SUMS OF LIKE POWERS AND SOME DENSE SETS

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Abstract. We introduce the notion of the P-sequences and apply their properties in studying representability of real numbers. Another application of P-sequences we find in generating the Prouhet–Tarry–Escott pairs.

1. Introduction

First we define the notion of P-sequence and establish their basic properties which will be used in later sections. In the next section we use the properties of P-sequences in the study of representability of real numbers by sequences of reals. Recall that a real number r is representable by a sequence $\langle a_n \mid n \in \mathbb{N}\rangle$ if there is $S \subseteq \mathbb{N}$ such that $r = \sum_{n\in S} a_n$.

The following result on representability of real numbers is due to Kakeya [**9**]: Suppose that $A = \langle a_n \mid n \in \mathbb{N}\rangle$ is a decreasing sequence of positive reals which converges to 0 and that $s = \sum a_n$, $0 < s \leqslant +\infty$. Then the following assertions are equivalent:

- Each $r \in (0, s]$ is representable by means of A;
- $a_n \leqslant \sum_{k=n+1}^{\infty} a_k$, for each n.

Though we do not use Kakeya's theorem in our proofs, some particular cases of our examples are its consequences. However, the most interesting cases cannot be obtained by it.

Finally, using the P-sequences, we obtain new methods of generating sums of like powers i.e., the Prouhet–Tarry–Escott pairs. For more information about the Prouhet–Tarry–Escott problem see [**1, 2, 4, 5, 6, 8**].

2. *P*-sequences

We use symbols $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}^+$, $\mathbb{R}$ and $\mathbb{R}^+$ to represent the sets of integers, nonnegative integers, positive integers, real numbers and positive real numbers, respectively. In addition, we also adopt the convention that $0^0 = 1$.

The notion of a P-sequence is recursively defined as follows:

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- $\langle 1, -1\rangle$ is a P-sequence;
- If $\langle a_0, \dots, a_k\rangle$ is a P-sequence and $a_0 = -a_k$, then $\langle a_0, \dots, a_k, a_k, \dots, a_0\rangle$ is also a P-sequence;
- If $\langle a_0, \dots, a_k\rangle$ is a P-sequence and $a_0 = a_k$, then $\langle a_0, \dots, a_{k-1}, 0, -a_{k-1}, \dots, -a_0\rangle$ is also a P-sequence;
- Each P-sequence can be obtained only by finite use of the above clauses.

We denote the n-th P-sequence by P_n (assuming that they are ordered by their increasing lengths). For instance,

$$P_1 = \langle 1, -1\rangle,\ P_2 = \langle 1, -1, -1, 1\rangle,\ P_3 = \langle 1, -1, -1, 0, 1, 1, -1\rangle \quad \text{etc.}$$

For an arbitrary positive integer n, the n-th P-sequence $P_n = \langle a_0, \dots, a_k\rangle$ and any integer $s \geqslant 0$ let us define a polynomial function $F_{n,s}(x)$ over $\mathbb{R}$ by

$$F_{n,s}(x) = \sum_{i=0}^{k} a_i (i + x)^s.$$

Lemma 1. *$F_{n,s} \equiv 0$ for $s = 0, \dots, n - 1$.*

Proof. Since

$$F_{n,s}(x) = ((n-1)\cdots(n-1-s))^{-1} F_{n,n-1}^{(n-1-s)}(x), \qquad 0 \leqslant s < n - 1,$$

where $F_{n,n-1}^{(n-1-s)}$ is the $(n-1-s)$-th derivative of $F_{n,n-1}$, it is sufficient to prove that

$$F_{n,n-1} \equiv 0. \tag{1}$$

We prove the lemma by induction on n. Trivially (1) is true for $n = 1$, so let us assume that for some $n \geqslant 1$ the equality (1) holds. We have the following two cases:

$n = 2m$. Assuming that $P_{2m} = \langle a_0, \dots, a_k\rangle$, we have that

$$\begin{aligned} P_{2m+1} &= \langle a_0, \dots, a_{k-1}, 0, -a_{k-1}, \dots, -a_0\rangle \\ F_{2m+1,2m}(x) &= \sum_{i=0}^{k-1} a_i (i+x)^{2m} - \sum_{i=0}^{k-1} a_i (2k - i + x)^{2m} \\ &= \sum_{i=0}^{k} a_i (i+x)^{2m} - \sum_{i=0}^{k} a_i (i - 2k - x)^{2m} \\ &= F_{2m,2m}(x) - F_{2m,2m}(-2k - x). \end{aligned}$$

Then

$$F'_{2m+1,2m}(x) = 2m \underbrace{F_{2m,2m-1}(x)}_{=0} + 2m \underbrace{F_{2m,2m-1}(-2k - x)}_{=0} = 0,$$

so $F_{2m+1,2m}$ is a constant function. Since $F_{2m+1,2m}(-k) = 0$, we conclude that $F_{2m+1,2m} \equiv 0$.

$n = 2m+1$. Similarly to the previous case one can easily check that $F_{2m+2,2m+1}$ is a constant function. Since $F_{2m+2,2m+1}(-(2k + 1)/2) = 0$, we conclude that $F_{2m+2,2m+1} \equiv 0$ as well. $\square$

THEOREM 1. *For $s \geqslant n$ the degree of $F_{n,s}$ is equal to $s - n$.*

PROOF. Clearly, it is sufficient to prove that

$$F_{n,n} \equiv \text{const.} \neq 0. \tag{2}$$

Observe that an immediate consequence of Lemma 1 is the fact that each $F_{n,n}$ is a constant function.

The proof goes by induction on n. $F_{1,1} \equiv -1$, so let us assume that for some $n \geqslant 1$ the relation (2) holds.

$n = 2m$. Assuming that $P_{2m} = \langle a_0, \dots, a_k \rangle$, we have

$$\begin{aligned} P_{2m+1} &= \langle a_0, \dots, a_{k-1}, 0, -a_{k-1}, \dots, -a_0 \rangle \\ F_{2m+1,2m+1}(x) &= \sum_{i=0}^{k-1} a_i (i+x)^{2m+1} - \sum_{i=0}^{k-1} a_i (2k - i + x)^{2m+1} \\ &= F_{2m,2m+1}(x) + F_{2m,2m+1}(-2k - x). \end{aligned}$$

$F_{2m+1,2m+1}$ is a constant function, so

$$F_{2m+1,2m+1}(x) = F_{2m+1,2m+1}(-k) = 2F_{2m,2m+1}(-k).$$

Since $a_i = a_{k-i}$, we have $F_{2m,2m+1}(-k) = -F_{2m,2m+1}(0)$. By the induction hypothesis $F_{2m,2m+1}$ is a linear function thus 1–1, hence $F_{2m,2m+1}(-k) \neq 0$.

$n = 2m + 1$. Similarly to the previous case one can easily deduce that

$$\begin{aligned} F_{2m+2,2m+2}(x) &= 2F_{2m+1,2m+2}(-(2k+1)/2) \\ F_{2m+1,2m+2}(-(2k+1)/2) &= -F_{2m+1,2m+2}(1/2), \end{aligned}$$

which combined with the induction hypothesis implies that

$$F_{2m+1,2m+2}(-(2k+1)/2) \neq 0. \qquad \square$$

COROLLARY 1. *Let $P_n = \langle a_0, \dots, a_k \rangle$ be a P-sequence. Then:*

(1) $\sum\limits_{i=1}^{k} a_i i^s = 0, \ s = 0, \dots, n-1$; (2) $\operatorname{sgn} \sum\limits_{i=1}^{k} a_i i^n = (-1)^n$.

PROOF. (1) is an immediate consequence of Lemma 1, while (2) can be obtained by induction, using the fact that $a_i = a_{k-i}$ for even n, and $a_i = -a_{k-i}$ for odd n. $\square$

3. Dense-expandable sequences

Through this section E will denote some denumerable sequence of positive real numbers, $E(n)$ its n-th member, $\sum E \overset{\text{def}}{=} \sum_{n=0}^{\infty} E(n)$, $\lim E \overset{\text{def}}{=} \lim_{n\to\infty} E(n)$, $r + sE \overset{\text{def}}{=} \langle r + sE(n) \mid n \in \mathbb{N} \rangle$ and μ_E is a measure on $\mathbb{N}$ defined by

$$\mu_E(S) = \sum_{n \in S} E(n), \quad S \subseteq \mathbb{N}.$$

We say that measure μ_E is continuous if for each $r \in [0, \infty]$ there is $S \subseteq \mathbb{N}$ such that $\mu_E(S) = r$.

Sequence E is *dense-expandable* if the set $X(E) \overset{\text{def}}{=} \big\{\sum_{n\in S} \varepsilon_n E(n) \mid S$ ranges over finite subsets of $\mathbb{N}$ and ε_n ranges over $\{-1,1\}\big\}$ is dense in $\mathbb{R}$.

If $\sum E$ is convergent, then $X(E)$ is bounded in $\mathbb{R}$, so it cannot be dense. On the other hand, if μ_E is continuous, then E is obviously dense-expandable.

Theorem 2. *Suppose that E is a sequence of positive real numbers such that $\lim E = 0$ and $\sum E = \infty$. Then μ_E is continuous.*

Proof. For the fixed positive real number c let $\mathcal{C}$ be the family of all subsets S of $\mathbb{N}$ such that $\mu_E(S) \leqslant c$. Note that $\mathcal{C} \neq \emptyset$ since $\lim E = 0$. $\langle \mathcal{C}, \subseteq\rangle$ is clearly a poset, so it has a maximal chain, say M. It is easy to see that $\bigcup M \in \mathcal{C}$. Suppose that $\mu_E(\bigcup M) = b < c$. Since $\mu_E(\mathbb{N}) = \infty$, $\mathbb{N} \smallsetminus \bigcup M$ is infinite. Now $\lim E = 0$ implies that there is an index $n \in \mathbb{N} \smallsetminus \bigcup M$ such that $b + E(n) < c$. But this implies that $\mu_E\big(\bigcup M\big) < \mu_E\big(\{n\} \cup \bigcup M\big) < c$, which contradicts the maximality of M. $\square$

A converse implication need not be true. Namely, the sequence

$$E = \langle 1, 2, 2^{-1}, 2^2, 2^{-2}, 2^3, 2^{-3}, \dots\rangle$$

clearly generates a continuous measure μ_E, but it is not convergent. However, its limes inferior $\underline{\lim}\, E$ is equal to 0. Note also that from $\underline{\lim}\, E = 0$ and $\sum E = \infty$ does not follow necessarily the continuity of μ_E.

Theorem 3. *Let $\varphi(n)$ be the Euler function. Then $A = \{\varphi(n)/n \mid n \in N\}$ is dense in the real interval* $[0,1]$.

Proof. First, we remind the reader that $\varphi(n)/n = (1 - 1/p_1)\cdots(1 - 1/p_k)$, where $p_1, \dots, p_k$ are all prime factors of n. Now, let $a_n = -\log(1 - 1/p_n)$, where $\langle p_n \mid n \in N\rangle$ is the sequence of elements of the set of all primes P. Then it is easy to see that a_n satisfies conditions of the previous theorem so for any $r \in R^+$ there is $S \subseteq N$ such that $\sum_{n\in S} -\log(1 - 1/p_n) = r$, i.e., $\prod_{n\in S}(1 - 1/p_n) = e^{-r}$. As r runs over R^+, e^{-r} takes all values in $[0,1]$, so for any $t \in [0,1]$, there is $S \subseteq P$ such that $\prod_{p\in S}(1 - 1/p) = t$. Thus $\lim_{n\in S} \varphi(n)/n = t$, hence A is dense in [0,1]. $\square$

The next result is useful in studying of dense-expandability.

Theorem 4. *Suppose that E is a sequence of positive real numbers such that $\lim E = 0$ and $\sum E = \infty$. Then for any nonnegative real number r the sequence $r + E$ is dense-expandable.*

Proof. For fixed $r \geqslant 0$ we want to prove that $\overline{X(r + E)} = \mathbb{R}$. Since $x \in X(r + E)$ iff $-x \in X(r + E)$, it is sufficient to prove that for any $c \geqslant 0$ and an arbitrary small $\varepsilon > 0$ open interval $(c - \varepsilon, c + \varepsilon)$ and $X(r + E)$ meet each other.

The assumed properties of E provide the existence of positive integers n and $m > n$ such that:

(1) $E(i) < \varepsilon/2$, for all $i \geqslant n$; (2) $\sum\limits_{i=n}^{m} E(i) \leqslant c < \sum\limits_{i=n}^{m+1} E(i) < c + \varepsilon$.

Let $q = \sum_{i=n}^{m+1} E(i) - c$, $l = m - n + 2$ and let $\delta = q/2l$. Since $\lim E = 0$, there is an integer $k > m$ such that $E(i) < \delta$ for all $i \geqslant k$. Then:

$$\begin{aligned} c + \varepsilon &> \sum_{i=n}^{m+1} (r + E(i)) - \sum_{i=k}^{k+l-1} (r + E(i)) \\ &= \sum_{i=n}^{m+1} E(i) - \sum_{i=k}^{k+l-1} E(i) \geqslant \sum_{i=n}^{m+1} E(i) - \frac{q}{2} > c. \end{aligned}$$

Finally,

$$\sum_{i=n}^{m+1} (r + E(i)) - \sum_{i=k}^{k+l-1} (r + E(i)) \in X(r + E),$$

so $X(r + E)$ is dense in $\mathbb{R}$. □

An immediate consequence of Theorem 4 is the fact that the property of being dense-expandable is not invariant to asymptotic equivalence. For instance, sequences $E_1 = \langle 1 \mid n \in \mathbb{N} \rangle$ and $E_2 = \langle 1 + \frac{1}{n+1} \mid n \in \mathbb{N} \rangle$ are asymptotically equivalent, but $\overline{X(E_1)} = \mathbb{Z}$ and $\overline{X(E_2)} = \mathbb{R}$.

In general, a cofinite subsequence of a dense-expandable sequence E need not be dense expandable. As we have mentioned earlier, the sequence

$$E = \langle 1, 2, 2^{-1}, 2^2, 2^{-2}, 2^3, 2^{-3}, \ldots \rangle$$

is dense-expandable, but its cofinite subsequence $E_1 = \langle E(n+2) \mid n \in \mathbb{N} \rangle$ is not since $X(E_1) \cap (1, 2) = \emptyset$.

The basic strategy in proving that a certain sequence E is dense-expandable is in choosing countably many pairwise disjoint finite subsets S_n of $\mathbb{N}$ and appropriate $\varepsilon_{n,i}$s such that the sequence $\left\langle \sum_{i \in S_n} \varepsilon_{n,i} E(i) \mid n \in \mathbb{N} \right\rangle$ satisfies conditions of theorem 4. As an illustration we will prove that sequence $E = \langle \ln n \mid n > 0 \rangle$ is dense-expandable. First, note that the sequence $\langle \ln(1 + \frac{1}{2n}) \mid n > 0 \rangle$ satisfies the conditions of Theorem 4, so it is dense-expandable. The sets $S_n = \{2n, 2n+1\}$, $n > 0$ are pairwise disjoint and $\ln(1 + 1/2n) = \ln(2n+1) - \ln 2n$, so $X\big(\langle \ln(1 + 1/2n) \mid n > 0 \rangle\big) \subseteq X(E)$. Hence E is dense-expandable.

THEOREM 5. *The sequence* $\langle n^\delta \mid n \in \mathbb{N}^+ \rangle$ *is dense-expandable if and only if* $\delta = -1$ *or* $\delta > -1$ *and* $\delta \notin \mathbb{Z}$.

PROOF. If $\delta < -1$, then $\sum_{n=1}^{\infty} n^\delta$ converges, so $X\big(\langle n^\delta \mid n \in \mathbb{N}^+ \rangle\big)$ is bounded in $\mathbb{R}$. By Theorem 4 sequence $\langle n^\delta \mid n \in \mathbb{N}^+ \rangle$ is dense-expandable for each $\delta \in [-1, 0)$. If δ is a positive integer, then $X\big(\langle n^\delta \mid n \in \mathbb{N}^+ \rangle\big) \subseteq \mathbb{Z}$. It remains to prove that $\langle n^\delta \mid n \in \mathbb{N}^+ \rangle$ is dense-expandable for any $\delta \in \mathbb{R}^+ \setminus \mathbb{Z}$.

Fix $\delta \in \mathbb{R}^+ \setminus \mathbb{Z}$. Then there is a unique positive integer m such that $m - 1 < \delta < m$. Let $E(n) \stackrel{\text{def}}{=} \sum_{i=0}^{k} a_i (n-i)^\delta$, $n > k$, where $P_m = \langle a_0, \ldots, a_k \rangle$ is the m-th P-sequence. Then:

$$\begin{aligned}
E(n) &= \sum_{i=0}^{k} a_i (n-i)^{\delta} = n^{\delta} \sum_{i=0}^{k} a_i \Big(1 - \frac{i}{n}\Big)^{\delta} \\
&= n^{\delta} \sum_{i=0}^{k} a_i \Big(\sum_{j=0}^{\infty} (-1)^j \binom{\delta}{j} i^j n^{-j} \Big) = n^{\delta} \sum_{j=0}^{\infty} (-1)^j n^{-j} \binom{\delta}{j} \Big(\sum_{i=0}^{k} a_i i^j \Big) \\
&= n^{\delta} \sum_{j=m}^{\infty} (-1)^j n^{-j} \binom{\delta}{j} \Big(\sum_{i=0}^{k} a_i i^j \Big) \qquad \text{(Corollary 1)} \\
&= (-1)^m n^{\delta - m} \binom{\delta}{m} \sum_{i=0}^{k} a_i i^m + o(n^{\delta - m}).
\end{aligned}$$

Since $\binom{\delta}{m} > 0$ and $\operatorname{sgn} \sum_{i=0}^{k} a_i i^m = (-1)^m$, sequence $\langle E(n) \mid n > k \rangle$ is ultimately positive. Now $-1 < \delta - m < 0$ implies that $\langle E(n) \mid n > k \rangle$ is dense-expandable (Theorem 4). The same is obviously true for the sequence $E = \langle E(kn) \mid n > k \rangle$.

Finally, sets $S_n = \{kn - i \mid i \in \{0, \dots, k\}\}$, $n > k$ are pairwise disjoint and each $E(kn)$ is equal to $\sum_{j \in S_n} \varepsilon_{n,j} j^{\delta}$, where $\varepsilon_{n,j}$ are the corresponding coordinates of P_m. Thus the sequence $\langle n^{\delta} \mid n > 0 \rangle$ is dense-expandable. □

Theorem 6. *Let $\langle p_n \mid n \in \mathbb{N} \rangle$ be the sequence of all prime numbers and let $E = \langle p_n^{\delta} \mid n \in \mathbb{N} \rangle$. Then:*

(1) *The Riemann hypothesis implies that E is dense-expandable for any $0 < \delta < 1/2$;*

(2) *Hypothesis $\lim_{n \to \infty} (\sqrt{p_{n+1}} - \sqrt{p_n}) = 0$ implies that E is dense-expandable for any $0 < \delta \leqslant 1/2$.*

Proof. In order to prove (1), assume the Riemann hypothesis. Then, the following relation holds for the consecutive primes:

$$p_{n+1} - p_n \ll \sqrt{p_n} \log p_n \tag{3}$$

Suppose that $0 < \delta < 1/2$. Then

$$\begin{aligned}
p_{n+1}^{\delta} - p_n^{\delta} &= p_{n+1}^{\delta} \left[1 - \Big(1 - \frac{p_{n+1} - p_n}{p_{n+1}} \Big)^{\delta} \right] \\
&= p_{n+1}^{\delta} \left[1 - \left(1 - \binom{\delta}{1} \frac{p_{n+1} - p_n}{p_{n+1}} + O\Big(\frac{p_{n+1} - p_n}{p_{n+1}} \Big) \right) \right] \\
&= \delta \frac{p_{n+1} - p_n}{p_{n+1}^{1-\delta}} + O\Big(\frac{p_{n+1} - p_n}{p_{n+1}^{1-\delta}} \Big) \ll \frac{\sqrt{p_n} \log p_n}{p_{n+1}^{1-\delta}} + o(1) \to 0
\end{aligned}$$

as $n \to \infty$ (see [**3**]). Taking $u_n = p_{n+1}^{\delta} - p_n^{\delta}$, $n \in N^+$, we see that for $0 < \delta < 1/2$,

$$\sum_{k \leqslant n} u_k = p_{n+1}^{\delta} - 2^{\delta} \to \infty, \qquad \text{as } n \to \infty,$$

$u_n > 0$ and $\lim_n u_n = 0$. Thus, by Theorem 4 E is dense-expandable.

In order to prove (2), let us assume that $\lim_{n\to\infty}\left(\sqrt{p_{n+1}}-\sqrt{p_n}\right)=0$. If $f(\delta)=x^\delta-y^\delta$, $0<\delta$ and $y<x$, then $f'(\delta)=x^\delta\ln(x)-y^\delta\ln(y)>0$, so $f(\delta)$ is increasing for $\delta>0$. Hence, if $0<\delta\leqslant 1/2$, then $0\leqslant p_{n+1}^\delta-p_n^\delta\leqslant\sqrt{p_{n+1}}-\sqrt{p_n}\to 0$, as $n\to\infty$, so by an argument as in (1), the assertion follows. □

4. Sums of like powers

Finite disjoint subsets U and V of $\mathbb{Z}$ will be called a Prouhet-Tarry-Escott pair for the given integer $n>1$ if they have the same cardinality and

$$\sum_{u\in U}u^s=\sum_{v\in V}v^s,\quad s=0,\dots,n-1,\quad\text{and}\quad\sum_{u\in U}u^n\neq\sum_{v\in V}v^n. \tag{4}$$

The sums satisfying the left-hand conjunct of (4) are also known as sums of like powers.

If $\langle U_1,V_1\rangle,\dots,\langle U_m,V_m\rangle$ are Prouhet-Tarry-Escott pairs for the given integer n and if sets $U_1,\dots,U_m,V_1,\dots,V_m$ are pairwise disjoint, then clearly sets $U=\bigcup_{i=1}^m U_i$ and $V=\bigcup_{i=1}^m V_i$ form another Prouhet-Tary-Escott pair for n.

Now let us describe how one can use the P-sequences in order to generate the Prouhet-Tarry-Escott pairs:

1. Let $n\geqslant 2$ be an arbitrary integer and let $P_n=\langle a_0,\dots,a_k\rangle$ be the n-th P-sequence. By lemma 1 we have that

$$\sum_{i=0}^k a_i(pi+l)^s=0,\quad s=0,\dots,n-1,\ l\in\mathbb{Z},\quad p\in\mathbb{Z}\smallsetminus\{0\}$$

$\big($observe that $\sum\_i=0^k a_i(pi+l)^s=p^sF_{n,s}(l/p)\big)$. Since each P-sequence has the same number of 1s and -1s, we have that sets $U_{p,l}$ and $V_{p,l}$ defined by

$$U_{p,l}=\{pi+l\mid 0\leqslant i\leqslant k\wedge a_i=-1\}\quad\text{and}\quad V_{p,l}=\{pi+l\mid 0\leqslant i\leqslant k\wedge a_i=1\}$$

form a Prouhet-Tarry-Escott pair for the given integer $n\geqslant 2$. Note that

$$\sum_{i=0}^k a_i(pi+l)^n\neq 0,$$

since $\sum_{i=0}^k a_i(pi+l)^n=p^nF_{n,n}(1/p)$ (see Theorem 1).

2. Let $P_n=\langle a_0,\dots,a_k\rangle$ be the n-th P-sequence ($n\geqslant 2$). We define the sequence $Q_n=\langle b_0,\dots,b_{k+2}\rangle$ as follows:

$$b_i=\begin{cases}a_i, & i\in\{0,1\}\\ a_i+a_{i-2}, & 1<i<k-1\\ a_{i-2}, & i\in\{k+1,k+2\}.\end{cases}$$

For example, we obtain Q_3 from $P_3=\langle 1,-1,-1,0,1,1,-1\rangle$ in the following manner:

P_3	1	−1	−1	0	1	1	−1		
			1	−1	−1	0	1	1	−1
Q_3	1	−1	0	−1	0	1	0	1	−1

By induction one can show that each $b_i \in \{-1, 0, 1\}$ and that each Q_n has the same number of 1s and -1s. Now for any non-negative integer $s < n$ we have that

$$\begin{aligned}\sum_{i=0}^{k} b_i(i+1)^s &= \sum_{i=0}^{k} a_i(i+1)^s \ + \ \sum_{i=0}^{k} a_i(i+3)^s = 0\\ \sum_{i=0}^{k} b_i(i+1)^n &= \sum_{i=0}^{k} a_i(i+1)^n \ + \ \sum_{i=0}^{k} a_i(i+3)^n\\ &= F_{n,n}(1) + F_{n,n}(3) = 2F_{n,n}(1) \neq 0,\end{aligned}$$

so $U = \{i+1 \mid b_i = 1\}$ and $V = \{i+1 \mid b_i = -1\}$ represents a Prouhet–Tarry–Escott pair.

3. For the n-th P-sequence $P_n = \langle a_0, \dots, a_k\rangle$ let

$$X_n = \{i \in \mathbb{N}^+ \mid i \leqslant k \wedge a_i = -1\} \text{ and } Y_n = \{i \in \mathbb{N}^+ \mid i \leqslant k \wedge a_i = 1\}.$$

Clearly, X_n and Y_n are disjoint and $|X_n| = |Y_n| + 1$. Furthermore, using the definition of the notion of a P-sequence one can easily check that $X_{2n+1} \subset X_{2n+2}$ and $Y_{2n+1} \subset Y_{2n+2}$, and the sets $U = X_{2n+2} \setminus X_{2n+1}$ and $V = Y_{2n+2} \setminus Y_{2n+1}$ are disjoint and have the same cardinality. Bearing in mind the corollary 1, we see that for each nonnegative integer $s \leqslant 2n$ holds

$$\sum_{i\in U} i^s = \sum_{i\in X_{2n+2}} i^s - \sum_{i\in X_{2n+1}} i^s = \sum_{i\in Y_{2n+2}} i^s - \sum_{i\in Y_{2n+1}} i^s = \sum_{i\in V} i^s.$$

For instance, if $n = 4$, then $X_4 = \{1, 2, 6, 7, 11, 12\}$, $Y_4 = \{4, 5, 8, 9, 13\}$, $U = \{7, 11, 12\}$, $V = \{8, 9, 13\}$ and

$$7^s + 11^s + 12^s = 8^s + 9^s + 13^s, \quad s = 1, 2.$$

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