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ON REGULARLY VARYING MOMENTS FOR POWER SERIES DISTRIBUTIONS

Slavko Simić

Abstract. For the power series distribution, generated by an entire function of finite order, we obtain the asymptotic behavior of its regularly varying moments. Namely, we prove that $E_w X^\alpha \ell(X) \sim (E_w X)^\alpha \ell(E_w X)$, $\alpha > 0$ $(w \to \infty)$, where $\ell(\cdot)$ is an arbitrary slowly varying function.

0. Introduction

0.1. Denote by A_ρ the class of transcendental entire functions with *positive* Taylor coefficients and of finite order ρ, $0 \leqslant \rho < \infty$.

Definition 1. Let $f(w) = \sum a_n w^n$, $f \in A_\rho$. A power series distribution with parameter $w > 0$, generated by f, is defined by (cf. [2])

$$P(X = n) := a_n w^n / f(w), \quad n = 0, 1, 2, \cdots$$

Our aim is to obtain the asymptotic behavior of the k-th moment $E_w X^k$ when $w \to \infty$, where

$$E_w X^k := \sum n^k P(X = n) = \sum n^k a_n w^n / f(w), \quad k = 1, 2, \cdots$$

Note that the expectation $E_w X$ is equal to

$$E_w X := \sum n a_n w^n / f(w) = w f'(w) / f(w). \tag{1}$$

For any k, consider the sequence of functions $f_k(w)$ defined recursively by

$$f_k(w) = w f'_{k-1}(w), \quad k = 1, 2, \cdots; \quad f_0(w) = f(w) = \sum a_n w^n.$$

Then $f_k(w) = \sum n^k a_n w^n \in A_\rho$ and

$$E_w X^k = f_k(w) / f(w), \quad k = 1, 2, \cdots \tag{2}$$

We shall derive the asymptotic behavior of $E_w X^k$ for large w by applying our following recent result:

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THEOREM 1. [5] *For an arbitrary* $f \in A_\rho$, *we have*

$$\frac{f(w)f''(w)}{(f'(w))^2} \to 1 \quad (w \to \infty).$$

independently of the order ρ.

0.2. Further generalization leads to the concept of *regularly varying moments* $E_w X^\alpha \ell(X)$ (cf. [1, p. 335]),

$$E_w X^\alpha \ell(X) := \sum n^\alpha \ell(n) a_n w^n / f(w), \tag{3}$$

where α is a positive *real* number and $\ell(\cdot)$ is a *slowly varying function*.

DEFINITION 2. A positive continuous function $\ell(\cdot)$, defined on $[x_0, \infty)$, is slowly varying if the asymptotic equivalence $\ell(tx) \sim \ell(x)$, $(x \to \infty)$, holds for each $t > 0$.

For $x \in [0, x_0)$ we can take $f(x) := f(x_0)$. Some examples of slowly varying functions are

$$\log^a x; \quad \log^b(\log x); \quad \exp(\log^c x); \quad \exp(\log x / \log\log x); \quad a, b \in R, \quad 0 < c < 1.$$

Functions $g(\cdot)$ of the form $g(x) = x^\mu \ell(x)$ are *regularly varying* with index $\mu \in R$ (cf. [1, p.18]). Each regularly varying function $x^\mu \ell(x)$ generates a regularly varying sequence of the form $\{n^\mu \ell(n)\}_{n=1}^\infty$.

The main tool for asymptotic estimation of regularly varying moments is the following theorem on matrix transforms with slowly varying sequences (cf. [4]).

THEOREM 2. *For a given complex-valued matrix* $(A_{nk})_{n,k=1}^\infty$ *define* $t_n(\rho) := \sum k^\rho |A_{nk}|$. *Suppose that for some positive constants* a, A, $t_n(\rho)$ *exists for* $-a \leqslant \rho \leqslant 1$ *and, for sufficiently large* n,

$$(i) \ \left|\sum A_{nk}\right| \geqslant A; \ (ii) \ t_n(0) \to 1; \ (iii) \ t_n(1) \to \infty; \ (iv) \ t_n(-a) = O((t_n(1))^{-a}).$$

Then the asymptotic relation

$$\sum A_{nk} \ell(k) = \ell(t_n(1)) \left(\sum A_{nk}\right) (1 + o(1)) \qquad (n \to \infty),$$

holds for all slowly varying sequences $\{\ell(k)\}_{k=1}^\infty$.

0.3. Here we quote some well-known assertions we shall need in the sequel.

LEMMA 1. *If* $a(x) \sim b(x) \to \infty$, *then* $\ell(a(x)) \sim \ell(b(x)) \quad (x \to \infty)$.

LEMMA 2. [3, Vol. I, p. 36]. *Let* $g(x) = \sum a_n x^n, h(x) = \sum b_n x^n$, $g, h \in A_\rho$. *If* $a_n \sim b_n$ $(n \to \infty)$, *then* $g(x) \sim h(x)$ $(x \to \infty)$.

LEMMA 3. *Jensen's inequality:* $EX^t \geqslant (EX)^t$, $t > 1$, *and vice versa for* $0 < t < 1$.

LEMMA 4. *Lyapunov moments inequality asserts that, for* $r > s > t > 0$,

$$(EX^s)^{r-t} \leqslant (EX^r)^{s-t} (EX^t)^{r-s}.$$

1. Results

1.1. The above Theorem 2 has many applications in real or complex analysis (cf. [4]). We shall apply it here to derive the following *theorem on regularly varying moments for discrete laws.*

THEOREM 3. *Let a discrete law G be given by $P(X_n = k) = p_{nk} \geqslant 0$, $\sum_k p_{nk} = 1$. If $EX_n \to \infty$ and $EX_n^\beta \sim C_\beta (EX_n)^\beta$ $(n \to \infty)$ for $\beta \in (0, B]$, $B > 1$, $C_\beta > 0$ then, for an arbitrary slowly varying function $\ell(\cdot)$, the asymptotic relation*

$$EX_n^\beta \ell(X_n) \sim C_\beta (EX_n)^\beta \ell(EX_n) \quad (n \to \infty),$$

holds

a. *for each $\beta \in (0, B-1]$;*

b. *for each $\beta \in (B-1, B]$, if EX_n^{B+1} exists and $EX_n^{B+1} = O((EX_n)^{B+1})$ $(n \to \infty)$.*

PROOF. Putting $A_{nk} := p_{nk} k^\beta / C_\beta (EX_n)^\beta$, we find out that conditions (i) and (ii) of Theorem 2 are satisfied. For $\beta \in (0, B-1]$, we obtain

$$t_n(1) = EX_n^{\beta+1} / C_\beta (EX_n)^\beta \sim (C_{\beta+1}/C_\beta) EX_n \to \infty \quad (n \to \infty).$$

Also,

$$t_n(-\beta/2) = EX_n^{\beta/2} / C_\beta (EX_n)^\beta \sim (C_{\beta/2}/C_\beta)(EX_n)^{-\beta/2} = O(t_n(1))^{-\beta/2} \ (n \to \infty).$$

Therefore, the conditions of Theorem 2 are satisfied with $A = 1$, $a = \beta/2$ and the result for $\beta \in (0, B-1]$ follows. □

For the case $\beta \in (B-1, B]$, we need the following

LEMMA 5. *Under the condition* b *of Theorem* 3, *we have*

$$EX_n^{\beta+1} = O((EX_n)^{\beta+1}) \quad (n \to \infty), \tag{4}$$

for each $\beta \in (B-1, B]$.

PROOF. Indeed, applying Lyapunov moments inequality (Lemma 4) with $r = B+1$, $s = \beta+1$, $t = B$, we get

$$\begin{aligned} EX_n^{\beta+1} &\leqslant (EX_n^B)^{B-\beta} (EX_n^{B+1})^{\beta+1-B} \\ &= O((EX_n)^{B(B-\beta)} (EX_n)^{(B+1)(\beta+1-B)}) = O((EX_n)^{\beta+1}). \end{aligned}$$

Now, by Jensen's inequality $EX_n^{\beta+1} \geqslant (EX_n)^{\beta+1}$ i.e., $t_n(1) \geqslant EX_n/C_\beta \to \infty$ $(n \to \infty)$.

Also, by (4),

$$t_n(-\beta/2) \sim (C_{\beta/2}/C_\beta)(EX_n)^{-\beta/2} = O((EX_n^{\beta+1}/(EX_n)^\beta)^{-\beta/2}) = O((t_n(1))^{-\beta/2}).$$

Therefore, the conditions of Theorem 2 are satisfied and we get

$$EX_n^\beta \ell(X_n) \sim C_\beta (EX_n)^\beta \ell(EX_n^{\beta+1}/C_\beta (EX_n)^\beta) \quad (n \to \infty),$$

for each $\beta \in (B-1, B]$.

But, since

$$EX_n/C_\beta \leqslant \frac{EX_n^{\beta+1}}{C_\beta(EX_n)^\beta} = O(EX_n) \quad (n \to \infty),$$

it follows by the uniform convergence theorem for slowly varying functions (cf. [1, p.6]), that

$$\ell\bigg(\frac{EX_n^{\beta+1}}{C_\beta(EX_n)^\beta}\bigg) \sim \ell(EX_n) \quad (n \to \infty). \qquad \square$$

1.2. We turn back now to the asymptotic evaluation of regularly varying moments for power series distributions. Using Theorem 3 above, it will be shown that this evaluation is equivalent to the following *theorem on moments of power series distributions.*

THEOREM 4. *For each $\alpha > 0$, we have $E_wX^\alpha \sim (E_wX)^\alpha \ \ (w \to \infty)$.*

For the generating entire function $f(w) = \sum a_k w^k \in A_\rho$, recall (1) and (2):

$$E_wX = \sum k a_k w^k / f(w) = w f'(w)/f(w);$$
$$E_wX^m = \sum k^m a_k w^k / f(w) = f_m(w)/f(w).$$

The proof of Theorem 4 requires some preliminary lemmas.

LEMMA 6. *The expectation E_wX is a monotone increasing and unbounded function in w.*

PROOF. Since

$$w\frac{d}{dw}(E_wX) = E_wX^2 - (E_wX)^2 > 0,$$

we conclude that E_wX is a monotone increasing function in w. If it is bounded, then there exists a $d > 0$ such that $E_wX < d$ for each $w > 0$. By (1) we get $f'(w)/f(w) < d/w$, and integrating we find $f(w) = O(w^d)$. Hence in this case f is a polynomial, which contradicts our assumption that f is a transcendental entire function. $\square$

LEMMA 7. *For $m \in N$, $\ f_m(w) \sim w^m f^{(m)}(w) \quad (w \to \infty)$.*

PROOF. Note that $f \in A_\rho$ implies $f^{(m)}, f_m \in A_\rho, \quad m = 1, 2, \cdots$. Since, for fixed $m \in N$,

$$f_m(w) = \sum k^m a_k w^k; \quad w^m f^{(m)}(w) = \sum_{k \geqslant m} k(k-1)\cdots(k-m+1) a_k w^k;$$
$$k(k-1)\cdots(k-m+1) \sim k^m \quad (k \to \infty),$$

the result follows by Lemma 2. $\square$

LEMMA 8. *For each $m \in N$ we have $E_wX^{m+1}/E_wX^m \sim E_wX \ \ (w \to \infty)$.*

PROOF. Applying Theorem 1, we obtain

$$\frac{E_w X^2}{(E_w X)^2} \to 1 \quad (w \to \infty), \tag{5}$$

because

$$\frac{E_w X^2}{(E_w X)^2} - \frac{1}{E_w X} = \frac{f(w) f''(w)}{(f'(w))^2} \to 1 \quad (w \to \infty),$$

and, by Lemma 6, $1/E_w X \to 0$.

Since Theorem 1 is valid for each $f \in A_\rho$ and $f^{(m)} \in A_\rho$, $m = 1, 2, \cdots$, replacing f by $f^{(m)}$, we get

$$\frac{f^{(m+1)}(w) f^{(m-1)}(w)}{(f^{(m)}(w))^2} \to 1 \;\text{ i.e. }\; \frac{f^{(m+1)}(w)}{f^{(m)}(w)} \sim \frac{f^{(m)}(w)}{f^{(m-1)}(w)} \quad (w \to \infty). \tag{6}$$

Hence by Lemma 7 and (6),

$$\frac{E_w X^{m+1}}{E_w X^m} = \frac{f_{m+1}(w)}{f_m(w)} \sim \frac{w^{m+1} f^{(m+1)}(w)}{w^m f^{(m)}(w)} \sim \frac{w^m f^{(m)}(w)}{w^{m-1} f^{(m-1)}(w)}$$
$$\sim \frac{f_m(w)}{f_{m-1}(w)} = \frac{E_w X^m}{E_w X^{m-1}}, \; n \in N.$$

Therefore,

$$\frac{E_w X^{m+1}(w)}{E_w X^m} \sim \frac{E_w X^m}{E_w X^{m-1}} \sim \cdots \sim \frac{E_w X^2}{E_w X} \sim E_w X \quad (w \to \infty). \qquad \square$$

A simple consequence of the previous lemma is the following:

LEMMA 9. *For each $m \in N$, we have $E_w X^m \sim (E_w X)^m$ $(w \to \infty)$.*

PROOF. Indeed,

$$E_w X^m = (E_w X) \prod_{k=1}^{m-1} (E_w X^{k+1}/E_w X^k) \sim (E_w X)^m \quad (w \to \infty).$$

For the rest of the proof of Theorem 4 we apply Lemma 4.

Let $m > \alpha > m-1$, $m \in N$. Then Lyapunov's inequality and Lemma 9 give

$$E_w X^\alpha \leqslant (E_w X^m)^{\alpha - m + 1} (E_w X^{m-1})^{n-\alpha} \sim (E_w X)^{m(\alpha - m + 1)} (E_w X)^{(m-1)(m-\alpha)}$$
$$= (E_w X)^\alpha.$$

Hence

$$\limsup_{w \to \infty} \frac{E_w X^\alpha}{(E_w X)^\alpha} \leqslant 1.$$

Now, let $r = m+1$, $s = m$, $t = \alpha$. We get $(E_w X^m)^{m+1-\alpha} \leqslant (E_w X^\alpha)(E_w X^{m+1})^{n-\alpha}$, i.e.,

$$E_w X^\alpha \geqslant (E_w X^m)^{m+1-\alpha} (E_w X^{m+1})^{\alpha - m} \sim (E_w X)^{m(m+1-\alpha)} (E_w X)^{(m+1)(\alpha - m)}$$
$$= (E_w X)^\alpha.$$

Therefore,

$$\liminf_{w\to\infty} \frac{E_w X^\alpha}{(E_w X)^\alpha} \geqslant 1,$$

and this concludes the proof of Theorem 4. $\square$

1.3. Combining the last two theorems, we finally obtain a *theorem on regularly varying moments for power series distributions.*

THEOREM 5. *For a power series distribution generated by an entire function* $f(w) = \sum a_k w^k \in A_\rho$, *we have*

$$E_w X^\alpha \ell(X) \sim (E_w X)^\alpha \ell(E_w x), \ \alpha > 0 \quad (w \to \infty),$$

i.e.,

$$\sum k^\alpha \ell(k) a_k w^k / f(w) \sim (wf'(w)/f(w))^\alpha \ell(wf'(w)/f(w)) \quad (w \to \infty),$$

where $\ell(\cdot)$ *is an arbitrary slowly varying function .*

As an example we take the well-known Poisson distribution. Applying Theorem 5, we obtain

THEOREM 6. *For the Poisson law defined by* $P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \ \lambda > 0, \ k = 0, 1, 2, \cdots$, *we have*

$$EX^\alpha \ell(X) := \sum k^\alpha \ell(k) \frac{\lambda^k}{k!} e^{-\lambda} \sim \lambda^\alpha \ell(\lambda), \ \alpha > 0 \quad (\lambda \to \infty).$$

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Matematički institut SANU
Knez Mihailova 35
11000 Beograd
Serbia
ssimic@mi.sanu.ac.yu

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