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# ON A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS HAVING SLOWLY VARYING SOLUTIONS

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*To the memory of Professor Tatjana Ostrogorski.*

Abstract. Functional differential equations with deviating arguments are studied for the first time in the framework of Karamata regularly varying functions. A sharp condition is established for the existence of slowly varying solutions for a class of second order linear equations of the form $x'' = q(t)x(g(t))$, both in the retarded and in the advanced case.

## 1. Introduction and results.

The theory of regular variation, which was initiated by Karamata in 1930, has provided a major tool for various branches of mathematical analysis including Abelian and Tauberian theorems, analytic number theory and complex analysis, and it is equally important for probability theory.

We recall that a measurable function $f : [0, \infty) \to (0, \infty)$ is said to be regularly varying of index $\rho \in \mathbf{R}$ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for any } \lambda > 0.$$

The totality of regularly varying functions of index $\rho$ is denoted by $\mathrm{RV}(\rho)$. The symbol SV is used to denote $\mathrm{RV}(0)$ and a member of $\mathrm{SV} = \mathrm{RV}(0)$ is referred to as a slowly varying function. If $f(t) \in \mathrm{RV}(\rho)$, then $f(t) = t^\rho L(t)$ for some $L(t) \in \mathrm{SV}$, and so the class of slowly varying functions is of fundamental importance in regular variation. In the later part of the paper, among many basic properties of slowly varying functions, we emphasize the representation theorem which asserts that $L(t) \in \mathrm{SV}$ if and only if it is expressible in the form

$$f(t) = c(t) \exp\left\{\int_a^t \frac{\varepsilon(s)}{s}\, ds\right\}, \quad t \geqslant a,$$

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for some $a > 0$ and some measurable functions $c(t)$ and $\varepsilon(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0,\infty) \quad \text{and} \quad \lim_{t\to\infty} \varepsilon(t) = 0.$$

For the most comprehensive exposition of regular variation and its applications, the reader is referred to the book of Bingham, Goldie and Teugels [2].

The history of the study how Karamata's theory intersects with the theory of differential equations began in 1947 by the seminal paper of Avakumović on the Thomas–Fermi equation, [1]. Linear equations were first studied by Omey in 1981, [13]. Systematic investigations in this direction started with a paper of Marić and Tomić [11] published in 1976. A complete survey of the results on differential equations, both linear and nonlinear, developed by means of regular variation is given in the monograph of Marić [10]. It is shown therein that the class of Karamata regularly varying functions is a well-suited framework for the asymptotic analysis of nonoscillatory solutions of second order linear and nonlinear differential equations. As an example for that statement we give the following theorem due to Marić and Tomić [12] (see also [10, Thm. 1.1]), which provides a sharp criterion for the existence of a slowly varying solution to the second order linear differential equation

$$x'' = q(t)x, \quad q(t) > 0, \tag{A}$$

where $q$ is continuous and integrable on some positive half-axis $(a,\infty)$.

THEOREM 1.1. *Equation (A) possesses a slowly varying solution $x(t)$ if and only if*

$$\lim_{t\to\infty} t\int_t^\infty q(s)\,ds = 0. \tag{1.1}$$

*It is decreasing and can be represented in the form*

$$x(t) = x(t_0)\exp\left\{\int_{t_0}^t \frac{v(s)-Q(s)}{s}\,ds\right\}, \quad t \geqslant t_0, \tag{1.2}$$

*for some $t_0 > a$, where*

$$Q(t) = t\int_t^\infty q(s)\,ds \tag{1.3}$$

*and $v(t)$ is a solution of the integral equation*

$$v(t) = t\int_t^\infty \left(\frac{v(s)-Q(s)}{s}\right)^2 ds, \quad t \geqslant t_0. \tag{1.4}$$

Since by (1.1), $Q(t) \to 0$, as $t \to \infty$, one can choose $t_1 > t_0$ so large that

$$8Q(t) \leqslant \theta < 1, \quad \text{for} \quad t \geqslant t_1. \tag{1.5}$$

Further study of equation (A) and its generalizations in the spirit of Theorem 1.1 has been carried out by Howard and Marić [5] and Jaroš and Kusano [6], [7].

A question naturally arises concerning the possibility of investigating the asymptotic behavior of functional differential equations with deviating arguments in the framework of Karamata functions. To the best of the authors' knowledge, nothing is known about this subject except for a paper of Grimm and Hall [3], in which the slowly varying character of positive decreasing solutions of some differential equations with advanced argument was discussed.

The present work was motivated by this observation and attempts to establish the existence of a slowly varying solution (an SV-solution for short) for equations of the type

$$x''(t) = q(t)x(g(t)), \quad q(t) > 0, \tag{$B$}$$

which is a companion functional differential equation to equation (A). Here again, $q$ is continuous and integrable on some positive half-axis $[t_0, \infty)$.

Our result pertinent to the retarded case is the following

THEOREM 1.2. *Suppose that $g : [0, \infty) \to \mathbf{R}^+$ is a continuous increasing function such that $g(t) \to \infty$, as $t \to \infty$, satisfying $g(t) < t$, for $t \geqslant t_0$ where $t_0$ is such that $g(t) \geqslant t_0 > 1$, for $t \geqslant t_1$ and*

$$\int_{g(t)}^{t} \frac{Q(s)}{s} ds \leqslant 1/e, \quad t \geqslant t_1. \tag{1.6}$$

*Then equation* (B) *possesses a slowly varying solution if and only if condition* (1.1) *is satisfied.*

Obviously, this solution is nonoscillatory since SV-functions are positive by definition.

The result pertinent to the advanced case is the following

THEOREM 1.3. *Suppose that $g : [0, \infty) \to \mathbf{R}^+$ is a continuous increasing function such that $g(t) \to \infty$, as $t \to \infty$, satisfying $g(t) > t$, for $t \geqslant t_1$, and*

$$\int_{t}^{g(t)} \frac{Q(s)}{s} ds \leqslant 1/e, \quad t \geqslant t_1. \tag{1.7}$$

*Then equation* (B) *possesses a slowly varying solution if and only if condition* (1.1) *is satisfied.*

To establish the existence of an SV-solution for (B) we proceed as follows. First, we form an infinite family of differential equations of the form (A) each of which possesses an SV-solution, and then, with the help of the Schauder–Tychonoff fixed-point theorem, look for the equation in the family whose SV-solution exactly gives birth to the desired solution of equation (B). To make this procedure feasible we need precise information about the structure of the SV-solutions of differential equations of the form (A) without functional argument. The proof of Theorem 1.1 will be given in Section 2 for completeness, and those of Theorems 1.2 and 1.3 will be presented in Section 3.

To conclude the introduction it should be mentioned that oscillation theory of functional differential equations including equation (B) has been the subject of intensive investigations for the past three decades and that there is a vast literature devoted to the study of the oscillatory and nonoscillatory behavior of a variety of such equations from diverse angles and viewpoints. See for example the books of Györi and Ladas [4], Koplatadze and Chanturia [8] and Ladde, Lakshmikantham and Zhang [9].

REMARK 1.1. For the linear equation (A), there exist results of a similar nature also when the limit in (1.1) is positive, [10, Thms. 1.2 and 1.11]. It seems plausible that such results hold for equation (B) under suitable assumptions on $g(t)$. This, however will be a subject of our future investigations.

## 2. Preliminaries

The proof of Theorem 1.1 given here is elementary and follows [10]. A different one is given by Jaroš and Kusano in [6]. It makes use of a fixed-point method and is applicable also for the case when $q(t)$ is of unrestricted sign.

The "only if" part. Let $x(t)$ be an SV (hence positive) solution of equation (A). Since it is convex due to $q(t) > 0$, it is monotone and in addition, because of [10, Prop. 9c], one has

$$\lim_{t\to\infty} \frac{tx'(t)}{x(t)} = 0. \tag{2.1}$$

Write equation (A) in the form

$$(x'(t)/x(t))' + (x'(t)/x(t))^2 = q(t),$$

integrate over $(t, \infty)$, use (2.1) and multiply throughout by $t$ to obtain

$$tx'(t)/x(t) + t\int_t^\infty (sx'/x)^2 s^{-2} ds = t\int_t^\infty q(s)\,ds.$$

Due to (2.1) the left-hand side integral, and so the one on the right-hand side, converge. Moreover, both sides tend to zero as $t \to \infty$.

Observe that here the convergence of the integral of $q(s)$ is a consequence, not a hypothesis.

The "if" part. It is known that equation (A) has a positive decreasing solution on $(a, \infty)$, [10, Lemma 1.1]; denote it again by $x(t)$. Then integrate on both sides of (A) over $(t, \infty)$. Since $x(t)$ is decreasing and convex, it is such that $x'(t) \to 0$, $t \to \infty$. This leads to

$$-x'(t) = \int_t^\infty q(s)x(s)\,ds,$$

and so

$$0 < -\frac{tx'(t)}{x(t)} \leqslant t\int_t^\infty q(s)\,ds.$$

The right-hand side tends to zero as $t \to \infty$ by hypothesis, whence (2.1) follows and consequently, $x(t)$ is SV, [10, Prop. 10].

Observe that $x'(t) \to c > 0$, as $t \to \infty$ cannot hold. For, this would imply $x(t) \sim cx$ contradicting the fact that $x(t)$ decreases.

Also, an SV-solution $x(t)$ cannot increase. For otherwise, due to the convexity, one would have eventually $x'(t) \geqslant k$, for some $k > 0$, or by integrating, $x(t) \geqslant kt+l$ which is impossible for an SV function [10, Prop. 4.ii].

To obtain the representation (1.2) put

$$\frac{x'(t)}{x(t)} = \frac{v(t) - Q(t)}{t}. \tag{2.2}$$

An integration over $(t_0, t)$ gives (1.2). Here $v(t)$ is indeed a solution of (1.4); as is well known the right-hand side of (2.2) satisfies the Riccati equation

$$\left(\frac{v(t)-Q(t)}{t}\right)' + \left(\frac{v(t)-Q(t)}{t}\right)^2 = q(t)$$

or, due to (1.3),

$$\left(\frac{v(t)}{t}\right)' + \left(\frac{v(t)-Q(t)}{t}\right)^2 = 0$$

which by integrating over $(t, \infty)$, since $\frac{v(t)}{t} \to 0$ due to (2.2), gives (1.4).

Observe that in view of (1.4), $v(t)$ is positive for $t \geqslant t_1$. Also since $x'(t) < 0$, $x(t)$ being decreasing, one obtains from (2.2), $v(t) \leqslant Q(t)$, $t \geqslant t_1$; i.e.

$$0 < v(t) \leqslant Q(t), \quad t \geqslant t_1. \tag{2.3}$$

## 3. Proofs

Our purpose here is to give proofs of Theorems 1.2 and 1.3 based on Theorem 1.1.

Proof of Theorem 1.2. Indeed, the "only if" part is a direct consequence of the later theorem. For, suppose that there exists an SV-solution $x(t)$ of equation (B) on $[t_0, \infty)$; then one writes it as

$$x''(t) = q_x(t)x(t), \quad t \geqslant t_0, \tag{3.1}$$

where $q_x(t) = q(t)x(g(t))/x(t)$. It follows, by Theorem 1.1, that $t\int_t^\infty q_x(s)\,ds \to 0$ as $t \to \infty$. This implies (1.1) since $x(g(t))/x(t) \geqslant 1$ by the decreasing nature of $x(t)$.

The proof of the "if" part. Suppose that (1.1) holds. Let us define $\Xi$ to be the set of positive, continuous nonincreasing functions $\xi(t)$ on $[t_0, \infty)$ such that

$$\xi(t) = 1, \quad \text{for } t_0 \leqslant t \leqslant t_1 \tag{3.2}$$

and

$$\frac{\xi(g(t))}{\xi(t)} \leqslant e \quad \text{for } t \geqslant t_1, \tag{3.3}$$

$t_1$ being defined by (1.5). We remark that $\Xi$ is a nonvoid set since it contains e.g., nonincreasing functions $\xi_\lambda(t)$, $\lambda \in (0, e]$, given by

$$\xi_\lambda(t) = 1, \quad t_0 \leqslant t \leqslant t_1, \quad \xi_\lambda(t) = \exp\left\{-\lambda \int_{t_1}^t \frac{Q(s)}{s}\,ds\right\}, \quad t \geqslant t_1.$$

To show that (3.3) also holds, notice that due to the properties of $g(t)$, there might exist an interval $t_1 \leqslant t \leqslant t_2$, where $g(t) \leqslant t_1$ and $g(t) \geqslant t_1$ for $t \geqslant t_2$. But then, due to (3.2), inequality (3.3) holds for $t_1 \leqslant t \leqslant t_2$ and for $t \geqslant t_2$, by (1.6), one has

$$\frac{\xi_\lambda(g(t))}{\xi_\lambda(t)} \leqslant \exp\left\{e \int_{g(t)}^t \frac{Q(s)}{s}\,ds\right\} \leqslant e.$$

Hence (3.3) holds for all $t \geqslant t_1$.

The set $\Xi$ is a closed convex subset of the locally convex space $C[t_0, \infty)$ of continuous functions on $[t_0, \infty)$ equipped with the metric topology of uniform convergence on compact subintervals of $[t_0, \infty)$.

The set $\Xi$ is clearly convex in $C[t_0, \infty)$. It is also closed; for let $\{\xi_n\}$ be a sequence in $\Xi$ converging to $\eta$ as $n \to \infty$ (i.e., $\xi_n(t)$ converging uniformly to $\eta(t)$ as $n \to \infty$) on compact subinterval of $[t_0, \infty)$. It is clear that $\eta(t)$ is continuous and to prove its positivity on $[t_0, \infty)$ one argues as follows: suppose on the contrary, that there exists a $T > t_1$ such that $\eta(t) > 0$ for $t_0 \leqslant t < T$ and $\eta(t) = 0$ for $t \geqslant T$. By (3.3) one has $\xi_n(g(T)) \leqslant e\xi_n(T)$, and letting $n \to \infty$, there follows $0 < \eta(g(T)) \leqslant e\eta(T) = 0$ which is impossible. This also implies (3.3) for $\eta(t)$.

For each $\xi \in \Xi$ consider the second order ordinary differential equation

$$x'' = q_\xi(t)x, \tag{3.4}$$

where $q_\xi(t)$ is given by

$$q_\xi(t) = q(t)\frac{\xi(g(t))}{\xi(t)}. \tag{3.5}$$

Define

$$Q_\xi(t) = t\int_t^\infty q_\xi(s)\,ds. \tag{3.6}$$

Since, due to (3.3) and (1.1), $Q_\xi(t) \leqslant eQ(t)$ and so $Q_\xi(t) \to 0$, $t \to \infty$ for all $\xi \in \Xi$, Theorem 1.1 ensures that equation (3.4) possesses for every $\xi \in \Xi$, an SV-solution $x_\xi(t)$ expressed in the form

$$x_\xi(t) = \exp\left\{\int_{t_1}^t \frac{v_\xi(s) - Q_\xi(s)}{s}\,ds\right\}, \quad t \geqslant t_1, \tag{3.7}$$

where $v_\xi(t)$ solves the integral equation

$$v_\xi(t) = t\int_t^\infty \left(\frac{v_\xi(s) - Q_\xi(s)}{s}\right)^2 ds, \quad t \geqslant t_1. \tag{3.8}$$

We denote by $\Phi$ a mapping which associates with each $\xi \in \Xi$ the function $\Phi\xi$ defined by

$$\Phi\xi(t) = 1 \quad \text{for } t_0 \leqslant t \leqslant t_1, \quad \Phi\xi(t) = x_\xi(t) \quad \text{for } t \geqslant t_1.$$

We will look for a fixed point of $\Phi$ with the help of the Schauder–Tychonoff fixed-point theorem. For that we need to prove that $\Phi$ is a self-map on $\Xi$, the relative compactness of the set $\Phi(\Xi)$ in $C[t_0, \infty)$ and the continuity of the mapping $\Phi$.

For any $\xi \in \Xi$, the function $\Phi\xi(t)$ is obviously positive and nonincreasing for $t \geqslant t_0$.

Furthermore, due to the definition of $\Phi$, arguing as before, we conclude that to prove the property (3.3) one needs only to consider the case $g(t) \geqslant t_1$ which leads to

$$\frac{\Phi\xi(g(t))}{\Phi\xi(t)} \leqslant \exp\left\{\int_{g(t)}^t \frac{Q_\xi(s) - v_\xi(s)}{s}\,ds\right\} \leqslant \exp\left\{e\int_{g(t)}^t \frac{Q(s)}{s}\,ds\right\} \leqslant e.$$

It follows that $\Phi\xi \in \Xi$, implying that $\Phi$ is a self-map on $\Xi$.

Since $\Phi(\Xi) \subset \Xi$, $\Phi(\Xi)$ is locally uniformly bounded on $[t_0, \infty)$, and since $\xi \in \Xi$ implies

$$0 \geqslant \frac{d}{dt}\Phi\xi(t) = \frac{d}{dt}x_\xi(t) = x_\xi(t)\frac{v_\xi(t) - Q_\xi(t)}{t} \geqslant -\frac{eQ(t)}{t}, \quad t \geqslant t_1,$$

$\Phi(\Xi)$ is locally equicontinuous on $[t_0, \infty)$. This guarantees via the Arzela–Ascoli lemma that $\Phi(\Xi)$ is relatively compact in $C[t_0, \infty)$.

Let $\{\xi_n\}$ be a sequence of functions in $\Xi$ converging to $\eta \in \Xi$ in $C[t_0, \infty)$. The continuity of $\Phi$ is guaranteed if it is shown that the sequence $\{\Phi\xi_n\}$ converges to $\Phi\eta$ in $C[t_0, \infty)$, or equivalently that $\{\Phi\xi_n(t)\}$ converges to $\Phi\eta(t)$ uniformly on compact subintervals of $[t_0, \infty)$. Using (3.7) and the mean value theorem, bearing in mind that the integrand is negative, we have for $t \geqslant t_1$

$$\begin{aligned} |\Phi\xi_n(t) - \Phi\eta(t)| &= |x_{\xi_n}(t) - x_\eta(t)| \\ &= \left|\exp\left\{\int_{t_1}^t \frac{v_{\xi_n}(s) - Q_{\xi_n}(s)}{s}\,ds\right\} - \exp\left\{\int_{t_1}^t \frac{v_\eta(s) - Q_\eta(s)}{s}\,ds\right\}\right| \\ &\leqslant \int_{t_1}^t \frac{|Q_{\xi_n}(s) - Q_\eta(s)| + |v_{\xi_n}(s) - v_\eta(s)|}{s}\,ds, \end{aligned} \tag{3.9}$$

where $Q_{\xi_n}(t)$ and $Q_\eta(t)$ are defined by (3.6) and $v_{\xi_n}(t)$ and $v_\eta(t)$ are the solutions of the integral equation (3.8) with $\xi$ replaced by $\xi_n$ and $\eta$, respectively. Consequently, to verify the continuity of $\Phi$ in the topology of $C[t_0, \infty)$ it suffices to prove that the integrand of the last integral in (3.9) converges to 0 uniformly on any compact subinterval of $[t_1, \infty)$. Since

$$\frac{|Q_{\xi_n}(t) - Q_\eta(t)|}{t} \leqslant \int_t^\infty q(s)\left|\frac{\xi_n(g(s))}{\xi_n(s)} - \frac{\eta(g(s))}{\eta(s)}\right| ds, \tag{3.10}$$

an application of the Lebesgue dominated convergence theorem ensures the uniform convergence $|Q_{\xi_n}(t) - Q_\eta(t)|/t \to 0$ on $[t_1, \infty)$ as $n \to \infty$. To estimate $|v_{\xi_n}(t) - v_\eta(t)|/t$ we proceed as follows. Using (3.8) we have

$$\begin{aligned} |v_{\xi_n}(t) - v_\eta(t)| &= t\left|\int_t^\infty \frac{(v_{\xi_n}(s) - Q_{\xi_n}(s))^2 - (v_\eta(s) - Q_\eta(s))^2}{s^2}\,ds\right| \\ &\leqslant t\int_t^\infty \frac{1}{s^2}\Big[\big(|v_{\xi_n}(s)| + |v_\eta(s)| + |Q_{\xi_n}(s)| + |Q_\eta(s)|\big) \\ &\qquad\qquad \times \big(|v_{\xi_n}(s) - v_\eta(s)| + |Q_{\xi_n}(s) - Q_\eta(s)|\big)\Big]\,ds \end{aligned}$$

for $t \geqslant t_1$, from which, noting that by (1.5),

$$|v_{\xi_n}(t)| + |v_\eta(t)| + |Q_{\xi_n}(t)| + |Q_\eta(t)| \leqslant 8Q(t) \leqslant \theta < 1, \quad t \geqslant t_1,$$

we obtain

$$|v_{\xi_n}(t) - v_\eta(t)| \leqslant \theta t\int_t^\infty \frac{|v_{\xi_n}(s) - v_\eta(s)| + |Q_{\xi_n}(s) - Q_\eta(s)|}{s^2}\,ds, \quad t \geqslant t_1. \tag{3.11}$$

For brevity we put

$$z_n(t) = \int_t^\infty \frac{|v_{\xi_n}(s) - v_\eta(s)|}{s^2}\,ds. \tag{3.12}$$

Then, (3.11) can be rewritten as

$$tz_n'(t) + \theta z_n(t) \geqslant -\theta \int_t^\infty \frac{|Q_{\xi_n}(s) - Q_\eta(s)|}{s^2}\,ds, \quad t \geqslant t_1,$$

or equivalently

$$(t^\theta z_n(t))' \geqslant \frac{-\theta}{t^{1-\theta}} \int_t^\infty \frac{|Q_{\xi_n}(s) - Q_\eta(s)|}{s^2}\,ds, \quad t \geqslant t_1. \tag{3.13}$$

Noting that $t^\theta z_n(t) \to 0$ as $t \to \infty$ and integrating (3.13) from $t$ to $\infty$, we obtain

$$t^\theta z_n(t) \leqslant \int_t^\infty \frac{|Q_{\xi_n}(s) - Q_\eta(s)|}{s^{2-\theta}}\,ds, \quad t \geqslant t_1. \tag{3.14}$$

Using (3.14) in (3.11), we conclude that

$$\begin{aligned} \frac{|v_{\xi_n}(t) - v_\eta(t)|}{t} \leqslant{}& \frac{1}{t^\theta} \int_t^\infty \frac{|Q_{\xi_n}(s) - Q_\eta(s)|}{s^{2-\theta}}\,ds \\ &+ \theta \int_t^\infty \frac{|Q_{\xi_n}(s) - Q_\eta(s)|}{s^2}\,ds, \quad t \geqslant t_1. \end{aligned} \tag{3.15}$$

Since the right-hand side of (3.15) converges uniformly on $[t_1, \infty)$ as $n \to \infty$, so does the function $|v_{\xi_n}(t) - v_\eta(t)|/t$. This, because of (3.9) and (3.10), establishes the continuity of the mapping $\Phi$.

Thus all the hypotheses of the Schauder–Tychonoff fixed-point theorem are fulfilled, and so there exists an element $\xi_0 \in \Xi$ such that $\xi_0 = \Phi\xi_0$. From the definition of $\Phi$ it follows that $\xi_0(t)$ satisfies the differential equation $\xi_0''(t) = q_\xi(t)\xi_0(t)$, for $t \geqslant t_1$, which because of (3.5) implies that $\xi_0''(t) = q(t)\xi_0(g(t))$ for $t \geqslant t_1$, that is, $\xi_0(t)$ is a solution of the functional differential equation (B) on $[t_1, \infty)$. That $\xi_0(t)$ is a slowly varying function follows from the fact that $\xi_0(t)$ coincides with $x_{\xi_0}(t)$ for $t \geqslant t_1$ which is an SV-solution of equation (3.4). This completes the proof of the "if" part of Theorem 1.2. □

Proof of Theorem 1.3. The "only if" part: As before, by supposing that equation (B) written in the form (3.1) has an SV-solution on $[t_0, \infty)$ one concludes that $t\int_t^\infty q_x(s)\,ds \to 0$ as $t \to \infty$. Here $q_x(t) = q(t)x(g(t))/x(t)$. Due to the representation (1.2) and condition (1.7) one has $x(g(t))/x(t) \geqslant 1/e$ for $t \geqslant t_1$ and condition (1.1) follows.

The proof of the "if" part. This time we define the set $\Xi$ as the set of positive, continuous, nonincreasing functions $\xi(t)$ on $[t_1, \infty)$ such that

$$\frac{\xi(t)}{\xi(g(t))} \leqslant e \quad \text{for} \quad t \geqslant t_1.$$

The same reasoning as before shows that the set $\Xi$ is a nonvoid convex and closed subset of the locally convex space $C[t_0, \infty)$.

Again, for each $\xi \in \Xi$, we consider equation (3.4) and use notations (3.5) and (3.6).

The mapping $\Phi$ is now defined as

$$\Phi\xi(t) = x_\xi(t) \quad \text{for} \quad t \geqslant t_1,$$

where $x_\xi(t)$ is a slowly varying solution of (3.4) whose existence is guaranteed by Theorem 1.1 bearing in mind that $\xi(g(t))/\xi(t) \leqslant 1$. It has the representation given by (3.7) and (3.8).

To show that mapping $\Phi$ fulfils the conditions of the Schauder–Tychonoff theorem one proceeds exactly as in the proof of Theorem B. This leads to the desired result. $\square$

Remark 3.1. Observe that slowly varying solutions of equation (B) cannot increase. This is obtained exactly as for the linear case (A). Moreover, all positive decreasing solutions of equation (B) in the case $g(t) \geqslant t$, provided that these exist, are slowly varying. Indeed if $x(t)$ is such a solution, then

$$-x'(t) = \int_t^\infty q(s)x(g(s))\,ds$$

and so

$$-x'(t) \leqslant x(t)\int_t^\infty q(s)\,ds$$

since $g(t) \geqslant t$ and $x(s)$ is decreasing. Hence, due to (1.1), one has $-tx'(t)/x(t) \to 0$, as $t \to \infty$ so that $x(t)$ is slowly varying (compare: Grimm and Hall [3]).

## 4. Examples and concluding remarks

We present some examples illustrating Theorems 1.2 and 1.3.

Example 4.1. Consider the equation

$$x''(t) = q_1(t)x(\lambda t), \quad t \geqslant e, \tag{4.1}$$

where $q_1(t)$ is defined by

$$q_1(t) = \frac{1}{2t^2\sqrt{\log t}}\left(1 + \frac{1}{2\sqrt{\log t}} + \frac{1}{2\log t}\right)\exp\left(\sqrt{\log t + \log\lambda} - \sqrt{\log t}\right).$$

The condition (1.1) is satisfied for this equation since

$$\int_t^\infty q_1(s)\,ds \sim \frac{1}{2t\sqrt{\log t}} \quad \text{as } t \to \infty,$$

where the symbol $\sim$ is used to denote the asymptotic equivalence

$$f(t) \sim g(t) \quad \text{as } t \to \infty \;\Leftrightarrow\; \lim_{t\to\infty}\frac{f(t)}{g(t)} = 1.$$

Equation (4.1) is retarded for $0 < \lambda < 1$ and advanced for $\lambda > 1$.

Notice also that here $g(t) = \lambda t$, satisfies the condition

$$\limsup_{t\to\infty}\frac{t}{g(t)} = \frac{1}{\lambda} \quad \text{for} \quad 0 < \lambda < 1, \tag{4.2a}$$

which implies (1.6) and

$$\limsup_{t\to\infty} \frac{g(t)}{t} = \lambda \quad \text{for} \quad \lambda > 1, \tag{4.2b}$$

which implies (1.7).

Therefore, equation (4.1) possesses a slowly varying solution by Theorem 1.2 or 1.3. It is easy to check that $x(t) = \exp\left(-\sqrt{\log t}\right)$ is one such solution.

An analogous reasoning holds for the case when in the considered equation (4.1) $x(\lambda t)$ is replaced by $x(t+\alpha)$. It is then retarded or advanced according as $\alpha < 0$ or $\alpha > 0$. Here the exponential factor in $q_1(t)$ should be replaced by $\exp((\log(t+\alpha)^{1/2}) - \exp(\log t)^{1/2}$ and repeat the argument.

An example of $\{q(t), g(t)\}$ satisfying (1.1) and (1.6) is given below.

Example 4.2. Consider the retarded equation

$$x''(t) = q_2(t)x(t^\theta), \quad t \geqslant 1, \tag{4.3}$$

where $0 < \theta < 1$ and $q_2(t)$ is defined by

$$q_2(t) = \frac{1}{2t^2\sqrt{\log t}}\left(1 + \frac{1}{2\sqrt{\log t}} + \frac{1}{2\log t}\right)\exp\left\{-\left(1-\sqrt{\theta}\right)\sqrt{\log t}\right\}.$$

Since

$$\int_t^\infty q_2(s)\,ds = o\left(\frac{1}{t(\log t)^m}\right) \quad \text{as } t\to\infty, \text{ for any } m \in \mathbf{N},$$

one can easily see that (1.1) and (1.6) are satisfied for the this equation, so that there exists an SV-solution of (4.3). In fact, (4.3) has such a solution $x(t) = \exp\left(-\sqrt{\log t}\right)$.

Remark 4.1. For the differential equation (A) it is known from Marić and Tomić [12] that the condition (1.1) is also a necessary and sufficient condition for the existence of a regularly varying solution of index 1; see also Marić [10] and Jaroš and Kusano [6]. From this fact we conjecture that (1.1) would provide a sharp condition for a class of retarded equations of the form (B) to have positive solutions belonging to the class of regularly varying solutions of index 1. We give below an example which might support the conjecture, but we are still far from its verification.

Example 4.3. Consider the equation

$$x''(t) = q_3(t)x(t/e), \quad t \geqslant e, \tag{4.4}$$

where

$$q_3(t) = \frac{1}{2t^2\sqrt{\log t}}\left(1 + \frac{1}{2\sqrt{\log t}} - \frac{1}{2\log t}\right)\exp\left(\sqrt{\log t} - \sqrt{\log t - 1}\right).$$

Since $q_3(t)$ satisfies (1.1) and (4.2a), the equation (4.4) possesses an SV-solution $x_0(t)$ by Theorem 1.2. A simple calculation shows that this equation has also the solution $x_1(t) = t\exp\left(\sqrt{\log t}\right)$ which is a regularly varying function of index 1.

EXAMPLE 4.4. Consider the equation

$$x''(t) = \exp\big(-(1-\gamma)t\big)x(\gamma t), \quad 0 < \gamma < 1, \tag{4.5}$$

with $q(t) = \exp\big(-(1-\gamma)t\big)$ and $g(t) = \gamma t$ satisfying (1.1) and (4.2a). Theorem 1.2 ensures the existence of an SV-solution $x_0(t)$ for (4.5). One sees that (4.5) has another solution $x_1(t) = \exp(-t)$. Note that $\exp(-t)$ is not a regularly varying function but is a rapidly varying one of index $-\infty$.

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