PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 80(94) (2006), 193–206 DOI:10.2298/PIM0694193K

ULTRAFAST SUBORDINATORS AND THEIR HITTING TIMES

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Abstract. Ultrafast subordinators are nondecreasing Lévy processes obtained as the limit of suitably normalized sums of independent random variables with slowly varying probability tails. They occur in a physical model of ultraslow diffusion, where the inverse or hitting time process randomizes the time variable. In this paper, we use regular variation arguments to prove that a wide class of ultrafast subordinators generate holomorphic semigroups. We then use this fact to compute the density of the hitting times. The density formula is important in the physics application, since it is used to calculate the solutions of certain distributed-order fractional diffusion equations.

1. Introduction

Ultrafast subordinators are connected with certain random walk models in physics. In these models, waiting times with power-law probability tails are randomized in terms of the power law exponent. The renewal process with these waiting times is then the inverse or hitting time process of the random walk with these jumps. The probability tails of these random variables are slowly varying, so that the random walk grows vary fast, and the renewal (inverse) process very slow. The random walk limits form an interesting new class of subordinators. The paper [**19**] develops the limit theory for these ultrafast subordinators, together with some results on their hitting times. The basic approach is to study the asymptotic behavior of the renewal process by first proving a limit theorem for the random walk, and then inverting. The random walk converges to a subordinator (a Lévy process with nondecreasing sample paths) and the renewal process converges to the inverse or hitting time (or first passage time) process of the subordinator. The paper [**19**] imposes a technical condition which is difficult to check. In this paper, we remove

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2000 *Mathematics Subject Classification*: Primary 60G50, 60F17; Secondary 60H30, 82C31.
Key words and phrases: regular variation, anomalous diffusion, subordinator, hitting time.
M. Kovács was partially supported by postdoctoral grant No. 623-2005-5078 of the Swedish Research Council and research grant (CZN-14/2005) of the Science and Technology Foundation. M.M. Meerschaert was partially supported by USA National Science Foundation grants DMS-0417869 and DMS-0139927.

that technical condition using a deep result from the theory of semigroups [**5, 10**] together with some regular variation arguments.

Regular variation is an asymptotic property of real functions that essentially imposes a power law growth condition [**12, 29**]. It has found many applications in probability theory [**9**] and other areas of pure and applied mathematics [**4**]. A real function $g(r)$ varies regularly with index α if $g(\lambda r) \sim \lambda^\alpha g(r)$ as $r \to \infty$ for any $\lambda > 0$. Regular variation for real-valued functions on $\mathbb{R}^d$ began with the work of Stam [**31**], de Haan and Resnick [**13**], Stadtmüller and Trautner [**30**] and Jakimiv [**15**]. Soon after this, Ostrogorski developed the theory of regularly varying functions on $\mathbb{R}^d$ even further, proving a representation theorem along with Abelian Theorems for Laplace transforms, Fourier transforms, and other integral transforms [**20, 21, 22, 23, 24, 25**]. A real-valued function $F(x)$ on $\mathbb{R}^d$ varies regularly if $F(\lambda x) \sim \lambda^\alpha F(x)$ as $\|x\| \to \infty$, uniformly on compact sets of x in some cone in $\mathbb{R}^d$. For some additional extensions and applications, see [**17**]. The main theorem in this paper, Theorem 2.2, depends on establishing a multivariable regular variation condition $I(re^{i\theta}) \sim g(r)e^{i\theta}$ uniformly in $|\theta| \leqslant \theta_0$ as $r \to \infty$ where $I(z)$ is a complex function and $g(r)$ is regularly varying. Separating the real and imaginary parts, and identifying the complex plane with the underlying two dimensional real vector space, this amounts to regular variation on a cone in $\mathbb{R}^2$. Hence the main technical tool in this paper is the multivariable theory of regular variation.

2. Ultrafast subordinators

We begin by explaining the ultrafast subordinator model as a limit of certain random walks. In this random walk model, we start with jumps having power-law probability tails, and then we randomize the power law exponent. Let $B_1, B_2, \dots$ be i.i.d. with density p, where $p(\beta)$ is a probability density on $0 < \beta < \alpha$ for some $0 < \alpha < 1$. For the purposes of this paper we make two regular variation assumptions to control the growth behavior of the density $p(\beta)$ at the endpoints $\beta = 0$ and $\beta = \alpha$. First we assume that $p(\beta)$ is regularly varying at $\beta = 0+$ with index $b > 0$, which means that

$$\lim_{\beta \downarrow 0} \frac{p(\lambda\beta)}{p(\beta)} = \lambda^b \quad \text{for all } \lambda > 0. \tag{2.1}$$

Next we assume that $p(\beta)$ is regularly varying at $\beta = \alpha-$ with index $a-1$ for some $a > 0$, so that

$$\lim_{\beta \downarrow 0} \frac{p(\lambda(\alpha-\beta))}{p(\alpha-\beta)} = \lambda^{a-1} \quad \text{for all } \lambda > 0. \tag{2.2}$$

We remark that if $f(x)$ varies regularly at infinity with index $c \in \mathbb{R}$, meaning that $f(\lambda x) \sim \lambda^c f(x)$ as $x \to \infty$ for all $\lambda > 0$, then it is easy to check that $F(x) = f(1/x)$ varies regularly at $x = 0+$ with index $-c$, and $F(y-x)$ varies regularly at $x = y-$ with index $-c$. Furthermore, it is also easy to check that $f(x) = x^c L(x)$ for some function L which is slowly varying at infinity, meaning that $L(\lambda x) \sim L(x)$ as $x \to \infty$ for all $\lambda > 0$. Then we also have $F(x) = x^{-c}L(1/x)$ and $F(y-x) = (y-x)^{-c}L(1/(y-x))$. Since for any $\varepsilon > 0$ we have $x^{-\varepsilon} < L(x) < x^{\varepsilon}$

for all x sufficiently large (see, e.g., [**9**, VIII.8, Lemma 2]), we see that regularly varying functions grow like power laws.

Recall that B_i are i.i.d. random variables taking values in $0 < \beta < \alpha$ for some $0 < \alpha < 1$. Given any $c \geqslant 1$ let $J_1^{(c)}, J_2^{(c)}, \ldots$ be nonnegative i.i.d. random variables such that for any $0 < \beta < \alpha$ we have

$$P\{J_i^{(c)} > u | B_i = \beta\} = \begin{cases} 1 & 0 \leqslant u < c^{-1/\beta} \\ c^{-1}u^{-\beta} & u \geqslant c^{-1/\beta} \end{cases}. \tag{2.3}$$

Then the density $\psi_c(u|\beta)$ of $J_i^{(c)}$ given $B_i = \beta$ is

$$\psi_c(u|\beta) = \begin{cases} 0 & 0 \leqslant u < c^{-1/\beta} \\ c^{-1}\beta u^{-\beta-1} & u \geqslant c^{-1/\beta} \end{cases}.$$

If we define for $0 < \beta < \alpha$

$$P\{J_1 > t | B_1 = \beta\} = \begin{cases} 1 & 0 \leqslant t < 1 \\ t^{-\beta} & t \geqslant 1 \end{cases}$$

we get by letting $u = c^{-1/\beta} t$ that

$$P\{c^{-1/\beta} J_1 > u | B_1 = \beta\} = \begin{cases} 1 & 0 \leqslant u < c^{-1/\beta} \\ c^{-1}u^{-\beta} & u \geqslant c^{-1/\beta} \end{cases}$$

so conditionally on $B_1 = \beta$ we have $J_1^{(c)} \stackrel{d}{=} c^{-1/\beta} J_1$. Moreover, for $t \geqslant 1$

$$P\{J_1 > t\} = \int_0^1 t^{-\beta} p(\beta) d\beta$$

where we define $p(\beta) = 0$ for $\beta \geqslant \alpha$. It turns out (see Remark 2.1) that $P\{J_1 > t\}$ is slowly varying at infinity, so that the unconditional waiting times have infinite moments of all orders. Limit theorems for renewal processes with these slowly varying waiting times were developed in [**18**] by first proving limit theorems for the associated random walk using nonlinear scaling, the usual approach for slowly varying tails [**7, 16, 32**], and then inverting. A different approach in [**19**] uses random rescaling. Let

$$T^{(c)}(0) = 0 \quad \text{and} \quad T^{(c)}(t) = \sum_{i=1}^{[t]} J_i^{(c)} \tag{2.4}$$

be a sequence of random walks depending on the parameter $c > 0$.

The following result is essentially contained in [**19**] but is summarized here for the convenience of the reader. The space $D([0,\infty),[0,\infty))$ consists of all nonnegative real valued functions $x(t)$ defined for $t \geqslant 0$ which are continuous from the right with left-hand limits. Elements x_n of this space converge to a limit x in the Skorokhod J_1-topology if the graph of $x_n(t)$ converges uniformly to the graph of $x(t)$ in a way that allows both the jump sizes and jump locations to vary with n, for a precise definition see [**3, 33**]. A Lévy process $X(t)$ is a random element of this space such that $X(0) = 0$ with probability one, $X(t_n) \Rightarrow X(t)$ in distribution when $t_n \to t$ (stochastically continuous), $X(t+s) - X(t)$ is independent of $X(t)$ (independent increments), and $X(t+s) - X(s)$ is identically distributed with $X(t)$

(stationary increments), for more information see [**2, 28**]. A Lévy process $X(t)$ is called a subordinator if $X(t)$ is a nondecreasing function of t with probability one.

THEOREM 2.1. *Define the triangular array* $\{J_i^{(c)} : 1 \leqslant i \leqslant [ct], c \geqslant 1\}$ *by* (2.3). *If* (2.1) *holds, then for the partial sum process* $\{T^{(c)}(t)\}_{t\geqslant 0}$ *defined by* (2.4) *we have*

$$\{T^{(c)}(ct)\}_{t\geqslant 0} \Rightarrow \{D(t)\}_{t\geqslant 0} \tag{2.5}$$

as $c \to \infty$ *in the* J_1*-topology on* $D([0,\infty),[0,\infty))$, *where* $D(t)$ *is a subordinator with* $\mathbb{E}(e^{-sD(t)}) = e^{-tI(s)}$ *and*

$$I(s) = \int_0^1 s^\beta p(\beta)\Gamma(1-\beta)d\beta. \tag{2.6}$$

PROOF. Theorem 3.4 and Corollary 3.5 in [**19**] show that (2.5) holds where $D(t)$ has Lévy representation $\mathbb{E}(e^{-sD(t)}) = \exp(-\int(e^{-su}-1)t\phi(du))$ with

$$\phi(u,\infty) = \int_0^1 u^{-\beta}p(\beta)d\beta \tag{2.7}$$

as long as p varies regularly at zero with some index $a-1$ for some $a > 0$ and additionally

$$J = \int_0^1 \frac{p(\beta)}{1-\beta}\,d\beta < \infty. \tag{2.8}$$

The Lévy representation uniquely determines the process, for more information see [**2, 17, 28**]. Assumption (2.1) implies that p varies regularly at zero with an index in the required range, and (2.8) follows from the fact that $p(\beta) = 0$ for $\beta > \alpha$ for some $\alpha < 1$. Then it follows from Theorem 30.1 of [**28**] along with (2.7) that the subordinator $D(t)$ has Laplace symbol

$$\begin{aligned} I(s) &= \int_0^\infty (1-e^{-su})\,\phi(du) = \int_0^1 \Big(\int_0^\infty (1-e^{-su})\beta u^{-\beta-1}\,du\Big)p(\beta)\,d\beta \\ &= \int_0^1 \Gamma(1-\beta)s^\beta p(\beta)\,d\beta. \end{aligned}$$

so that (2.6) also holds. □

Let $F(x,t) = P\{D(t) \leqslant x\}$ denote the family of distribution functions of the subordinator $D(t)$. Consider the Banach space $X = C_0(\mathbb{R})$ of continuous functions $u : \mathbb{R} \to \mathbb{C}$ that satisfy $u(x) \to 0$ as $|x| \to \infty$, with the supremum norm $\|u\| = \sup\{|u(x)| : x \in \mathbb{R}\}$. Let $\mathcal{B}(X)$ denote the Banach space of bounded linear operators on X endowed with the operator norm $\|T\|_{\mathcal{B}(X)} := \sup_{||u||\leqslant 1} ||Tu||$. The continuous convolution (Feller) semigroup $\{T(t)\}_{t\geqslant 0} \subset \mathcal{B}(X)$ associated with the subordinator $D(t)$ is defined for $u \in X$ by

$$[T(t)u](y) = \int_0^\infty u(y-x)F(dx,t) = \mathbb{E}[u(y-D(t))]. \tag{2.9}$$

It is well known that $\{T(t)\}_{t\geqslant 0}$ defined in (2.9) is a strongly continuous semigroup on $X = C_0(\mathbb{R})$ [**14**, Example 4.1.3]. The generator A of the semigroup $\{T(t)\}_{t\geqslant 0}$ is defined as follows. If for $u \in C_0(\mathbb{R})$ there is $g \in C_0(\mathbb{R})$ such that

$$\lim_{t\downarrow 0}\left\|\frac{T(t)u-u}{t}-g\right\| = \lim_{t\downarrow 0}\sup_{y\in\mathbb{R}}\left|\frac{[T(t)u](y)-u(y)}{t}-g(y)\right| = 0,$$

then we define $Au := g$. The domain $\mathcal{D}(A)$ of A consists of all $u \in X$ such that the Banach space limit $\lim_{t\downarrow 0} t^{-1}(T(t)u-u)$ exists. Since in $C_0(\mathbb{R})$ the norm is the supremum norm, if $u \in \mathcal{D}(A)$, then $[Au](y)$ is given by the pointwise limit

$$[Au](y) = \lim_{t\downarrow 0}\frac{[T(t)u](y)-u(y)}{t} = g(y),\ \ y \in \mathbb{R}. \tag{2.10}$$

In this case it can be shown (as a special case of [**27**, Theorem 4.3]) that

$$\mathcal{D} := \{u \in C_0(\mathbb{R}) :\ u \text{ is differentiable and } u' \in C_0(\mathbb{R})\} \subset \mathcal{D}(A) \tag{2.11}$$

and

$$[Au](y) = \int_0^\infty (u(y-z)-u(y))\ \phi(dz),\ \ u \in \mathcal{D}. \tag{2.12}$$

Note that the term under the limit in (2.10) has Laplace transform

$$\frac{e^{-tI(s)}-1}{t}\ \tilde{u}(s) \to -I(s)\tilde{u}(s) \text{ as } t\downarrow 0$$

where $\tilde{u}(s) = \int_0^\infty e^{-sy}u(y)\,dy$ denotes the Laplace transform of the function u. To motivate the generator formula (2.12), note that formally we can invert the Laplace transform

$$-I(s)\,\tilde{u}(s) = \int_0^\infty (e^{-sz}-1)\,\phi(dz)\,\tilde{u}(s)$$

term by term using the fact that $e^{-sz}\tilde{u}(s)$ is the Laplace transform of the function $y \mapsto u(y-z)$.

The semigroup defined in (2.9) is a special case of the following general construction. Let X be a Banach space and let $\{S(t)\}_{t\geqslant 0}$ be a uniformly bounded strongly continuous semigroup on X. Then the subordinated semigroup

$$T(t)u = \int_0^\infty S(x)u\,F(dx,t),\ \ u \in X, \tag{2.13}$$

where the integral can be understood either in the Riemann–Stieltjes or in the Lebesgue–Stieltjes (Bochner) sense, is strongly continuous on X (see, for example, [**14**, Theorem 4.3.1] and [**27**]). If $X = C_0(\mathbb{R})$ and $[S(x)u](y) = u(y-x)$, then $\{T(t)\}_{t\geqslant 0}$ coincide with the semigroup defined in (2.9) since the translation semigroup is strongly continuous on $C_0(\mathbb{R})$.

Next we introduce a special class of strongly continuous semigroups. Let X be a complex Banach space and $\Omega \subset \mathbb{C}$ be an open set. A function $f : \Omega \to X$ is *holomorphic* if the Banach space limit

$$f'(w_0) := \lim_{\substack{h\to 0\\ h\in\mathbb{C}\setminus\{0\}}}\frac{f(w_0+h)-f(w_0)}{h}$$

exists for all $w_0 \in \Omega$. A strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ is called a *bounded holomorphic semigroup* if for some $\theta \in (0, \frac{\pi}{2}]$ the semigroup $\{T(t)\}_{t\geqslant 0}$ has a bounded holomorphic extension to a sectorial region $\Sigma_{\theta'} := \{w \in \mathbb{C} \setminus \{0\} : |\arg w| < \theta'\}$ for all $\theta' \in (0, \theta)$. That is, if for some $\theta \in (0, \frac{\pi}{2}]$ there exists a family $\{T(w)\}_{w\in\Sigma_\theta} \subset \mathcal{B}(X)$ that extends $\{T(t)\}_{t\geqslant 0}$ such that the function $\Sigma_\theta \ni w \mapsto T(w) \in \mathcal{B}(X)$ is holomorphic in Σ_θ and for all $\theta' \in (0, \theta)$ there exists $M_{\theta'}$ such that $\sup_{w\in\Sigma_{\theta'}} ||T(w)||_{\mathcal{B}(X)} \leqslant M_{\theta'}$ (see, [**1**, Definition 3.7.3]).

Carasso and Kato [**5**] proved that if $\mathcal{B}$ denotes the Banach algebra of complex valued functions of bounded total variation on $[0, \infty)$ (or complex Borel measures) with convolution as multiplication and with the total variation norm $|| \cdot ||_{TV}$ on $[0, \infty)$, then the following conditions are equivalent:

(1) The semigroup $\{T(t)\}_{t\geqslant 0}$ defined in (2.13) is analytic for every uniformly bounded strongly continuous semigroup $\{S(t)\}_{t\geqslant 0}$ on X,

(2) $F(dx, t)$, $t > 0$, is differentiable as a function $(0, \infty) \to \mathcal{B}$ and

$$\limsup_{t\to 0+} \left[t \cdot \left\| \frac{\partial F}{\partial t} \right\|_{TV} \right] < \infty.$$

In [**5**] the following, unfortunately only *necessary* condition is given. If (2) holds, then the Laplace symbol $I(s)$ of $F(dx, t)$ maps $\mathrm{Re}\, s > 0$ into a sector $\{s \in \mathbb{C},\ |\arg s| \leqslant (\pi - \omega)/2\}$ for some $\omega < \pi$, and there is a constant K and $0 < \gamma < 1$ such that $|I(s)| \leqslant K|s|^\gamma$ for $|s| \geqslant 1$ and $\mathrm{Re}\, s \geqslant 0$.

Fujita in [**10**], gives a checkable set of *sufficient* conditions for the Carasso–Kato Theorem to hold, stated in terms of regular variation. For $0 < \alpha < 1$ let $\theta_\alpha = \pi/(1 + \alpha)$, choose $\theta_\alpha < \Theta < \pi$, and define the sectorial region $\Sigma = \{w \in \mathbb{C} \setminus \{0\} : \arg w < \Theta\}$. Let $\bar{\Sigma}$ denote the closure of this region. The main theorem in [**10**] states that Condition (2) holds if for some $0 < \alpha < 1$ the following three conditions hold:

(A1) $I(s)$ has a holomorphic extension to Σ and is continuous on $\bar{\Sigma}$;

(A2) For some function $g(r)$ that varies regularly at infinity with index α we have $I(re^{i\theta}) \sim g(r)e^{i\alpha\theta}$ uniformly in $|\theta| \leqslant \theta_\alpha$ as $r \to \infty$;

(A3) The functions $I(re^{i\theta})/r$ are integrable on some neighborhood of $0+$ for every $|\theta| \leqslant \theta_\alpha$.

With the above preparation we come to the main technical result of this paper. In the next section, we will use this result to prove a general formula for the hitting time density of the ultrafast subordinator $D(t)$. The result may also be useful in other applications, since it shows that these subordinators are smoothing. For example, when time-discretizing the Cauchy problem $\dot{u}(t) = Au(t)$, $u_0 = u(0)$ using rational approximation schemes, the fact that $\{T(t)\}_{t\geqslant 0}$ is analytic implies stability of the scheme and faster convergence of the method for less smooth initial data, see for example [**6**] and [**26**].

THEOREM 2.2. *Under the regular variation conditions* (2.1) *and* (2.2)*, the semigroup* $\{T(t)\}_{t\geqslant 0}$ *defined by* (2.9) *is holomorphic on* $C_0(\mathbb{R})$.

PROOF. If conditions (A1)–(A3) are satisfied, then the semigroup $\{T(t)\}_{t\geqslant 0}$ defined by (2.9) is holomorphic on $C_0(\mathbb{R})$ by (2.13) and the Carasso–Kato theorem.

First we show that (A1) holds. To do this recall Montel's theorem (see, for example, [**11**, Theorem 6.5.3]), namely, if $\mathcal{F} := \{f_\lambda\}_{\lambda\in\Lambda}$ is a family of holomorphic functions on an open set $U \subset \mathbb{C}$ such that $|f_\lambda(w)| \leqslant M$ $(M > 0)$ for all $w \in U$ and $f_\lambda \in \mathcal{F}$, then every sequence $\{f_j\} \subset \mathcal{F}$ has a subsequence $\{f_{n_j}\}$ that converges uniformly on compact subsets of U to a holomorphic function f_0. Let $q(\beta) := \Gamma(1-\beta)p(\beta)$ and $Q(\beta) := \int_0^\beta q(\beta')\,d\beta'$ for $0 \leqslant \beta \leqslant 1$. Then, since the function $\beta \mapsto r^\beta$ is continuous on $[0,1]$,

$$I(r) = \int_0^1 r^\beta q(\beta)\,d\beta = \int_0^1 r^\beta\,dQ(\beta)$$

where the latter integral is a Riemann–Stieltjes integral.

The functions $w \mapsto w^\beta$, $0 \leqslant \beta \leqslant 1$, are holomorphic on $\mathbb{C}\setminus(-\infty,0]$ and hence so are the functions, defined by the Riemann–Stieltjes sums,

$$f_n(w) := \sum_{j=1}^n w^{j/n}\Big[Q\Big(\frac{j}{n}\Big) - Q\Big(\frac{j-1}{n}\Big)\Big].$$

Let $w_0 \in \mathbb{C}\setminus(-\infty,0]$ and choose an open disc $D(w_0,r)$ with center w_0 and radius $r > 0$ such that $\overline{D(w_0,r)}\cap(-\infty,0] = \emptyset$. Then $\mathcal{F} := \{f_n\}_{n\in\mathbb{N}}$ consists of holomorphic functions on $U := D(w_0,r)$ and

$$\begin{aligned}|f_n(w)| \leqslant \sum_{j=1}^n |w|^{j/n}\left|Q\Big(\frac{j}{n}\Big) - Q\Big(\frac{j-1}{n}\Big)\right| &\leqslant \sup_{\substack{0\leqslant\beta\leqslant 1\\ w\in D(w_0,r)}} |w|^\beta \operatorname{Var}[Q]_0^1\\ &\leqslant \max(1,2|w_0|)\int_0^1 q(\beta)\,d\beta.\end{aligned}$$

Therefore, by Montel's theorem, there is a subsequence $\{f_{n_j}\} \subset \{f_n\}_{n\in\mathbb{N}}$ that converges uniformly on compact subsets of $D(w_0,r)$ to a holomorphic function f_0. But $f_n(w) \to \int_0^1 w^\beta\,dQ(\beta) = \int_0^1 w^\beta q(\beta)\,d\beta$ as $n\to\infty$ for all $w \in D(w_0,r)$ and hence $\int_0^1 w^\beta q(\beta)\,d\beta = f_0(w)$ is holomorphic on $D(w_0,r)$ and hence on $\mathbb{C}\setminus(-\infty,0]$. Clearly, the function $\int_0^1 w^\beta q(\beta)\,d\beta$ is an extension of $I(s)$ and it follows that (A1) holds for arbitrary $\theta_\alpha < \Theta < \pi$.

Next we show that (A2) holds in the special case $\theta = 0$. A change of variables $\beta' = \alpha - \beta$ shows that

$$\begin{aligned}r^{-\alpha}I(r) &= \int_0^1 r^{\beta-\alpha}q(\beta)\,d\beta = \int_0^\alpha r^{\beta-\alpha}q(\beta)\,d\beta\\ &= \int_0^\alpha r^{-\beta'}q(\alpha-\beta')\,d\beta' = \int_0^1 r^{-\beta}h(\beta)\,d\beta\end{aligned} \tag{2.14}$$

where $h(\beta) = q(\alpha-\beta) = \Gamma(1+\beta-\alpha)p(\alpha-\beta)$. Since (2.2) holds we can write $p(\beta) = (\alpha-\beta)^{a-1}L(\alpha-\beta)$ for some function L that is slowly varying at $0+$. Then we have $h(\beta) \sim \Gamma(1-\alpha)\beta^{a-1}L(\beta)$ is regularly varying with index $a-1$ at $\beta = 0+$.

Now define $H(\beta) = \int_0^\beta h(u)du$ the distribution of the finite Borel measure with density $h(u)$. A change of variables $v = 1/u$ shows that

$$H(1/x) = \int_x^\infty h(1/v)\frac{dv}{v^2}$$

where $h(1/v)$ varies regularly at infinity with index $1 - a$. Then the Lemma on [**9**, p. 280, VIII.9] implies that $H(1/x)$ varies regularly at infinity with index $-a$, and hence $H(\beta)$ varies regularly at $0+$ with index $a > 0$. Furthermore, [**9**, Theorem 1, VIII.9] implies that

$$\frac{x^{-1}h(1/x)}{H(1/x)} \to a \quad \text{as } x \to \infty$$

which implies that $aH(\beta) \sim \beta h(\beta)$ as $\beta \to 0+$, and hence $H(\beta) \sim \beta^a L_0(\beta)$ as $\beta \downarrow 0$ where $L_0(\beta) = a^{-1}\Gamma(1-\alpha)L(\beta)$. Since the Laplace transform

$$\tilde{h}(r) = \int e^{-r\beta} h(\beta)\, d\beta = \int e^{-r\beta} H(d\beta)$$

we can apply Karamata's Tauberian Theorem (see, e.g., [**9**, Theorem 3, XIII.5]) to conclude that $\tilde{h}(r)$ varies regularly as $r \to \infty$ with index $-a$ and furthermore we have $\tilde{h}(r) \sim \Gamma(1+a)r^{-a}L_0(r^{-1})$. Then in view of (2.14) we have

$$r^{-\alpha}I(r) = \int_0^1 r^{-\beta}h(\beta)\, d\beta = \tilde{h}(\log r) \sim \Gamma(1+a)(\log r)^{-a}L_0(1/\log r) \tag{2.15}$$

and hence $I(r) \sim r^\alpha \Gamma(1+a)(\log r)^{-a}L_0(1/\log r)$ varies regularly with index α as $r \to \infty$. This proves (A2) for the case $\theta = 0$ and shows that in fact we can take $g(r) = I(r)$.

Next we consider an arbitrary $|\theta| \leqslant \theta_\alpha$ for some $0 < \alpha < 1$ and we write

$$\begin{aligned}
I(re^{i\theta}) &= r^\alpha e^{i\alpha\theta} \int_0^1 r^{\beta-\alpha} e^{i(\beta-\alpha)\theta} q(\beta)\, d\beta \\
&= r^\alpha e^{i\alpha\theta} \left(\int_0^1 r^{\beta-\alpha} \cos((\beta-\alpha)\theta) q(\beta)\, d\beta + i \int_0^1 r^{\beta-\alpha} \sin((\beta-\alpha)\theta) q(\beta)\, d\beta \right) \\
&= r^\alpha e^{i\alpha\theta}(J_1 + iJ_2)
\end{aligned}$$

and we substitute $\beta' = \alpha - \beta$ as before to get

$$\begin{aligned}
J_1 &= \int_0^\alpha r^{\beta-\alpha} \cos((\beta-\alpha)\theta) q(\beta)\, d\beta = \int_0^\alpha r^{-\beta'} \cos(\beta'\theta) q(\alpha - \beta')\, d\beta' \\
&= \int_0^\alpha r^{-\beta} h_1(\beta)\, d\beta = \tilde{h}_1(\log s)
\end{aligned}$$

with $h_1(\beta) = \cos(\beta\theta)q(\alpha - \beta) = \cos(\beta\theta)p(\alpha-\beta)\Gamma(1-\alpha+\beta)$, and similarly

$$J_2 = \int_0^\alpha r^{-\beta} h_2(\beta)\, d\beta$$

with $h_2(\beta) = -\sin(\beta\theta)p(\alpha-\beta)\Gamma(1-\alpha+\beta)$. As $\beta \downarrow 0$ we have $h_1(\beta) \sim h(\beta) \sim \Gamma(1-\alpha)\beta^{a-1}L(\beta)$ regularly varying with index $a - 1$ at $\beta = 0+$. If $\theta\alpha < \pi/2$ then it follows as in the case $\theta = 0$ that $\tilde{h}_1(r) \sim \Gamma(1+a)r^{-a}L_0(r^{-1})$ is regularly varying as $r \to \infty$, so that $J_1 \sim r^{-\alpha}I(r)$ by (2.15). If not, then we write

$$J_1 = J_{11} + J_{12} = \int_0^{\beta_1} r^{-\beta} h_1(\beta)\, d\beta + \int_{\beta_1}^{\alpha} r^{-\beta} h_1(\beta)\, d\beta$$

where $\beta_1 = \pi/(2\theta)$. Since $h_1(\beta) \geqslant 0$ on $0 < \beta < \beta_1$ the same argument as before yields $J_{11} \sim r^{-\alpha}I(r)$ as $r \to \infty$. Noting that $h_2(\beta) < 0$ on $\beta_1 < \beta < \alpha$ we have $|J_{12}| \leqslant Cr^{-\beta_1}$ for all $r \geqslant 1$ where $C = \int_{\beta_1}^{\alpha} h_1(\beta) d\beta$. Then $J_{12}/J_{11} \to 0$ as $r \to \infty$ and again it follows that $J_1 \sim r^{-\alpha}I(r)$ as $r \to \infty$. Furthermore, since J_1 is an even function of θ that is monotone on $0 < \theta < \theta_\alpha$ this statement holds true uniformly in $|\theta| \leqslant \theta_\alpha$.

As for the imaginary part, note that $|\sin(\beta\theta)| \leqslant \beta|\theta| \leqslant \beta\theta_\alpha$ for all $\beta > 0$ and $|\theta| \leqslant \theta_\alpha$. Hence we have $|J_2| \leqslant \int_0^{\alpha} r^{-\beta} h_3(\beta)\, d\beta$ for all $|\theta| \leqslant \theta_\alpha$ where $h_3(\beta) = \beta\theta_\alpha h(\beta) \sim \theta_\alpha \Gamma(1-\alpha)\beta^a L(\beta)$ is regularly varying with index a as $\beta \downarrow 0$. Define a finite Borel measure $H_3(\beta) = \int_0^{\beta} h_3(u) du$, and argue as before that $H_3(\beta) \sim \beta^{a+1} L_3(\beta)$ as $\beta \downarrow 0$ where $L_3(\beta) = \theta_\alpha (a+1)^{-1}\Gamma(1-\alpha)L(\beta)$. Then the Laplace transform $\tilde{h}_3(r) \sim \Gamma(1+a)r^{-a-1}L_3(r^{-1})$ is regularly varying as $r \to \infty$, so that

$$|J_2| \leqslant \tilde{h}_3(\log r) \sim \Gamma(1+a)(\log r)^{-a-1} L_3(1/\log r) \quad \text{ as } r \to \infty.$$

Then it follows that $J_2/J_1 \to 0$ and $J_1 + iJ_2 \sim r^{-\alpha}I(r)$ as $r \to \infty$, uniformly in $|\theta| \leqslant \theta_\alpha$. Hence $I(re^{i\theta}) = r^{\alpha}e^{i\alpha\theta}(J_1 + iJ_2) \sim e^{i\alpha\theta}I(r)$ as $r \to \infty$ uniformly in $|\theta| \leqslant \theta_\alpha$ where $I(r)$ varies regularly with index α, which completes the proof of (A2).

Finally we establish (A3) by considering the real and imaginary parts separately. Using (2.1) we can write $p(\beta) = \beta^b L_1(\beta)$ where L_1 varies slowly at zero. Fix $\theta \in [-\theta_\alpha, \theta_\alpha]$ and substitute $t = 1/r$ to obtain

$$I(re^{i\theta}) = \int_0^1 r^{\beta} e^{i\beta\theta} q(\beta)\, d\beta = \int_0^1 t^{-\beta} e^{i\beta\theta} q(\beta)\, d\beta = I_1 + iI_2$$

where

$$I_j = \int_0^1 t^{-\beta} q_j(\beta)\, d\beta \quad \text{for } j = 1, 2$$

with $q_1(\beta) = \cos(\beta\theta)q(\beta)$, while $q_2(\beta) = \sin(\beta\theta)q(\beta)$. In either case we have $|q_j(\beta)| \leqslant q(\beta)$ so that $|I_j| \leqslant I_0 = \int_0^1 t^{-\beta} q(\beta)\, d\beta$ where $q(\beta) = p(\beta)\Gamma(1-\beta) \sim \beta^b L_1(\beta)$ as $\beta \downarrow 0$. The function $Q(\beta) = \int_0^{\beta} q(u) du$ varies regularly with index $b+1$ at zero with $Q(\beta) \sim (b+1)^{-1}\beta^{b+1}L_1(\beta)$ as $\beta \downarrow 0$, and then Karamata's Tauberian theorem implies that $\tilde{q}(s) \sim \Gamma(b+2)(b+1)^{-1}s^{-b-1}L_1(1/s)$ as $s \to \infty$. Then we have $I_0 = \tilde{q}(\log t) \sim \Gamma(b+1)(\log t)^{-b-1}L_1(1/\log t)$ as $t \to \infty$ so that, recalling that $t = 1/r$, we have

$$I_0 \sim \Gamma(b+1)(-\log r)^{-b-1} L_1(-1/\log r) \quad \text{as } r \downarrow 0.$$

Hence for any $\varepsilon > 0$ and any $\delta > 0$, for some $C > 0$ we have $I_0 \leqslant C(-\log r)^{\delta-b-1}$ for all $0 < r < \varepsilon$. Substituting $u = -\log r$ we conclude that the absolute values of the real and imaginary parts of the integral $\int_0^\varepsilon r^{-1}I(re^{i\theta})dr$ are each bounded above by

$$\int_0^\varepsilon r^{-1}I_0\,dr \leqslant \int_0^\varepsilon r^{-1}C(-\log r)^{\delta-b-1}\,dr = C\int_{-\log\varepsilon}^\infty u^{\delta-b-1}\,du < \infty$$

as long as $0<\delta<b$ and $0<\varepsilon<1$. This shows that (A3) holds, which concludes the proof of Theorem 2.2. □

REMARK 2.1. Recall that the ultrafast subordinator is the limit of a random walk whose jumps J_i satisfy $P\{J_i>t\}=\int_0^1 t^{-\beta}p(\beta)d\beta$ for $t>1$. If assumption (2.1) holds, then it follows as in the proof of condition (A3) of Theorem 2.2 that this probability tail is slowly varying, and furthermore, if we write $p(\beta)=\beta^b L_1(\beta)$ where L_1 varies slowly at zero then $P\{J_i>t\}\sim\Gamma(b+1)(\log t)^{-b-1}L_1(1/\log t)$ as $t\to\infty$.

REMARK 2.2. It is clear from the proof of condition (A3) in Theorem 2.2 that we can weaken the assumption (2.1) to say that for some $\beta_1>0$, $C>0$ and $b>0$ we have $p(\beta)\leqslant C\beta^b$ for all $0<\beta<\beta_1$. We cannot relax the assumption that $p(\beta)$ vanishes in a neighborhood of $\beta=1$ since, as we have already mentioned, $|I(s)|\leqslant C|s|^\alpha$ for all $|s|\geqslant 1$ with $\operatorname{Re}s\geqslant 0$ for some $\alpha<1$ is a necessary condition for the semigroup to be holomorphic, see [**5**, Theorem 4].

COROLLARY 2.1. *Under the assumptions of Theorem 2.2, if $D(t)$ has a density $f_t(x):=f(x,t)$ and $f_t\in C_0(\mathbb{R})$, then $f_t\in\mathcal{D}(A^n)$ for all $t>0$ and $n\in\mathbb{N}$. Moreover, the function $t\mapsto f(x,t)$ is C^∞ for all $x\in\mathbb{R}$ and*

$$\frac{\partial^n f(x,t)}{\partial t^n}=[A^n f_t](x),\ x\in\mathbb{R},\ t>0. \tag{2.16}$$

PROOF. Since $\{T(t)\}_{t\geqslant 0}$ is holomorphic it follows that $T(t)u\in\mathcal{D}(A^n)$ for all $u\in C_0(\mathbb{R})$, $t>0$ and $n\in\mathbb{N}$ and

$$\frac{d^n}{dt^n}[T(t)u]=A^nT(t)u, \tag{2.17}$$

see, for example [**8**, p. 104]. Note that (2.17) means that $T(t)u\in\mathcal{D}(A^n)$ for all $u\in C_0(\mathbb{R})$, $t>0$, $n\in\mathbb{N}$ and

$$\lim_{h\to 0+}\left\|\frac{A^{n-1}T(t+h)u-A^{n-1}T(t)u}{h}-A^nT(t)u\right\|=0. \tag{2.18}$$

Let $t>0$ and choose $0<\varepsilon<t$. Then

$$f_t(x)=f(x,t)=[T(t-\varepsilon)f_\varepsilon](x) \tag{2.19}$$

and hence $f_t\in\mathcal{D}(A^n)$ for all $n\in\mathbb{N}$. We will show (2.16) by induction. If $s:=t-\varepsilon$, then

$$\frac{d}{dt}f_t=\frac{d}{dt}[T(t-\varepsilon)f_\varepsilon]=\frac{d}{ds}[T(s)f_\varepsilon]=AT(s)f_\varepsilon=AT(t-\varepsilon)f_\varepsilon=Af_t.$$

That is,

$$\lim_{h\to 0}\sup_{x\in\mathbb{R}}\left|\frac{f_{t+h}(x)-f_t(x)}{h}-[Af_t](x)\right|=0\quad\text{for all } t>0 \tag{2.20}$$

and hence

$$[Af_t](x) = \lim_{h\to 0} \frac{f_{t+h}(x) - f_t(x)}{h} = \lim_{h\to 0} \frac{f(x,t+h) - f(x,t)}{h} = \frac{\partial f(x,t)}{\partial t},$$

for $x \in \mathbb{R},\ t > 0$. This shows (2.16) for $n = 1$. Assume that

$$\frac{\partial^k f(x,t)}{\partial t^k} = [A^k f_t](x),\ x \in \mathbb{R},\ t > 0. \tag{2.21}$$

Then, by (2.18), for any $s > 0$ we have

$$\lim_{h\to 0+} \sup_{x\in\mathbb{R}} \left| \frac{[A^k T(s+h)u](x) - A^k[T(s)u](x)}{h} - [A^{k+1}T(s)u](x) \right| = 0.$$

Let $u := f_\varepsilon$ with $0 < \varepsilon$. Then, by (2.19),

$$\lim_{h\to 0+} \sup_{x\in\mathbb{R}} \left| \frac{[A^k f_{s+\varepsilon+h}](x) - A^k[f_{s+\varepsilon}](x)}{h} - [A^{k+1} f_{s+\varepsilon}](x) \right| = 0$$

which yields, by the induction hypothesis (2.21) and the change of variable $t := s+\varepsilon$,

$$\lim_{h\to 0+} \sup_{x\in\mathbb{R}} \left| \frac{1}{h} \left[\frac{\partial^k f(x,t+h)}{\partial t^k} - \frac{\partial^k f(x,t)}{\partial t^k} \right] - [A^{k+1} f_t](x) \right| = 0. \tag{2.22}$$

Since $0 < \varepsilon$ is arbitrary, (2.22) yields

$$\frac{\partial^{k+1} f(x,t)}{\partial t^{k+1}} = [A^{k+1} f_t](x),\ x \in \mathbb{R},\ t > 0,$$

which finishes the proof. □

3. Hitting times for ultrafast subordinators

If $D(t)$ is an ultrafast subordinator from Theorem 2.1 we define the inverse (or hitting time, or first passage time) process $E(x) = \inf\{t \geqslant 0 : D(t) > x\}$. Then it is easy to see that for $t, x \geqslant 0$

$$\{E(x) \leqslant t\} = \{D(t) \geqslant x\}. \tag{3.1}$$

Since the subordinator $D(t)$ grows at a very fast rate, the inverse process grows very slowly. The process $E(x)$ plays a crucial role in physical models of ultraslow diffusion [**19**]. The next result shows that the hitting time process $E(x)$ has a smooth density for all $x > 0$, and shows how this density can be explicitly computed in terms of the generator of the semigroup associated with the ultrafast subordinator $D(t)$. This resolves an open problem in [**19**].

Theorem 3.1. *If $D(t)$ is an ultrafast subordinator from Theorem 2.1 with distribution function $F(x,t) = P\{D(t) \leqslant x\}$, and if A is the generator (2.10) of the associated semigroup (2.9) on $C_0(\mathbb{R})$, then the inverse process $E(x)$ has a C^∞ density $h(t,x)$ for any $x > 0$ and furthermore this density can be computed from the formula*

$$h(t,x) = -\int_0^x [Af_t](y)dy = \int_0^1 \int_0^x (x-y)^\beta f(y,t)\,dy\,p(\beta)\,d\beta \tag{3.2}$$

where $f_t(x) := f(x,t) = (\partial/\partial x)F(x,t)$ is the density of $D(t)$.

PROOF. It follows from [**19**, Corollary 3.7] that $D(t)$ has a density $f_t(x) = f(x,t) = \partial F(x,t)/\partial x$ for any $t > 0$ and that $f_t \in C_0(\mathbb{R})$. Theorem 2.2 shows that the semigroup (2.9) is holomorphic on $C_0(\mathbb{R})$, and hence, by Corollary 2.1, the function $t \mapsto f(x,t)$ is C^∞ for all $x \in \mathbb{R}$. We can use (3.1) to write

$$P\{E(x) \leqslant t\} = P\{D(t) \geqslant x\} = \int_x^\infty f(y,t)dy = 1 - F(x,t). \tag{3.3}$$

By (3.3) and Corollary 2.1,

$$\begin{aligned} h(t,x) &= \frac{\partial}{\partial t} P\{E(x) \leqslant t\} \ = \frac{\partial}{\partial t}\left(1 - F(x,t)\right) \\ &= -\frac{\partial}{\partial t}\int_0^x f(y,t)dy = -\int_0^x \frac{\partial}{\partial t} f(y,t)dy = -\int_0^x [Af_t](y)dy \end{aligned} \tag{3.4}$$

where we use the bounded convergence theorem to move $\frac{\partial}{\partial t}$ inside the integral. Indeed, the difference quotient $(f_{t+h}(y) - f_t(y))/h = (f(y,t+h) - f(y,t))/h$ converges uniformly in $y \in \mathbb{R}$ to $[Af_t](y)$ by (2.20). By Corollary 2.1, $Af_t \in C_0(\mathbb{R})$ and hence is uniformly bounded for $y \in [0,x]$. This certainly implies the boundedness of the difference quotient for $h \leqslant h_0$ and $y \in [0,x]$.

By a similar argument, using Corollary 2.1 and (2.22), it follows that

$$\frac{\partial^k}{\partial t^k} h(t,x) = -\int_0^x \frac{\partial^{k+1}}{\partial t^{k+1}} f(y,t)dy = -\int_0^x [A^k f_t](y)dy$$

for all $x > 0$ and $t > 0$, so that $t \mapsto h(t,x)$ is C^∞ for all $x > 0$.

Define $f_1 = \partial f/\partial x$. Corollary 3.7 in [**19**] also shows that the function $x \mapsto f(x,t)$ is C^∞ for all $t > 0$ and $x \mapsto \frac{\partial^n}{\partial x^n} f(x,t) \in C_0(\mathbb{R})$ for all $n \in \mathbb{N}$. Therefore, we may apply (2.12) to the integrand in (3.4), noting that $f(x,t) = 0$ for $x < 0$, integrate by parts, and apply (2.7) to obtain

$$\begin{aligned} h(t,x) &= \int_0^x \int_0^y \left(f(y,t) - f(y-z,t)\right)\phi(dz)\,dy \ = \int_0^x \int_0^y f_1(y-z,t)\phi(z,\infty)\,dz\,dy \\ &= \int_0^x \int_z^x f_1(y-z,t)\phi(z,\infty)\,dy\,dz = \int_0^x f(x-z,t)\phi(z,\infty)\,dz \\ &= \int_0^x \int_0^1 f(x-z,t) z^{-\beta} p(\beta)\,d\beta\,dz = \int_0^1 \int_0^x (x-y)^{-\beta} f(y,t)\,dy\,p(\beta)\,d\beta \end{aligned}$$

which agrees with (3.2). Note that $|f(y,t) - f(y-z,t)| \leqslant Cz$ uniformly in $y \in [0,x]$ since f_1 is bounded on this compact set. Also $\phi(z,\infty) \leqslant z^{-\alpha}$ so it is easy to see that $[f(y,t) - f(y-z,t)]\phi(z,\infty) \to 0$ as $z \to 0+$ which justifies the first integration by parts. □

REMARK 3.1. The integral formula in (3.2) was first derived in [**19**, (5.3)]. Theorem 3.1 strengthens that result by removing the technical assumption [**19**, (5.7)], and by showing that the density is C^∞ in $t > 0$ for all $x > 0$.

REMARK 3.2. It is not hard to show that Theorems 2.2 and 3.1 also hold, with the obvious change in notation, under the somewhat weaker assumption that the random variables $0 < B_i < \alpha$ in (2.3) are i.i.d. with distribution function

$P(\beta) = P\{B_i \leqslant \beta\}$ where $P(\beta)$ varies regularly at zero with index $b+1$ and $1-P(\beta)$ varies regularly at $\beta = \alpha-$ with index a. Then (3.2) also holds with $p(\beta)d\beta$ replaced by $P(d\beta)$.

Remark 3.3. The fractional derivative $D^\beta u(y)$ is defined as the function with Laplace transform $s^\beta \tilde{u}(s)$. Since a β-stable subordinator has Laplace transform e^{-ts^β}, fractional derivatives of order $0 < \beta < 1$ are the negative generators of the associated semigroups. Similarly, the negative generator of the semigroup associated with an ultrafast subordinator is a distributed-order fractional derivative $D_p u(y)$, defined as the function whose Laplace transform is

$$I(s)\tilde{u}(s) = \int_0^1 s^\beta p(\beta)\Gamma(1-\beta)d\beta\ \tilde{u}(s).$$

The density function $g(y,t) \equiv g_t(y)$ of the stable subordinator solves the initial value problem $\partial g(x,t)/\partial t = -D^\beta g_t(y),\ g(y,t_0) = g_{t_0}(y),\ y \in \mathbb{R}$, and similarly the density $f(y,t) \equiv f_t(y)$ of an ultrafast subordinator solves the initial value problem $\partial f(y,t)/\partial t = -D_p f_t(y),\ f(y,t_0) = f_{t_0}(y),\ y \in \mathbb{R}$. Laplace transform arguments in [**19**] show that densities $h(t,y) \equiv h_t(y)$ of the ultraslow hitting time process solve the boundary value problem $\partial h(t,y)/\partial t = -D_p h_t(y),\ h(t_0,y) = h_{t_0}(y),\ y \geqslant 0$, noting that here the roles of t and y are reversed.

Remark 3.4. A somewhat different proof of the density formula (3.2) can be obtained without using Theorem 2.2. Assume as in [**19**] that p varies regularly at zero with index $a-1$ for some $a > 0$ and (2.8) holds. Then [**19**, Corollary 3.7] shows that the density $f_t(x)$ of $D(t)$ and its first derivative in x are elements of $C_0(\mathbb{R})$ for any $t > 0$, and hence (2.11) implies that $f_t \in \mathcal{D}(A)$ for any $t > 0$. Then (2.20) holds, and (3.2) follows as in the proof of Theorem 3.1. However, this approach does not establish the smoothness of $t \mapsto f_t(x)$.

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(Received 08 06 2006)
(Revised 21 09 2006)

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