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CONVOLUTIONS AND MEAN SQUARE ESTIMATES OF CERTAIN NUMBER-THEORETIC ERROR TERMS

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Dedicated to the memory of Tatjana Ostrogorski

Abstract. We study the convolution function

$$C[f(x)] := \int_1^x f(y) f\Big(\frac{x}{y}\Big)\frac{dy}{y}$$

when $f(x)$ is a suitable number-theoretic error term. Asymptotics and upper bounds for $C[f(x)]$ are derived from mean square bounds for $f(x)$. Some applications are given, in particular to $|\zeta(\frac12+ix)|^{2k}$ and the classical Rankin–Selberg problem from analytic number theory.

1. Convolution functions

Motivated by considerations from analytic number theory, the author investigated in [10] the following class of convolution functions. Let $\mathcal{M}_a$ denote the set of functions $f(x) \in L^1(a,\infty)$ for a given $a > 0$, for which there exists a constant $\alpha_f \geqslant 0$ such that

$$f(x) \ll_\varepsilon x^{\alpha_f+\varepsilon}. \tag{1.1}$$

Actually it is more precise to define α_f as the infimum of the constants for which (1.1) holds. Here and later $\varepsilon > 0$ denotes arbitrarily small constants, not necessarily the same ones at each occurrence. The notation $A \ll_\varepsilon B$ (same as $A = O_\varepsilon(B)$) means that $|A| \leqslant C(\varepsilon)B$ for some positive constant $C(\varepsilon)$, which depends only on ε. We define the convolution of functions $f, g \in \mathcal{M}_1$ as

$$(f \odot g)(x) := \int_1^x f(y) g\Big(\frac{x}{y}\Big)\frac{dy}{y},$$

which is the special case $a = 1$ of the more general convolution function

$$(f \odot g)_a(x) := \int_a^{x/a} f(y) g\Big(\frac{x}{y}\Big)\frac{dy}{y} \qquad (a > 0; f, g \in \mathcal{M}_a).$$

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Of special interest is the function, for $f \in \mathcal{M}_1$,

$$C[f(x)] := (f \odot f)(x) = \int_1^x f(y) f\Big(\frac{x}{y}\Big)\frac{dy}{y} \qquad (x \geqslant 1),$$

or more generally

$$C_a[f(x)] := (f \odot f)_a(x) = \int_a^{x/a} f(y) f\Big(\frac{x}{y}\Big)\frac{dy}{y} \qquad (x \geqslant a,\ f \in \mathcal{M}_a). \tag{1.2}$$

Furthermore, the iterates of $C[f(x)]$ are defined as

$$C^{(1)}[f(x)] \equiv C[f(x)], \quad C^{(k)}[f(x)] := C[C^{(k-1)}[f(x)]] \quad (x \geqslant 1,\ k \geqslant 2).$$

Obviously we have, in view of (1.1),

$$C^{(k)}[f(x)] \ll_{\varepsilon,k} x^{\alpha_f+\varepsilon}, \tag{1.3}$$

and in [10] the bound (1.3) was improved in case when $f(x)$ represents several well-known number theoretic error terms. In particular this includes the mean square and biquadrate of $|\zeta(\frac{1}{2}+it)|$ and the error terms in the corresponding asymptotic formulas, $\Delta_k(x)$, the error term in the (generalized) Dirichlet divisor problem and the problems involving the distribution of non-isomorphic Abelian groups and the Rankin–Selberg convolution of holomorphic cusp forms. Relevant definitions and notions are to be found in [10].

One of the reasons for the study of the convolution functions $(f \odot g)(x)$ is that they appear naturally in the context of (modified) Mellin transforms

$$F^*(s) \equiv m[f(x)] := \int_1^\infty f(x)x^{-s}dx \qquad (s = \sigma + it;\ \sigma, t \in \mathbb{R})$$

by means of the formula, which holds under suitable conditions,

$$m[(f \odot g)(x)] = m[f(x)]\, m[g(x)]. \tag{1.4}$$

The application to the summatory function $A(x) := \sum_{n \leqslant x} a_n$ of the sequence $\{a_n\}_{n=1}^\infty$ was given in [10]. Let $A(x)$ be of the form

$$A(x) := \sum_{i=1}^{k}\sum_{j=0}^{M_i} c_{i,j} x^{\alpha_i}\log^j x + u(x), \tag{1.5}$$

where the $c_{i,j}$'s are real constants with $c_{1,M_1} > 0$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_k > 0$, and $u(x)$ $(= o(x^{\alpha_k})$ as $x \to \infty)$ is the error term in the asymptotic formula for $A(x)$. If $u(x)$ satisfies the mean square estimate

$$\int_0^X u^2(x)dx \ll X^{1+2\beta} \qquad (0 \leqslant \beta < \alpha_k),$$

then the following result was proved in [10].

Theorem 1. *Let the above hypotheses on $A(x)$ and $u(x)$ hold, and suppose that the function $\mathcal{A}(s) = \sum_{n=1}^\infty a_n n^{-s}$ admits analytic continuation to the region*

$\operatorname{Re} s > 0$, *where it is regular except for the poles at* $s = \alpha_1, \alpha_2, \ldots, \alpha_k$ *which are of order* $M_1 + 1, M_2 + 1, \ldots, M_k + 1$, *respectively. If*

$$\int_T^{2T} |\mathcal{A}(\sigma_1 + it)|^2 dt \ll T^{2-\delta}$$

holds for some $\delta > 0$ *and* $0 < \sigma_1 < \alpha_k$, *then we have*

$$C[u(x)] = \int_1^x u(y) u\Big(\frac{x}{y}\Big) \frac{dy}{y} \ll x^{\sigma_1}.$$

2. The asymptotics of the convolution function

The asymptotic formula for $C[f(x)]$ is not easy to obtain, even if a sharp formula for $f(x)$ (or its integral) is known. For example, it is well known (see [5]) that

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^2 dt = T \log\Big(\frac{T}{2\pi}\Big) + (2\gamma - 1)T + E(T), \tag{2.1}$$

where $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ ($\operatorname{Re} s > 1$) is the Riemann zeta-function, $\gamma = -\Gamma'(1)$ is Euler's constant, and for the error term $E(T)$ one has the asymptotic formula

$$\int_0^T E^2(t)\, dt = CT^{3/2} + O(T \log^4 T) \qquad (C > 0). \tag{2.2}$$

It seems difficult to obtain an asymptotic formula for $C[|\zeta(\frac{1}{2} + ix)|^2]$, even with the precise information contained in (2.1) and (2.2). We shall return to this problem in Section 4.

In number theory one often encounters, as error terms in asymptotic formulas, *regularly varying* functions. These are functions $h(x)$ which are positive, continuous (or, more generally, measurable) for $x \geqslant x_0\,(> 0)$, for which there exists $\rho \in \mathbb{R}$ (called the index of $h(x)$) such that

$$\lim_{x\to\infty} \frac{h(cx)}{h(x)} = c^\rho, \quad \text{for all } \ c > 0. \tag{2.3}$$

We shall denote the set of all regularly varying functions by $\mathcal{R}$. We shall also denote by $\mathcal{L}$ the set of *slowly varying* (or *slowly oscillating*) functions, namely those functions $h(x)$ in $\mathcal{R}$ for which the index $\rho = 0$. It is easy to show that if $h \in \mathcal{R}$, then there exists $L \in \mathcal{L}$ such that $h(x) = x^\rho L(x)$, with ρ being the index of h.

For a comprehensive account of regularly varying functions the reader is referred to the monographs of Bingham et al. [1] and E. Seneta [22]. By a fundamental result of J. Karamata [16], who founded the theory of regular variation, the limit in (2.3) is uniform for $0 < a \leqslant c \leqslant b < \infty$ and any $0 < a < b$. This is known as the *uniform convergence theorem*. It is used to show that any slowly varying function $L(x)$ (for $x \geqslant x_0\,(> 0)$) is necessarily of the form

$$L(x) = A(x) \exp\left(\int_{x_0}^x \eta(t) \frac{dt}{t}\right), \ \lim_{x\to\infty} A(x) = A > 0, \ \lim_{x\to\infty} \eta(x) = 0, \tag{2.4}$$

so that $x^{-\varepsilon} \ll L(x) \ll x^{\varepsilon}$ always holds. If $h(x) \in \mathcal{R}$ with index ρ, then

$$C[h(x)] = \int_1^x h(u)h\Big(\frac{x}{u}\Big)\frac{du}{u} = x^{\rho}\int_1^x L(u)L\Big(\frac{x}{u}\Big)\frac{du}{u},$$

where $L(x)$ is a slowly varying function. Hence the problem of the asymptotic evaluation of $C[h(x)]$ is in this case reduced to the evaluation of

$$C[L(x)] = \int_1^x L(u)L\Big(\frac{x}{u}\Big)\frac{du}{u} \qquad (L(x) \in \mathcal{L}). \tag{2.5}$$

In some cases it is possible to evaluate the integral in (2.5) explicitly, but in the general case it is not an easy task. For example, let $L(x) = (\log x)^{\alpha}$ with $\alpha > -1$ a given constant. Then we have, with the change of variable $t = \log u/\log x$,

$$\begin{aligned} C[(\log x)^{\alpha}] &= \int_1^x (\log u)^{\alpha}(\log x - \log u)^{\alpha}\frac{du}{u} \\ &= (\log x)^{2\alpha+1}\int_0^1 t^{\alpha}(1-t)^{\alpha}\,dt \\ &= \frac{\Gamma^2(\alpha+1)}{\Gamma(2\alpha+2)}(\log x)^{2\alpha+1}. \end{aligned} \tag{2.6}$$

We note that in (2.6) the resulting function is again slowly varying. This is also true in general, when we consider $C_a[h(x)]$ (cf. (1.2)) for sufficiently large a (if (2.4) holds, then $a = x_0$ may be taken). The result is

Theorem 2. *If $h(x) \in \mathcal{R}$ with index ρ, then for sufficiently large a we have*

$$C_a[h(x)] = x^{\rho}\int_a^{x/a} L(u)L\Big(\frac{x}{u}\Big)\frac{du}{u} = x^{\rho}C_a[L(x)],$$

where $C_a[L(x)]$ is a slowly varying function.

Proof. Let $L(x) \in \mathcal{L}$, $a > 0$. The result follows from the uniform convergence theorem that, uniformly for $k_1 \leqslant c \leqslant k_2$, $L(x) \in \mathcal{L}$ and any $k_2 > k_1 > 0$ we have $L(cx)/L(x) \to 1$ as $x \to \infty$. Namely, if $B > 0$ is a large constant, then we have

$$\begin{aligned} C_a[L(x)] &= \int_a^{x/a} L(u)L\Big(\frac{x}{u}\Big)\frac{du}{u} \geqslant \int_{\sqrt{x}/B}^{\sqrt{x}} L(u)L\Big(\frac{x}{u}\Big)\frac{du}{u} \\ &= (1+o(1))L^2(\sqrt{x}\,)\int_{\sqrt{x}/B}^{\sqrt{x}}\frac{du}{u} \geqslant \frac{\log B}{2}L^2(\sqrt{x}\,), \end{aligned}$$

so that

$$L^2(\sqrt{x}\,) \ll C_a[L(x)]/\log B \tag{2.7}$$

for $x \geqslant x(B)$. On the other hand, if $c \geqslant 1$ is a given constant, then

$$\begin{aligned} C_a[L(cx)] &= 2\int_a^{\sqrt{cx}} L(u)L\Big(\frac{cx}{u}\Big)\frac{du}{u} = (2+o(1))\int_a^{\sqrt{cx}} L(u)L\Big(\frac{x}{u}\Big)\frac{du}{u} \\ &= (1+o(1))C_a[L(x)] + (2+o(1))\int_{\sqrt{x}}^{\sqrt{cx}} L(u)L\Big(\frac{x}{u}\Big)\frac{du}{u} \\ &= (1+o(1))C_a[L(x)] + O(L^2(\sqrt{x}))\int_{\sqrt{x}}^{\sqrt{cx}} \frac{du}{u} \\ &= (1+o(1))C_a[L(x)] + O(C_a[L(x)]/\log B), \end{aligned} \tag{2.8}$$

since (2.7) holds. But B can be arbitrarily large, and consequently (2.8) implies that $C_a[L(cx)] \sim C_a[L(x)]$ as $x \to \infty$. This proves the assertion, since $c \geqslant 1$ may be assumed without loss of generality. □

3. Mean square bounds

In case when it is difficult to obtain an asymptotic formula for $C_a[f(x)]$ one has to be content with upper bound estimates. In this direction we have

THEOREM 3. *Suppose that $f \in L^2(1,\infty)$ and that for some $\theta \geqslant 0$ and $D \geqslant 0$ we have*

$$\int_1^X f^2(x)dx \ll X^{1+2\theta}(\log X)^D. \tag{3.1}$$

Then, for any $a \geqslant 1$,

$$C_a[f(x)] \ll x^\theta(\log x)^{c(\theta)}, \quad c(\theta) = \begin{cases} D+1, & \text{if } \theta > 0, \\ D+2, & \text{if } \theta = 0. \end{cases} \tag{3.2}$$

PROOF. We note that

$$\begin{aligned} C_a[f(x)] &= \int_a^{\sqrt{x}} f(u)f\Big(\frac{x}{u}\Big)\frac{du}{u} + \int_{\sqrt{x}}^{x/a} f(u)f\Big(\frac{x}{u}\Big)\frac{du}{u} \\ &= 2\int_{\sqrt{x}}^{x/a} f(u)f\Big(\frac{x}{u}\Big)\frac{du}{u}. \end{aligned} \tag{3.3}$$

The last integral is split into $\ll \log x$ subintegrals of the form

$$I(x,T) := \int_T^{T'} f(u)f\Big(\frac{x}{u}\Big)\frac{du}{u} \qquad (\sqrt{x} \leqslant T < T' \leqslant 2T \leqslant 2x/a).$$

An application of the Cauchy–Schwarz inequality for integrals and (3.1) gives

$$\begin{aligned}
I(x,T) &\leqslant \left(\int_T^{T'} f^2(u)\frac{du}{u}\int_T^{T'} f^2\Big(\frac{x}{u}\Big)\frac{du}{u}\right)^{1/2} \\
&= \left(\int_T^{T'} f^2(u)\frac{du}{u}\int_{x/T'}^{x/T} f^2(u)\frac{du}{u}\right)^{1/2} \\
&\ll (T^{2\theta}(\log T)^{c(\theta)-1}\cdot(x/T)^{2\theta}(\log x/T)^{c(\theta)-1})^{1/2} \ll x^\theta(\log x)^{c(\theta)-1},
\end{aligned}$$

from which (3.2) follows. □

As an application of Theorem 3, we consider the distribution of non-isomorphic Abelian groups. As usual, let $a(n)$ denote the number of non-isomorphic Abelian groups with n elements (see e.g., [5, Section 14.5] for an extensive account). This is a multiplicative function and its generating series is

$$\sum_{n=1}^{\infty} a(n)n^{-s} = \zeta(s)\zeta(2s)\zeta(3s)\cdots \qquad (\operatorname{Re} s > 1).$$

If one sets (this is (1.5) with $k = 6,\ M_i \equiv 0$)

$$A(x) := \sum_{n\leqslant x} a(n) = \sum_{j=1}^{6} A_j x^{1/j} + R_0(x), \quad A_j := \prod_{k=1,k\neq j}^{\infty} \zeta(k/j),$$

then $R_0(x)$ can be thought of as the error term in the asymptotic formula for the summatory function of $a(n)$. The author obtained

$$C[R_0(x)] \ll_\varepsilon x^{1/6+\varepsilon} \tag{3.4}$$

in [10] directly, by using results on power moments of $\zeta(s)$. A slight improvement of (3.4) follows from Theorem 3 (with $\theta = 1/6,\ D = 89$) and the bound

$$\int_1^X R_0^2(x)dx \ll X^{4/3}(\log X)^{89}, \tag{3.5}$$

of D.R. Heath-Brown [3], namely

$$C[R_0(x)] \ll x^{1/6}(\log x)^{90}.$$

Incidentally, the bound in (3.5) is best possible up to a power of the logarithm, since the author [6] proved that

$$\int_1^X R_0^2(x)dx = \Omega(X^{4/3}\log X),$$

where as usual $f(x) = \Omega(g(x))$ means that $f(x) = o(g(x))$ does not hold as $x \to \infty$. It seems reasonable to conjecture that

$$C[R_0(x)] \sim x^\rho L(x) \qquad (L(x) \in \mathcal{L},\ 0 \leqslant \rho \leqslant 1/6,\ x \to \infty).$$

4. The second and fourth fourth power of the zeta-function

In this section we shall consider the asymptotic evaluation of $C[|\zeta(\frac{1}{2}+ix)|^{2k}]$ when $k=1$ and $k=2$. Naturally, the values $k>2$ could be also considered, but the problem then becomes much more difficult, since our knowledge on the $2k$-th moment of $|\zeta(\frac{1}{2}+ix)|$ when $k>2$ is rather modest (see [5, Chapter 8]).

It was proved in [10] that

$$C[|\zeta(\tfrac{1}{2}+ix)|^2] \ll_\varepsilon x^\varepsilon \tag{4.1}$$

and that

$$C[|\zeta(\tfrac{1}{2}+ix)|^4] \ll_\varepsilon x^\varepsilon \min(x^{4\mu(\frac{1}{2})},\, x^{2\mu(\frac{1}{2})+\frac{1}{4}},\, x^{\frac{1}{3}}), \tag{4.2}$$

where for a given $\sigma\in\mathbb{R}$

$$\mu(\sigma) = \limsup_{t\to\infty} \frac{\log|\zeta(\sigma+it)|}{\log t} \tag{4.3}$$

is the Lindelöf function. If the famous (hitherto unproved) Lindelöf conjecture that $\mu(\sigma)=0$ for $\sigma\geqslant\frac{1}{2}$ (or equivalently that $\zeta(\frac{1}{2}+it)\ll_\varepsilon |t|^\varepsilon$) is true, then we have trivially

$$C[|\zeta(\tfrac{1}{2}+ix)|^{2k}] \ll_{\varepsilon,k} x^\varepsilon,$$

and in any case

$$C[|\zeta(\tfrac{1}{2}+ix)|^{2k}] \ll_{\varepsilon,k} x^{2k\mu(\frac{1}{2})+\varepsilon}$$

does hold. Heuristically, one expects $C[|\zeta(\frac{1}{2}+ix)|^{2k}] \sim L(x)$ $(\in\mathcal{L})$ to hold as $x\to\infty$. More precisely, I conjecture that for $k=1,2$ there exists a constant $A_k>0$ such that

$$C[|\zeta(\tfrac{1}{2}+ix)|^{2k}] \sim A_k(\log x)^{2k^2+1} \qquad (x\to\infty), \tag{4.4}$$

and (4.4) probably also holds at least for $k=3$ and $k=4$. If true, this conjecture is certainly beyond reach at present. The heuristic motivation for (4.4) is given shortly as follows. For $k\geqslant 1$ a fixed integer let

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt = T\,P_{k^2}(\log T) + E_k(T), \tag{4.5}$$

where for some suitable coefficients $a_{j,k}$ $(a_{k^2,k}>0)$ one has

$$P_{k^2}(y) = \sum_{j=0}^{k^2} a_{j,k} y^j, \tag{4.6}$$

and in particular it is known that $P_1(y) = y + 2\gamma - 1 - \log(2\pi)$ holds (cf. (2.1)). One hopes that

$$E_k(T) = o(T) \qquad (T\to\infty) \tag{4.7}$$

will hold for every fixed integer $k\geqslant 1$, but so far this is known to be true only in the cases $k=1$ and $k=2$, when $E_k(T)$ is a true error term in (4.5) (see [5] and [8]). Recently (see Conrey et al. [2]) plausible heuristic arguments have been given, by employing the techniques of random matrix theory, to produce explicit values of the coefficients $a_{j,k}$ in (4.6). Nevertheless, the author in [8] expressed doubts that (4.5)–(4.6) will, in general, hold for $k > 4$. Regardless of the moment conjecture, it certainly seems plausible that, for some index $\rho = \rho(k) \geqslant 0$, one has

$$C[|\zeta(\tfrac{1}{2}+ix)|^{2k}] \sim x^\rho L(x) \in \mathcal{R} \qquad (x \to \infty).$$

If (4.5)–(4.7) holds, then for $\sigma > 1$ and some constants $d_{j,k}$ we have

$$\begin{aligned} \mathcal{Z}_k(s) := \int_1^\infty |\zeta(\tfrac{1}{2}+ix)|^{2k} x^{-s}dx &= \int_1^\infty (x\,P_{k^2}(\log x) + E_k(x))' x^{-s}dx \\ &= \sum_{j=0}^{k^2+1} \frac{d_{j,k}}{(s-1)^j} + s\int_1^\infty E_k(x)x^{-s-1}dx. \end{aligned} \tag{4.8}$$

Thus we obtain analytic continuation of the Mellin transform $\mathcal{Z}_k(s)$ to the region $\sigma \geqslant 1$ (at least). From (1.4) it follows that

$$C[|\zeta(\tfrac{1}{2}+ix)|^{2k}] = \frac{1}{2\pi i}\int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} \mathcal{Z}_k^2(s)x^{s-1}ds. \tag{4.9}$$

We shift the line of integration in (4.9) to $\operatorname{Re} s = c$ for some suitable $0 < c < 1$, passing over the pole of $\mathcal{Z}_k^2(s)$ of order $2k^2+2$. By the residue theorem (4.4) follows, provided of course that we can make this procedure rigorous.

By the method of proof of Theorem 3 and (4.5)–(4.7) with $k = 2$ one can easily improve (4.1) to

$$C[|\zeta(\tfrac{1}{2}+ix)|^2] \ll (\log x)^5.$$

Any further improvements seem difficult, but nevertheless we can prove an asymptotic formula for the integral of $C[|\zeta(\frac{1}{2}+ix)|^2]$, which supports the conjectural (4.4) when $k = 1$. This is

THEOREM 4. *There exist effectively computable constants $A\,(= 1/6), B, C, D$ such that*

$$\int_1^X C[|\zeta(\tfrac{1}{2}+ix)|^2]dx = (A\log^3 X + B\log^2 X + C\log X + D)X + O_\varepsilon(X^{1/2+\varepsilon}). \tag{4.10}$$

PROOF. Integrating (4.9) when $k = 1$ we obtain

$$\int_1^X C[|\zeta(\tfrac{1}{2}+ix)|^2]dx = \frac{1}{2\pi i}\int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} \mathcal{Z}_1^2(s)\frac{X^s-1}{s}\,ds. \tag{4.11}$$

We note (see the author's paper [11]) that the function $\mathcal{Z}_1(s)$ continues meromorphically to $\mathbb{C}$, having only a double pole at $s = 1$, and simple poles at $s = -1, -3, \dots$. The principal part of its Laurent expansion at $s = 1$ is

$$\frac{1}{(s-1)^2} + \frac{2\gamma - \log(2\pi)}{s-1}.$$

In (4.11) we shift the line of integration to $\operatorname{Re} s = \frac{1}{2}+\varepsilon$, passing over the pole $s = 1$ of the integrand of order four. By the residue theorem, the main term in (4.10) comes from this pole. The integral over the line $\operatorname{Re} s = \frac{1}{2} + \varepsilon$ is $\ll_\varepsilon x^{1/2+\varepsilon}$, if one uses the mean square bound

$$\int_1^T |\mathcal{Z}_1(\sigma + it)|^2 dt \ \ll_\varepsilon \ T^{2-2\sigma+\varepsilon} \qquad (\tfrac{1}{2} \leqslant \sigma \leqslant 1),$$

proved in [13] by M. Jutila, Y. Motohashi and the author. The value $A = 1/6$ easily follows by calculating the residue at $s = 1$ of the integrand in (4.11). □

The function $C[|\zeta(\frac{1}{2} + ix)|^4]$ is more difficult to deal with than $C[|\zeta(\frac{1}{2} + ix)|^2]$. The results that we obtain in this case are contained in the following

Theorem 5. *We have*

$$C[|\zeta(\tfrac{1}{2} + ix)|^4] \ \ll_\varepsilon \ \min(x^{2\mu(\frac{1}{2})+\varepsilon},\, x^{\frac{1}{4}}(\log x)^{23/2}), \tag{4.12}$$

and with suitable constants A_j $(j = 0, \dots, 9)$ we have

$$\int_1^X C[|\zeta(\tfrac{1}{2} + ix)|^4] dx = X \sum_{j=0}^{9} A_j \log^j X + O_\varepsilon(X^{5/6+\varepsilon}). \tag{4.13}$$

Proof. First note that

$$\begin{aligned}
C[|\zeta(\tfrac{1}{2} + ix)|^4] &= \int_1^x |\zeta(\tfrac{1}{2} + it)|^4 |\zeta(\tfrac{1}{2} + i\frac{x}{t})|^4 \,\frac{dt}{t} \\
&\ll_\varepsilon \int_1^x \Big|\zeta(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + i\frac{x}{t})\Big|^2 \, t^{2\mu(1/2)+\varepsilon} (x/t)^{2\mu(1/2)+\varepsilon} \,\frac{dt}{t} \\
&= x^{2\mu(1/2)+\varepsilon} \int_1^x |\zeta(\tfrac{1}{2} + it)|^2 \,\Big|\zeta(\tfrac{1}{2} + i\frac{x}{t})\Big|^2 \,\frac{dt}{t} \\
&= x^{2\mu(1/2)+\varepsilon} C[|\zeta(\tfrac{1}{2} + ix)|^2] \ll_\varepsilon x^{2\mu(1/2)+\varepsilon}
\end{aligned}$$

because (4.1) holds. This establishes the first bound in (4.12). The second one follows from Theorem 3 (with $\theta = 1/4$, $D = 21/2$) and the bound

$$\begin{aligned}
\int_0^T |\zeta(\tfrac{1}{2} + it)|^8 dt &= \int_0^T |\zeta(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it)|^6 dt \\
&\leqslant \left(\int_0^T |\zeta(\tfrac{1}{2} + it)|^4 dt \int_0^T |\zeta(\tfrac{1}{2} + it)|^{12} dt \right)^{1/2} \\
&\ll (T \log^4 T \cdot T^2 \log^{17} T)^{1/2} = T^{3/2} \log^{21/2} T,
\end{aligned}$$

where the well-known bounds (see e.g., [5, Chapter 8])

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^4 dt \ll T \log^4 T, \qquad \int_0^T |\zeta(\tfrac{1}{2} + it)|^{12} dt \ll T^2 \log^{12} T$$

were used. Note that the sharpest known result at present (see M. N. Huxley [4]) is $\mu(1/2) \leqslant 32/205 = 0.156097\dots$, hence unconditionally we have the bound

$$C[|\zeta(\tfrac{1}{2} + ix)|^4] \ll x^{1/4} (\log x)^{23/2}. \tag{4.14}$$

The proof of (4.13) is analogous to the proof of (4.10). Note that we have, similarly to (4.11),

$$\int_1^X C[|\zeta(\tfrac12+ix)|^4]dx = \frac{1}{2\pi i}\int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} \mathcal{Z}_2^2(s)\frac{X^s-1}{s}\,ds, \tag{4.15}$$

where $\mathcal{Z}_2(s)$ is given by (4.8) with $k=2$. This function is regular for $\sigma > \frac12$, except for pole $s=1$ of order five (see [13]). Moreover we have the mean square bound (see the author's paper [12])

$$\int_1^T |\mathcal{Z}_2(\sigma+it)|^2 dt \ll_\varepsilon T^{\frac{15-12\sigma}{5}+\varepsilon} \quad (\tfrac56 \leqslant \sigma \leqslant \tfrac54). \tag{4.16}$$

Thus (4.13) follows if we shift the line of integration in (4.15) to $\mathrm{Re}\, s = 5/6+\varepsilon$ and use (4.16); the main term in (4.13) comes from the residue of the integrand at $s=1$. One can show that $A_9 = 1/(2520\pi^2)$ and evaluate also explicitly the remaining constants A_j $(j=0,\dots,8)$. □

If the eighth moment bound holds for $|\zeta(\frac12+it)|$ (cf. Theorem 3 with $\theta=\varepsilon$), then the right-hand side of (4.14) can be replaced by x^ε. Moreover, in this case the exponent in (4.16) will be $4-4\sigma+\varepsilon$ for $\frac12 < \sigma \leqslant 1$, giving the exponent $3/4+\varepsilon$ in the error term in (4.13).

5. The Rankin–Selberg problem

This work will be concluded by analyzing estimates of convolution functions in the classical Rankin–Selberg problem. In this section we shall make a digression and consider the problem itself by means of a complex integration technique, while mean square bounds will be dealt with in the last section. The Rankin–Selberg problem consists of the estimation of the error term function

$$\Delta(x) = \sum_{n\leqslant x} c_n - Cx, \tag{5.1}$$

where the notation is as follows (see e.g., R.A. Rankin's monograph [18]). Let $\varphi(z)$ be a holomorphic cusp form of weight κ with respect to the full modular group $SL(2,\mathbb{Z})$, and denote by $a(n)$ the n-th Fourier coefficient of $\varphi(z)$. We suppose that $\varphi(z)$ is a normalized eigenfunction for the Hecke operators $T(n)$, that is, $a(1)=1$ and $T(n)\varphi = a(n)\varphi$ for every $n\in\mathbb{N}$. In (5.1) $C>0$ is a suitable constant (see e.g., [14] for its explicit expression), and c_n is the convolution function defined by

$$c_n = n^{1-\kappa}\sum_{m^2\mid n} m^{2(\kappa-1)}\left|a\Big(\frac{n}{m^2}\Big)\right|^2.$$

The classical Rankin–Selberg bound of 1939 is

$$\Delta(x) = O(x^{3/5}), \tag{5.2}$$

hitherto unimproved. In their works, done independently, R. A. Rankin [17] derives (5.2) from a general result of E. Landau, while A. Selberg [20] states the result with no proof. We shall estimate now $\Delta(x)$ by the complex integration technique.

The key fact in this approach is that, for $s = \sigma + it$ with $\sigma > 1$, one has the decomposition

$$Z(s) := \sum_{n=1}^{\infty} c_n n^{-s} = \zeta(s) \sum_{n=1}^{\infty} b_n n^{-s} = \zeta(s)B(s), \tag{5.3}$$

say, where $B(s)$ belongs to the Selberg class of Dirichlet series of degree three, and $B(s)$ is holomorphic for $\operatorname{Re} s > 0$. This follows from G. Shimura [23] (see also A. Sankaranarayanan [19], who used (5.3) to obtain mean square bounds for $Z(s)$). The coefficients b_n satisfy $b_n \ll_\varepsilon n^\varepsilon$ (see [19], actually the coefficients b_n are bounded by a log-power in mean square, but this is not needed here). For the definition and properties of the Selberg class of L-functions the reader is referred to A. Selberg [21] and the survey paper of Kaczorowski–Perelli [15].

On using classical Perron's formula (see e.g., the Appendix of [5]) and the convexity bound $Z(s) \ll_\varepsilon |t|^{2-2\sigma+\varepsilon}$ $(0 \leqslant \sigma \leqslant 1,\ |t| \geqslant 1)$, it follows that

$$\Delta(x) = \frac{1}{2\pi i}\int_{\frac12 - iT}^{\frac12 + iT} \frac{Z(s)}{s} x^s ds + O_\varepsilon\left(x^\varepsilon\left(x^{1/2} + \frac{x}{T}\right)\right) \quad (1 \ll T \ll x). \tag{5.4}$$

If we suppose that

$$\int_X^{2X} |B(\tfrac12 + it)|^2 dt \ll_\varepsilon X^{\theta+\varepsilon} \qquad (\theta \geqslant 1), \tag{5.5}$$

and use the elementary fact (see [5, Chapter 8] for the results on the moments of $|\zeta(\frac12 + it)|$) that

$$\int_X^{2X} |\zeta(\tfrac12 + it)|^2 dt \ll X \log X, \tag{5.6}$$

then from (5.3)–(5.6) and the Cauchy–Schwarz inequality for integrals we obtain

$$\Delta(x) \ll_\varepsilon x^\varepsilon (x^{1/2} T^{\theta/2 - 1/2} + xT^{-1}) \ll_\varepsilon x^{\frac{\theta}{\theta+1}+\varepsilon} \tag{5.7}$$

with $T = x^{1/(\theta+1)}$. Thus we have proved the following

THEOREM 6. *If* (5.5) *holds, then we have*

$$\Delta(x) \ \ll_\varepsilon \ x^{\frac{\theta}{\theta+1}+\varepsilon}.$$

As $B(s)$ belongs to the Selberg class of degree three, then $B(\frac12 + it)$ in (5.5) can be written as a sum of two Dirichlet polynomials (e.g., by the reflection principle discussed in [5, Chapter 4]), each of length $\ll X^{3/2}$. Thus by the mean value theorem for Dirichlet polynomials (op. cit.) we have $\theta \leqslant 3/2$, giving (with unimportant ε) the Rankin–Selberg bound $\Delta(x) \ll_\varepsilon x^{3/5+\varepsilon}$. Clearly improvement will come from better values of θ. Note that the best possible value of θ in (5.5) is $\theta = 1$, which follows from general results on Dirichlet series (see e.g., [5, Chapter 9]). It gives $1/2 + \varepsilon$ as the exponent in the Rankin–Selberg problem, which is the limit of the method (the author's conjectural exponent $3/8 + \varepsilon$ (see [7]) is out of reach). To attain this improvement one faces essentially the same problem as in proving the sixth moment for $|\zeta(\frac12 + it)|$, namely $\int_0^T |\zeta(\frac12 + it)|^6 dt \ll_\varepsilon T^{1+\varepsilon}$. In fact the present problem is even more difficult, because the properties of the coefficients b_n are even less known than the properties of the divisor coefficients

$$d_3(n) = \sum_{abc=n;a,b,c\in\mathbb{N}} 1,$$

generated by $\zeta^3(s)$, which occur in the investigations relating to the sixth moment of $|\zeta(\frac{1}{2}+it)|$. If we knew the analogue of the strongest sixth moment bound

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^6 dt \ll T^{5/4}\log^C T \qquad (C>0),$$

namely $\theta = 5/4$ in (5.5), then (5.7) would yield $\Delta(x) \ll_\varepsilon x^{5/9+\varepsilon}$, improving substantially (5.2).

6. Mean square and convolution in the Rankin–Selberg problem

In [14] the explicit formula for $\Delta(x)$ was derived. This is

(6.1)
$$\Delta(x) = \frac{x^{3/8}}{2\pi}\sum_{k\leqslant K} c_k k^{-5/8}\sin\left(8\pi(kx)^{1/4}+\frac{3\pi}{4}\right) + O_\varepsilon(x^\varepsilon((Kx)^{1/4}+x^{3/4}K^{-1/4})),$$

where K is a parameter which satisfies $1 \ll K \ll x$.

If we use (6.1) with $K = x$, square and integrate, then by the first derivative test (see e.g., [5, Lemma 2.1]) it follows that

$$\int_1^X \Delta^2(x)dx \ \ll_\varepsilon\ X^{1+2\beta+\varepsilon} \tag{6.2}$$

holds with $\beta = 1/2$. But as we have (see [14, eq. (3.5)])

$$\Delta(X) = H^{-1}\int_{X-H}^{X+H}\Delta(x)dx + O(H) \qquad (X^\varepsilon \leqslant H \leqslant \tfrac{1}{2}X),$$

it follows by the Cauchy–Schwarz inequality that

$$\Delta^2(X) \ll H^{-1}\int_{X-H}^{X+H}\Delta^2(x)dx + H^2 \qquad (X^\varepsilon \leqslant H \leqslant \tfrac{1}{2}X). \tag{6.3}$$

Hence (6.2) with $\beta = 1/2$ and (6.3) give (5.2) with the (poor) exponent $2/3+\varepsilon$, and any exponent $\beta < 2/5$ would lead to an improvement of the Rankin–Selberg exponent $3/5$. Although we cannot at present attain such an improvement from a mean square bound, we can improve on the value $\beta = 1/2$. Namely, let as before $\mu(\sigma)$ denote the Lindelöf function (see (4.3)). Then we have the following

Theorem 7. *We have* (6.2) *with*

$$\beta = \frac{2}{5-2\mu(\frac{1}{2})}. \tag{6.4}$$

PROOF. From the analogy with the divisor problem (see e.g., [5, Chapter 13]) it follows that (6.4) will be proved if we can show that

$$\int_T^{2T} |Z(\sigma+it)|^2 dt \ll T^{2-\delta}$$

holds with $\sigma > \frac{2}{5-2\mu(\frac{1}{2})}$ and some small $\delta\,(>0)$, with $Z(s)$ given by (5.3). Note that we have the functional equation

$$Z(s) = \mathcal{X}(s)Z(1-s), \quad \mathcal{X}(\sigma+it) \asymp |t|^{2-4\sigma} \quad (0<\sigma<1), \tag{6.5}$$

since $Z(s)$ is in the Selberg class of degree four. Furthermore, we have the mean square bound, proved by the author in [10, eq. (9.27)],

$$\int_T^{2T} |Z(\sigma+it)|^2 dt \ll_\varepsilon \; T^{2\mu(1/2)(1-\sigma)+\varepsilon}(T+T^{3(1-\sigma)}) \qquad (\tfrac{1}{2} \leqslant \sigma \leqslant 1).$$

Therefore we obtain

$$\begin{aligned}\int_T^{2T} |Z(\sigma+it)|^2 dt &\ll T^{4-8\sigma}\int_T^{2T} |Z(1-\sigma+it)|^2 dt \\ &\ll_\varepsilon T^{4-8\sigma+2\mu(1/2)\sigma+\varepsilon}(T+T^{3\sigma}) \qquad (0<\sigma\leqslant \tfrac{1}{2}),\end{aligned}$$

and for $1/3 \leqslant \sigma \leqslant 1/2$ the last quantity is $\ll T^{2-\delta}$ if $\sigma = (2+\delta+\varepsilon)/(5-2\mu(\frac{1}{2}))$, proving the assertion of Theorem 7. Note that with the sharpest result (see M. N. Huxley [4]) $\mu(1/2) \leqslant 32/205$ we obtain $\beta = 410/961 = 0.426638917\ldots$. The limit is the value $\beta = 2/5$ if the Lindelöf hypothesis (that $\mu(\frac{1}{2}) = 0$) is true. Of course, improving the value $\theta = 3/2$ in (5.5) would be another way to improve on the value of β. □

The merit of the value of β in (6.4) is that is strictly less than one half. As already mentioned, if we square out and integrate (6.1), all that follows is $\beta \leqslant \frac{1}{2}$. Incidentally, this bound follows in the general case of the mean square bound for an L-function of degree four in the Selberg class. Thus Theorem 7 shows that the finer information that we have in the Rankin–Selberg problem (the product representation (5.3)) can be put to advantage. As a consequence of Theorem 7 and Theorem 3 we obtain that

$$C[\Delta(x)] \;\ll_\varepsilon\; x^{\frac{2}{5-2\mu(1/2)}+\varepsilon}. \tag{6.6}$$

The bound (6.6) was obtained in [10] by a direct, more involved technique. With some more effort one can replace 'ε' in (6.6) by an explicit power of the logarithm. If one considers averages of $C[\Delta(x)]$, then even more cancellations occur. In this direction we shall prove

THEOREM 8. *For any given $\varepsilon > 0$ we have*

$$\int_1^X C[\Delta(x)]dx \;\ll_\varepsilon\; X^{5/4+\varepsilon}. \tag{6.7}$$

PROOF. From (5.1) and (5.3) we obtain, for $\mathrm{Re}\, s > 1$,

$$Z(s) = \int_{1-0}^{\infty} x^{-s} d\Big(\sum_{n\leqslant x} c_n\Big) = \int_{1-0}^{\infty} x^{-s}\Big(C\,dx + d\Delta(x)\Big)$$
$$= \frac{Cs}{s-1} + s\int_1^{\infty} \Delta(x) x^{-s-1} dx,$$

since $\Delta(1-0) = -C$. From (1.4) it follows that

$$C[\Delta(x)] = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{U^2(s)}{s^2} x^s ds,$$

where (5.3) shows that the function

$$U(s) := Z(s) - \frac{Cs}{s-1}$$

is regular in the region $\mathrm{Re}\, s > 0$. By integration we have

$$\int_1^X C[\Delta(x)]dx = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{U^2(s)}{s^2}\cdot\frac{X^{s+1}-1}{s+1}\, ds. \tag{6.8}$$

Now we shift the line of integration in the last integral to the line $\mathrm{Re}\, s = \frac14 + \varepsilon$. We note that (6.5) holds, and we obtain that the right-hand side of (6.8) is

$$\ll_\varepsilon X^{5/4+\varepsilon}\left(1 + \int_{-\infty}^{\infty} (|t|+1)^{-1-8\varepsilon} |Z(\tfrac34 - \varepsilon + it)|^2 dt\right) \ll_\varepsilon X^{5/4+\varepsilon}. \tag{6.9}$$

Namely $Z(s)$ is of degree four in the Selberg class, and consequently by (6.5) and the mean value theorem for Dirichlet polynomials one obtains without difficulty

$$\int_1^T |Z(\sigma+it)|^2 dt \ll_\varepsilon T^\varepsilon (T + T^{4-4\sigma}) \qquad (\tfrac12 \leqslant \sigma \leqslant 1). \tag{6.10}$$

Then we obtain ($T \gg 1$), on using (6.10),

$$\int_T^{2T} (|t|+1)^{-1-8\varepsilon} |Z(\tfrac34 - \varepsilon + it)|^2 dt \ll_\varepsilon T^{-1-8\varepsilon} T^{1+5\varepsilon} = T^{-3\varepsilon},$$

which means that the integral in (6.9) converges, and (6.7) follows. □

Finally we note that (6.10) can be sharpened to an asymptotic formula which improves Theorem 3 of the author's paper [9]. This is

THEOREM 9. *If β is given by (6.4), then for fixed σ satisfying $\frac12 < \sigma \leqslant 1$ we have*

$$\int_1^T |Z(\sigma+it)|^2 dt = T\sum_{n=1}^{\infty} c_n^2 n^{-2\sigma} + O_\varepsilon(T^{(2-2\sigma)/(1-\beta)+\varepsilon}). \tag{6.11}$$

PROOF. We proceed as in the proof of Theorem 3 of [9]. The only difference is that, instead of using (p. 174 of [9]) the bound

$$\int_T^{2T} |E|^2 dt \ll_\varepsilon X^{2-2\sigma+\varepsilon} + T^2 X^{1-2\sigma+\varepsilon},$$

which corresponds to (6.2) with $\beta = \frac{1}{2}$, we can use a better bound. This is (6.2) with β given by (6.4), so that the above bound becomes

$$\int_T^{2T} |E|^2 dt \ll_\varepsilon X^{2-2\sigma+\varepsilon} + T^2 X^{2\beta-2\sigma+\varepsilon},$$

where β is given by (6.4) and satisfies $\frac{2}{5} \leqslant \beta < \frac{1}{2}$. Instead of the exponent $4-4\sigma+\varepsilon$ that appears in (4.2) of [9], we obtain now the better exponent $(2-2\sigma)/(1-\beta)+\varepsilon$ in (6.11). This ends the discussion on Theorem 9, with the remark that its use instead of (6.10) does not lead to a better exponent on the right-hand side of (6.7). □

References

[1] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation,* CUP, Cambridge, 1987.
[2] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith, *Integral moments of L-functions,* Proc. London Math. Soc. (3) **91** (2005), 33–104.
[3] D. R. Heath-Brown, *The number of Abelian groups of order at most x,* Astérisque **198-199-200** (1991), 153–163.
[4] M. N. Huxley, *Exponential sums and the Riemann zeta-function V,* Proc. London Math. Soc. (3) **90** (2005), 1–41.
[5] A. Ivić, *The Riemann zeta-function,* John Wiley & Sons, New York, 1985 (2nd ed., Dover, Mineola, N.Y., 2003).
[6] A. Ivić, *The general divisor problem,* J. Number Theory **26** (1987), 73–91.
[7] A. Ivić, *Large values of certain number-theoretic error terms,* Acta Arith. **56** (1990), 135–159.
[8] A. Ivić, *On some results concerning the Riemann Hypothesis, in "Analytic Number Theory"* (Kyoto, 1996) ed. Y. Motohashi, LMS LNS **247**, Cambridge University Press, Cambridge, 1997, pp. 139–167.
[9] A. Ivić, *On mean values of zeta-functions in the critical strip,* J. Théorie des Nombres de Bordeaux **15** (2003), 163–173.
[10] A. Ivić, *Estimates of convolutions of certain number-theoretic error terms,* Inter. J. Math. Math. Sci. **1** (2004), 1–23.
[11] A. Ivić, *The Mellin transform of the square of Riemann's zeta-function,* International J. of Number Theory **1** (2005), 65–73.
[12] A. Ivić, *On the estimation of some Mellin transforms connected with the fourth moment of* $|\zeta(\frac{1}{2}+it)|$, in Proc. ELAZ2004 Conf. (Mainz, 2004), eds. W. Schwarz and J. Steuding, Franz Steiner Verlag, 2006, pp. 77–88.
[13] A. Ivić, M. Jutila and Y. Motohashi, *The Mellin transform of powers of the Riemann zeta-function,* Acta Arith. **95** (2000), 305–342.
[14] A. Ivić, K. Matsumoto and Y. Tanigawa, *On Riesz mean of the coefficients of the Rankin–Selberg series,* Math. Proc. Camb. Phil. Soc. **127** (1999), 117–131.
[15] A. Kaczorowski and A. Perelli, *The Selberg class: a survey,* in "Number Theory in Progress, Proc. Conf. in honour of A. Schinzel (K. Györy et al. eds)", de Gruyter, Berlin, 1999, pp. 953–992.
[16] J. Karamata, *Sur un mode de croissance régulière des fonctions,* Mathematica (Cluj) **4** (1930), 38–53.
[17] R. A. Rankin, *Contributions to the theory of Ramanujan's function* $\tau(n)$ *and similar arithmetical functions. II, The order of Fourier coefficients of integral modular forms,* Math. Proc. Cambridge Phil. Soc. **35** (1939), 357–372.
[18] R. A. Rankin, *Modular Forms,* Ellis Horwood Ltd., Chichester, England, 1984.
[19] A. Sankaranarayanan, *Fundamental properties of symmetric square L-functions I,* Illinois J. Math. **46** (2002), 23–43.
[20] A. Selberg, *Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist,* Arch. Math. Naturvid. **43** (1940), 47–50.

[21] A. Selberg, *Old and new conjectures and results about a class of Dirichlet series,* in "Proc. Amalfi Conf. Analytic Number Theory 1989 (E. Bombieri et al. eds.)", University of Salerno, Salerno, 1992, pp. 367–385.

[22] E. Seneta, *Regularly Varying Functions,* LNM **508**, Springer-Verlag, Berlin–Heidelberg–New York, 1976.

[23] G. Shimura, *On the holomorphy of certain Dirichlet series,* Proc. London Math. Soc. **31** (1975), 79–98.

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