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# REGULAR VARIATION FOR MEASURES ON METRIC SPACES

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Abstract. The foundations of regular variation for Borel measures on a complete separable space $\mathbf{S}$, that is closed under multiplication by nonnegative real numbers, is reviewed. For such measures an appropriate notion of convergence is presented and the basic results such as a Portmanteau theorem, a mapping theorem and a characterization of relative compactness are derived. Regular variation is defined in this general setting and several statements that are equivalent to this definition are presented. This extends the notion of regular variation for Borel measures on the Euclidean space $\mathbf{R}^d$ to more general metric spaces. Some examples, including regular variation for Borel measures on $\mathbf{R}^d$, the space of continuous functions $\mathbf{C}$ and the Skorohod space $\mathbf{D}$, are provided.

## 1. Introduction

In many areas of applied probability one encounters a Borel measure $\nu$ on $\mathbf{R}^d$ with the following asymptotic scaling property: for some $\alpha > 0$ and all $\lambda > 0$

$$\lim_{t\to\infty} \frac{\nu(\lambda tA)}{\nu(tE)} = \lambda^{-\alpha} \lim_{t\to\infty} \frac{\nu(tA)}{\nu(tE)}, \tag{1.1}$$

where $E$ is a fixed reference set such as $E = \{x \in \mathbf{R}^d : |x| \geqslant 1\}$, $tE = \{tx : x \in E\}$, and $A$ may vary over the Borel sets that are bounded away from the origin 0 (the closure of $A$ does not contain 0). Typically $\nu$ is a probability measure but other classes of measures, such as Lévy- or intensity measures for infinitely divisible distributions or random measures, with this property appear frequently in the probability literature. The asymptotic scaling property implies that the function $c(t) = 1/\nu(tE)$ is regularly varying (at $\infty$) with index $\alpha$ and that $c(t)\nu(tA)$ converges to a finite constant $\mu(A)$ as $t \to \infty$. Then (1.1) translates into the scaling property $\mu(\lambda A) = \lambda^{-\alpha}\mu(A)$ for $\lambda > 0$. A measure satisfying (1.1) is called regularly varying (see Section 3 for a precise definition). The name is motivated by $t \mapsto \nu(tA)$ being a regularly varying function (with index $-\alpha$) whenever $\mu(A) > 0$. To motivate the approach we suggest for defining regular variation for measures on general spaces, it is useful to first consider the most commonly encountered way to

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define regular variation for measures on $\mathbf{R}^d$. In order to use the well established notion of vague convergence (see e.g. [**17**]), instead of $\mathbf{R}^d$ one considers the space $[-\infty,\infty]^d\setminus\{0\}$. The reason is that sets that are bounded away from 0 in $\mathbf{R}^d$ become topologically bounded (or relatively compact) in $[-\infty,\infty]^d\setminus\{0\}$. This means that regular variation for a measure $\nu$ on $\mathbf{R}^d$ can be defined as the vague convergence $\nu(t\cdot)/\nu(tE)\to\mu(\cdot)$ as $t\to\infty$, where $\mu$ is a nonzero measure on $[-\infty,\infty]^d\setminus\{0\}$. There exist definitions of regular variation for measures on other spaces that are not locally compact. For instance, on the Skorohod space $\mathbf{D}[0,1]$, regular variation is formulated using polar coordinates and a notion of convergence for boundedly finite measures (see [**13**] and [**12**]). Also in this case the original space is changed by introducing "points at infinity" in order to turn sets bounded away from 0 in the original space into metrically bounded sets. It is the aim of this paper to review the foundations of regular variation for measures on general metric spaces without considering modifications of the original space of the types explained above.

The first step towards a general formulation of regular variation is to find an appropriate notion of convergence of measures. Recall that a sequence of bounded (or totally finite) measures $\mu_n$ on a separable metric space $\mathbf{S}$ converges weakly to a bounded measure $\mu$ as $n\to\infty$ if $\lim_{n\to\infty}\int f\,d\mu_n=\int f\,d\mu$ for all bounded and continuous real functions $f$. By the Portmanteau theorem, an equivalent formulation is that $\lim_{n\to\infty}\mu_n(A)=\mu(A)$ for all Borel sets $A$ with $\mu(\partial A)=0$, where $\partial A$ denotes the boundary of $A$. A convenient way to modify weak convergence to fit into a regular variation context is to define convergence $\mu_n\to\mu$ by $\lim_{n\to\infty}\int f\,d\mu_n=\int f\,d\mu$ for all bounded and continuous real functions $f$ that vanish in a neighborhood of a fixed point $s_0\in\mathbf{S}$ (the origin). The foundations of this notion of convergence, including a Portmanteau theorem, a mapping theorem and characterizations of relative compactness, are presented in Section 2.

To define regular variation for measures on $\mathbf{S}$, the space $\mathbf{S}$ has to be closed under multiplication by nonnegative real numbers $\lambda\in\mathbf{R}_+$. Moreover, the map $(\lambda,x)\mapsto\lambda x$ from $\mathbf{R}_+\times\mathbf{S}$ into $\mathbf{S}$ should be continuous, there should exist an element $0\in\mathbf{S}$ so that $0x=0$ for all $x\in\mathbf{S}$, and the metric $d$ on $\mathbf{S}$ should satisfy $d(0,\lambda_1x)<d(0,\lambda_2x)$ for all $\lambda_1,\lambda_2\in\mathbf{R}_+$ with $\lambda_1<\lambda_2$ and all $x\in\mathbf{S}\setminus\{0\}$. The last assumption means that for $x\in\mathbf{S}\setminus\{0\}$ the distance to the origin for a point on the ray $\{\lambda x:\lambda\in\mathbf{R}_+\}$ is strictly increasing in $\lambda$. Under these additional assumptions the notion of regular variation for measures on $\mathbf{S}$ is introduced in Section 3. An alternative approach to define regular variation for measures on $\mathbf{S}$ would be to identify $\mathbf{S}$ with a product space and use polar coordinates. However, we do not pursue such alternative approaches here. It is worth noticing that the formulation of regular variation proposed here is equivalent to usual formulations of regular variation for measures on $\mathbf{R}^d$, $\mathbf{C}[0,1]$ and $\mathbf{D}[0,1]$ found in e.g. [**3, 20, 21, 12, 13**].

The advantage of the construction proposed here is two-fold. Firstly, it provides a general framework for measures on metric spaces. In particular, it is irrelevant whether the space is locally compact. Secondly, there is no need to introduce artificial compactification points. These points lead to annoying difficulties that blur the elegant mathematics underlying the theory. For example, any mappings $h:\mathbf{S}\to\mathbf{S}'$ would have to be modified to cope with the compactification points.

In Section 2 we introduce the space $\mathbf{M}_0$ of measures on a complete separable metric space $\mathbf{S}$. The measures in $\mathbf{M}_0$ assign finite mass to sets bounded away from $s_0$, i.e. $s_0$ is not contained in the closure of the set, where $s_0$ is a fixed element in $\mathbf{S}$. Convergence in $\mathbf{M}_0$ is closely related to, but not identical to, weak convergence. We derive relevant fundamental results and characterize relative compactness in $\mathbf{M}_0$. In Section 3 we consider regular variation for a sequence of measures and for a single measure in $\mathbf{M}_0$. A number of statements that are equivalent to the definition of regular variation for a measure in $\mathbf{M}_0$ is presented. In Section 4 we give examples that illustrate the framework developed in Sections 2 and 3, and provide references to related work. In particular we derive results for $\mathbf{R}^d$, for the space of continuous functions, and for the space of càdlàg functions. All the proofs are found in Section 5.

Regular variation conditions for probability measures on $\mathbf{R}^d$ appear frequently in the literature. Natural examples are the studies of the asymptotic behavior of partial sums of independent and identically distributed terms (the general central limit theorem, see e.g. [**22**], [**19**]), of componentwise partial maxima and point processes (extreme value theory, see e.g. [**20**]) and of solutions to random difference equations (see e.g. [**16**]). Regular variation also appears naturally in necessary and sufficient conditions in central limit theorems in Banach spaces, see e.g. [**1**] and [**10**] and the references therein. In the space $\mathbf{C}([0,1];\mathbf{R}^d)$ of continuous functions it is used to characterize max-stable distributions and convergence in distribution of normalized partial maxima (see [**11**] and [**12**]). In the space $\mathbf{D}([0,1];\mathbf{R}^d)$ of càdlàg functions the framework of regularly varying measures has been employed to study the extremal behavior of heavy-tailed stochastic processes and the tail behavior of functionals of their sample paths (see [**13**] and [**15**]).

## 2. Convergence of measures in the space $\mathbf{M}_0$

Let $(\mathbf{S},d)$ be a complete separable metric space. We write $\mathscr{S}$ for the Borel $\sigma$-algebra on $\mathbf{S}$ and $B_{x,r} = \{y \in \mathbf{S} : d(x,y) < r\}$ for the open ball centered at $x$ with radius $r$.

Let $\mathscr{C}_b$ denote the class of real-valued, bounded and continuous functions on $\mathbf{S}$, and let $\mathbf{M}_b$ denote the class of finite Borel measures on $\mathscr{S}$. A basic neighborhood of $\mu \in \mathbf{M}_b$ is a set of the form $\{\nu \in \mathbf{M}_b : |\int f_i\,d\nu - \int f_i\,d\mu| < \varepsilon,\ i = 1,\dots,k\}$, where $\varepsilon > 0$ and $f_i \in \mathscr{C}_b$ for $i = 1,\dots,k$. Thus, $\mathbf{M}_b$ is equipped with the weak topology. The convergence $\mu_n \to \mu$ in $\mathbf{M}_b$, weak convergence, is equivalent to $\int f\,d\mu_n \to \int f\,d\mu$ for all $f \in \mathscr{C}_b$. See e.g. Sections 2 and 6 in [**5**] for details. Fix an element $s_0 \in \mathbf{S}$, called the origin, and set $\mathbf{S}_0 = \mathbf{S} \setminus \{s_0\}$. The subspace $\mathbf{S}_0$ is a metric space in the relative topology with $\sigma$-algebra $\mathscr{S}_0 = \{A : A \subset \mathbf{S}_0, A \in \mathscr{S}\}$. Let $\mathscr{C}_0$ denote the real-valued bounded and continuous functions $f$ on $\mathbf{S}_0$ such that for each $f$ there exists $r > 0$ such that $f$ vanishes on $B_{0,r}$; we use the notation $B_{0,r}$ for the ball $B_{s_0,r}$. Let $\mathbf{M}_0$ be the class of Borel measures on $\mathbf{S}_0$ whose restriction to $\mathbf{S} \setminus B_{0,r}$ is finite for each $r > 0$. A basic neighborhood of $\mu \in \mathbf{M}_0$ is a set of the form $\{\nu \in \mathbf{M}_0 : |\int f_i\,d\nu - \int f_i\,d\mu| < \varepsilon,\ i = 1,\dots,k\}$, where $\varepsilon > 0$ and $f_i \in \mathscr{C}_0$

for $i = 1, \dots, k$. Similar to weak convergence, the convergence $\mu_n \to \mu$ in $\mathbf{M}_0$ is equivalent to $\int f\,d\mu_n \to \int f\,d\mu$ for all $f \in \mathscr{C}_0$.

THEOREM 2.1. *$\mu_n \to \mu$ in $\mathbf{M}_0$ if and only if $\int f\,d\mu_n \to \int f\,d\mu$ for each $f \in \mathscr{C}_0$.*

The proof of this and all subsequent results are found in Section 5.

For $\mu \in \mathbf{M}_0$ and $r > 0$, let $\mu^{(r)}$ denote the restriction of $\mu$ to $\mathbf{S} \setminus B_{0,r}$. Then $\mu^{(r)}$ is finite and $\mu$ is uniquely determined by its restrictions $\mu^{(r)}$, $r > 0$. Moreover, convergence in $\mathbf{M}_0$ has a natural characterization in terms of weak convergence of the restrictions to $\mathbf{S} \setminus B_{0,r}$.

THEOREM 2.2. (i) *If $\mu_n \to \mu$ in $\mathbf{M}_0$, then $\mu_n^{(r)} \to \mu^{(r)}$ in $\mathbf{M}_b(\mathbf{S} \setminus B_{0,r})$ for all but at most countably many $r > 0$.*
(ii) *If there exists a sequence $\{r_i\}$ with $r_i \downarrow 0$ such that $\mu_n^{(r_i)} \to \mu^{(r_i)}$ in $\mathbf{M}_b(\mathbf{S} \setminus B_{0,r_i})$ for each $i$, then $\mu_n \to \mu$ in $\mathbf{M}_0$.*

Weak convergence is metrizable (for instance by the Prohorov metric, see e.g. p. 72 in [**5**]) and the close relation between weak convergence and convergence in $\mathbf{M}_0$ in Theorem 2.2 indicates that the topology in $\mathbf{M}_0$ is metrizable too. Theorem 2.3 shows that, with minor modifications of the arguments in [**7**, pp. 627–628], we may choose the metric

$$d_{\mathbf{M}_0}(\mu, \nu) = \int_0^\infty e^{-r} p_r(\mu^{(r)}, \nu^{(r)})[1 + p_r(\mu^{(r)}, \nu^{(r)})]^{-1}\,dr, \tag{2.1}$$

where $\mu^{(r)}, \nu^{(r)}$ are the finite restriction of $\mu, \nu$ to $\mathbf{S} \setminus B_{0,r}$, and $p_r$ is the Prohorov metric on $\mathbf{M}_b(\mathbf{S} \setminus B_{0,r})$.

THEOREM 2.3. *The metric $d_{\mathbf{M}_0}$ makes $\mathbf{M}_0$ a complete separable metric space.*

Many useful applications of weak convergence rely on the Portmanteau theorem and the Mapping theorem. Next we derive the corresponding versions of these results for convergence in $\mathbf{M}_0$. A more general version of the Portmanteau theorem below can be found in [**2**].

For $A \subset \mathbf{S}$, let $A^\circ$ and $A^-$ denote the interior and closure of $A$, and let $\partial A = A^- \setminus A^\circ$ be the boundary of $A$.

THEOREM 2.4. (Portmanteau theorem) *Let $\mu, \mu_n \in \mathbf{M}_0$. The following statements are equivalent.*

(i) *$\mu_n \to \mu$ in $\mathbf{M}_0$ as $n \to \infty$,*
(ii) *$\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for all $A \in \mathscr{S}$ with $\mu(\partial A) = 0$ and $0 \notin A^-$,*
(iii) *$\limsup_{n\to\infty} \mu_n(F) \leqslant \mu(F)$ and $\liminf_{n\to\infty} \mu_n(G) \geqslant \mu(G)$ for all closed $F \in \mathscr{S}$ and open $G \in \mathscr{S}$ with $s_0 \notin F$ and $s_0 \notin G^-$.*

REMARK 2.1. Consider statement (iii) above. In contrast to weak convergence of probability measures neither of the two statements implies the other. Take $c \in (0, \infty) \setminus \{1\}$ and $x \in \mathbf{S}_0$. Let $\mu_n = \delta_x$ and $\mu = c\,\delta_x$, where $\delta_x(A) = 1$ if $x \in A$ and 0 otherwise. If $c < 1$, then the second statement in (iii) holds but not the first. If $c > 1$, then the first statement in (iii) holds but not the second. See also Remarks 3 and 4 in [**2**].

We conclude this section with a mapping theorem. Let $(\mathbf{S},\mathscr{S})$ and $(\mathbf{S}',\mathscr{S}')$ be complete separable metric spaces. We denote by $s_0$ and $s_0'$ the origin in $\mathbf{S}$ and $\mathbf{S}'$, respectively. The open ball in $\mathbf{S}'$ centered at $s_0'$ with radius $r$ is denoted by $B_{0',r}$. For a measurable mapping $h:(\mathbf{S},\mathscr{S})\to(\mathbf{S}',\mathscr{S}')$, let $D_h\subset\mathbf{S}$ be the set of discontinuity points of $h$. Notice that $D_h\in\mathscr{S}$, see e.g. p. 243 in [**5**].

THEOREM 2.5. (Mapping theorem) *Let $h:(\mathbf{S},\mathscr{S})\to(\mathbf{S}',\mathscr{S}')$ be a measurable mapping. If $\mu_n\to\mu$ in $\mathbf{M}_0(\mathbf{S})$, $\mu(D_h\cap\mathbf{S}_0)=0$, $h(s_0)=s_0'$ and $s_0\notin D_h$, then $\mu_nh^{-1}\to\mu h^{-1}$ in $\mathbf{M}_{0'}(\mathbf{S}')$.*

Consider the following statements for a measurable mapping $h:\mathbf{S}\to\mathbf{S}'$:

(i) The mapping $h$ is continuous at $s_0$ and $h(s_0)=s_0'$.
(ii) For every $A\in\mathscr{S}'$ with $s_0'\notin A^-$ it holds that $s_0\notin h^{-1}(A)^-$ in $\mathbf{S}$.
(iii) For every $\varepsilon>0$ there exists $\delta>0$ such that $B_{0,\delta}\subset h^{-1}(B_{0',\varepsilon})$.

By Lemma 2.1 below, they are all equivalent. Hence, we could have chosen to formulate the mapping theorem with any one of them.

LEMMA 2.1. *The statements* (i)–(iii) *are equivalent.*

**2.1. Relative compactness in $\mathbf{M}_0$.** Since we are interested in convergence of measures in $\mathbf{M}_0$ it is essential to give an appropriate characterization of relative compactness. A subset of a topological space is said to be relatively compact if its closure is compact. A subset of a metric space is compact if and only if it is sequentially compact. Hence, $M\subset\mathbf{M}_0$ is relatively compact if and only if every sequence $\{\mu_n\}$ in $M$ contains a convergent subsequence.

For $\mu\in M\subset\mathbf{M}_0$ and $r>0$, let $\mu^{(r)}$ be the restriction of $\mu$ to $\mathbf{S}\setminus B_{0,r}$ and $M^{(r)}=\{\mu^{(r)}:\mu\in M\}$. By Theorem 2.2 we have the following characterization of relative compactness.

THEOREM 2.6. *A subset $M\subset\mathbf{M}_0$ is relatively compact if and only if there exists a sequence $\{r_i\}$ with $r_i\downarrow 0$ such that $M^{(r_i)}$ is relatively compact in $\mathbf{M}_b(\mathbf{S}\setminus B_{0,r_i})$ for each $i$.*

Relative compactness in the weak topology is characterized by Prohorov's theorem. This translates to the following characterization of relative compactness in $\mathbf{M}_0$.

THEOREM 2.7. *$M\subset\mathbf{M}_0$ is relatively compact if and only if there exists a sequence $\{r_i\}$ with $r_i\downarrow 0$ such that for each $i$*

$$\sup_{\mu\in M}\mu(\mathbf{S}\setminus B_{0,r_i})<\infty, \tag{2.2}$$

*and for each $\eta>0$ there exists a compact set $C_i\subset\mathbf{S}\setminus B_{0,r_i}$ such that*

$$\sup_{\mu\in M}\mu(\mathbf{S}\setminus(B_{0,r_i}\cup C_i))\leqslant\eta. \tag{2.3}$$

A convenient way to prove the convergence $\mu_n\to\mu$ in $\mathbf{M}_0$ is to show that $\{\mu_n\}$ is relatively compact and show convergence $\mu_n(A)\to\mu(A)$ for sets $A$ in a determining class $\mathscr{A}$. The next standard result (e.g. Theorem 2.3 in [**5**]) is useful for identifying a determining class of sets in $\mathscr{S}_0$.

Theorem 2.8. *Suppose that $\mathscr{A}$ is a $\pi$-system of sets in $\mathscr{S}_0$ and, for each $x \in \mathbf{S}_0$ and $\varepsilon > 0$, there exists an $A \in \mathscr{A}$ for which $x \in A^\circ \subset A \subset B_{x,\varepsilon}$. If $\mu, \nu \in \mathbf{M}_0$ and $\mu = \nu$ on $\mathscr{A}$, then $\mu = \nu$ on $\mathscr{S}_0$.*

## 3. Regularly varying sequences of measures

In the first part of this section we introduce the notion of regularly varying sequences of measures in $\mathbf{M}_0$, only assuming the general setting presented in Section 2. Then we define regular variation for a single measure in $\mathbf{M}_0$. In order to formulate this definition we need to assume further properties of the space $\mathbf{S}$, essentially we assume that $\mathbf{S}$ has the structure of a cone. For a measure $\nu \in \mathbf{M}_0$ we provide equivalent statements which are all equivalent to $\nu$ being regularly varying. These statements extend the corresponding equivalent definitions of regular variation for Borel measures on $\mathbf{R}^d$ to the general setting considered here.

Recall from e.g. [**6**] that a positive measurable function $c$ defined on $(0, \infty)$ is regularly varying with index $\rho \in \mathbf{R}$ if $\lim_{t\to\infty} c(\lambda t)/c(t) = \lambda^\rho$ for all $\lambda > 0$. Similarly, a sequence $\{c_n\}_{n\geqslant 1}$ of positive numbers is regularly varying with index $\rho \in \mathbf{R}$ if $\lim_{n\to\infty} c_{[\lambda n]}/c_n = \lambda^\rho$ for all $\lambda > 0$ (here $[\lambda n]$ denotes the integer part of $\lambda n$).

Definition 3.1. A sequence $\{\nu_n\}_{n\geqslant 1}$ in $\mathbf{M}_0$ is regularly varying with index $-\alpha < 0$ if there exists a sequence $\{c_n\}_{n\geqslant 1}$ of positive numbers which is regularly varying with index $\alpha > 0$, and a nonzero $\mu \in \mathbf{M}_0$ such that $c_n\nu_n \to \mu$ in $\mathbf{M}_0$ as $n \to \infty$.

The choice of terminology is motivated by the fact that $\{\nu_n(A)\}_{n\geqslant 1}$ is a regularly varying sequence for each set $A \in \mathscr{S}$ with $0 \notin A^-$, $\mu(\partial A) = 0$ and $\mu(A) > 0$.

We will now define regular variation for a single measure in $\mathbf{M}_0$. In order to formulate this definition we need to assume further properties of the space $\mathbf{S}$. Suppose that there is an element $0 \in \mathbf{S}$ and let $s_0 = 0$ in the definitions of $\mathbf{S}_0$ and $\mathbf{M}_0$ in Section 2. Suppose that the space $\mathbf{S}$ is closed under multiplication by nonnegative real numbers $\lambda \in \mathbf{R}_+$ and that the map $(\lambda, x) \mapsto \lambda x$ from $\mathbf{R}_+ \times \mathbf{S}$ into $\mathbf{S}$ is continuous. In particular, we have $0x = 0$ for all $x \in \mathbf{S}$. Suppose further that the metric $d$ on $\mathbf{S}$ satisfies $d(0, \lambda_1 x) < d(0, \lambda_2 x)$ for all $\lambda_1, \lambda_2 \in \mathbf{R}_+$ with $\lambda_1 < \lambda_2$ and all $x \in \mathbf{S}_0$, i.e. the distance to the origin $s_0 = 0$ for a point on the ray $\{\lambda x : \lambda \in \mathbf{R}_+\}$ is strictly increasing in $\lambda$.

Definition 3.2. A measure $\nu \in \mathbf{M}_0$ is regularly varying if the sequence $\{\nu(n\cdot)\}_{n\geqslant 1}$ in $\mathbf{M}_0$ is regularly varying.

In particular, a probability measure $P$ on $\mathscr{S}$ is regularly varying if the sequence $\{P(n\cdot)\}_{n\geqslant 1}$ in $\mathbf{M}_0$ is regularly varying.

There are many possible equivalent ways to formulate regular variation for a measure $\nu \in \mathbf{M}_0$. Consider the following statements.

(i) There exist a nonzero $\mu \in \mathbf{M}_0$ and a regularly varying sequence $\{c_n\}_{n\geqslant 1}$ of positive numbers such that $c_n\nu(n\cdot) \to \mu(\cdot)$ in $\mathbf{M}_0$ as $n \to \infty$.

(ii) There exist a nonzero $\mu \in \mathbf{M}_0$ and a regularly varying function $c$ such that $c(t)\nu(t\cdot) \to \mu(\cdot)$ in $\mathbf{M}_0$ as $t \to \infty$.

(iii) There exist a nonzero $\mu \in \mathbf{M}_0$ and a set $E \in \mathscr{S}$ with $0 \notin E^-$ such that $\nu(tE)^{-1}\nu(t\cdot) \to \mu(\cdot)$ in $\mathbf{M}_0$ as $t \to \infty$.
(iv) There exists a nonzero $\mu \in \mathbf{M}_0$ such that $\nu(t[\mathbf{S} \setminus B_{0,1}])^{-1}\nu(t\cdot) \to \mu(\cdot)$ in $\mathbf{M}_0$ as $t \to \infty$.
(v) There exist a nonzero $\mu \in \mathbf{M}_0$ and a sequence $\{a_n\}_{n\geqslant 1}$ of positive numbers such that $n\nu(a_n\cdot) \to \mu(\cdot)$ in $\mathbf{M}_0$ as $n \to \infty$.

THEOREM 3.1. (a) *Each of the statements* (i)–(v) *above implies that* $\mu(\lambda A) = \lambda^{-\alpha}\mu(A)$ *for some* $\alpha > 0$ *and all* $A \in \mathscr{S}_0$ *and* $\lambda > 0$. (b) *The statements* (i)–(v) *are equivalent.*

Several equivalent formulations of regular variation for measures on $\mathbf{R}^d$, similar to those above, can be found in e.g. [**3**] and [**21**]. Theorem 3.1 extends some of them to measures on general metric spaces. One could also identify $\mathbf{S}$ with a product space and formulate regular variation in terms of polar coordinates. However, we have not pursued this approach here.

On $\mathbf{R}^d$ statements equivalent to regular variation for probability measures have appeared at numerous places in the vast literature on domains of attraction for sums and maxima. The notion of regular variation for measures on $\mathbf{R}^d$ first appeared in [**18**], where it was used for multivariate extensions of results in [**8**] on characterizations of domains of attractions. See Chapter 6 in [**19**] for a more recent account on this topic. The definition of regular variation for a measure on $\mathbf{R}^d$ in [**18**] differs from the one considered here and those in e.g. [**21**] in the sense that the limiting measure $\mu$ in the above statements (i)–(v) may be supported in a proper subspace of $\mathbf{S}$.

## 4. Examples

In this section we provide some examples of metric spaces on which regularly varying measures are natural in applications. We consider the Euclidean space $\mathbf{R}^d$, the space of continuous functions, and the space of càdlàg functions. We review some conditions to check relative compactness in $\mathbf{M}_0$ for measures on these spaces and provide conditions for determining if a given measure is regularly varying.

**4.1. The Euclidean space $\mathbf{R}^d$.** A fundamental example of a metric space $\mathbf{S}$ is the Euclidean space $\mathbf{R}^d$ with the usual Euclidean norm $|\cdot|$. The characterization of relative compactness in $\mathbf{M}_0$ simplifies considerably if $\mathbf{M}_0 = \mathbf{M}_0(\mathbf{R}^d)$. Since the unit ball is relatively compact in $\mathbf{R}^d$, Theorem 2.7 implies that $M \subset \mathbf{M}_0(\mathbf{R}^d)$ is relatively compact if and only if $\sup_{\mu\in M} \mu(\mathbf{R}^d \setminus B_{0,r}) < \infty$ for each $r > 0$, and $\lim_{R\to\infty} \sup_{\mu\in M} \mu(\mathbf{R}^d \setminus B_{0,R}) = 0$.

Regular variation for measures on $\mathbf{R}^d$ is often proved by showing convergence to a measure $\mu \in \mathbf{M}_0(\mathbf{R}^d)$ for an appropriate convergence determining class of subsets of $\mathbf{R}^d$. If $V_{u,S} = \{x \in \mathbf{R}^d : |x| > u, x/|x| \in S\}$ for $u > 0$ and Borel sets $S \subset \{x \in \mathbf{R}^d : |x| = 1\}$, then the collection of such sets satisfying $\mu(\partial V_{u,S}) = 0$ form a convergence determining class (see [**3**]).

THEOREM 4.1. *Let* $\nu, \mu \in \mathbf{M}_0(\mathbf{R}^d)$ *be nonzero and let* $\{c_n\}$ *be a regularly varying sequence with index* $\alpha > 0$. *If* $c_n\nu(nV_{u,S}) \to \mu(V_{u,S})$ *as* $n \to \infty$ *for each* $u > 0$

*and Borel set $S \subset \{x \in \mathbf{R}^d : |x| = 1\}$ with $\mu(\partial V_{u,S}) = 0$, then $c_n\nu(n\cdot) \to \mu(\cdot)$ in $\mathbf{M}_0$ as $n \to \infty$ and $\nu$ is regularly varying.*

The sets of the form $A_x = [0,\infty)^d \setminus \{[0,x_1]\times\cdots\times[0,x_d]\}$, for $x = (x_1,\dots,x_d) \in [0,\infty)^d$, form a convergence determining class for regular variation for measures on $[0,\infty)^d \setminus \{0\}$. This is well known, see e.g. [**20**].

THEOREM 4.2. *Let $\nu,\mu \in \mathbf{M}_0([0,\infty)^d)$ be nonzero and let $\{c_n\}$ be a regularly varying sequence with index $\alpha > 0$. If $c_n\nu(nA_x) \to \mu(A_x)$ as $n \to \infty$ for each $x \in [0,\infty)^d \setminus \{0\}$, then $c_n\nu(n\cdot) \to \mu(\cdot)$ in $\mathbf{M}_0$ as $n \to \infty$ and $\nu$ is regularly varying.*

To show regular variation in function spaces is typically less straight-forward. Similar to weak convergence of probability measures on these spaces, convergence is typically shown by showing relative compactness and convergence for finite dimensional projections of the original measures. In the following two sections we will exemplify applications of the framework set up in Sections 2 and 3 by considering regular variation for measures on the space $\mathbf{C}([0,1];\mathbf{R}^d)$ of continuous functions with the uniform topology and the space $\mathbf{D}([0,1];\mathbf{R}^d)$ of càdlàg functions with the Skorohod $J_1$-topology.

**4.2. The space $\mathbf{C}([0,1];\mathbf{R}^d)$.** Let $\mathbf{S}$ be the space $\mathbf{C} = \mathbf{C}([0,1];\mathbf{R}^d)$ of continuous functions $[0,1] \to \mathbf{R}^d$ with the uniform topology given by the supremum norm $|\cdot|_\infty$. Tightness conditions for weak convergence on $\mathbf{C}$ are well known [**5**, p. 82] and translate naturally to conditions for relative compactness in $\mathbf{M}_0(\mathbf{C})$. For $x : [0,1] \to \mathbf{R}^d$ the modulus of continuity is given by $w_x(\delta) = \sup_{|s-t|\leqslant\delta} |x(s)-x(t)|$.

THEOREM 4.3. *A set $M \subset \mathbf{M}_0(\mathbf{C})$ is relatively compact if and only if for each $r > 0$ and each $\varepsilon > 0$*

$$\sup_{\mu\in M} \mu(\mathbf{S} \setminus B_{0,r}) < \infty, \tag{4.1}$$

$$\lim_{R\to\infty} \sup_{\mu\in M} \mu(x : |x(0)| > R) = 0, \tag{4.2}$$

$$\lim_{\delta\to 0} \sup_{\mu\in M} \mu(x : w_x(\delta) \geqslant \varepsilon) = 0. \tag{4.3}$$

To prove that a measure $\nu \in \mathbf{M}_0(\mathbf{C})$ is regularly varying we typically need to show that for some regularly varying sequence $\{c_n\}$ of positive numbers, (i) $\{c_n\nu(n\cdot) : n \geqslant 1\}$ is relatively compact, and (ii) any two subsequential limits of $\{c_n\nu(n\cdot)\}$ coincide. The point (ii) holds, similar to the case for weak convergence, if the subsequential limits have the same finite dimensional projections. For $(t_1,\dots,t_k) \in [0,1]^k$ denote by $\pi_{t_1,\dots,t_k}$ the map $\mathbf{C} \ni x \mapsto (x(t_1),\dots,x(t_k)) \in \mathbf{R}^{dk}$. The finite dimensional projections of $\nu \in \mathbf{M}_0(\mathbf{C})$ are measures of the form $\nu\pi^{-1}_{t_1,\dots,t_k}$.

THEOREM 4.4. *Let $\nu,\mu \in \mathbf{M}_0(\mathbf{C})$ be nonzero and let $\{c_n\}$ be a regularly varying sequence of positive numbers. Then $c_n\nu(n\cdot) \to \mu(\cdot)$ in $\mathbf{M}_0(\mathbf{C})$ as $n \to \infty$ if and only if for each integer $k \geqslant 1$ and $(t_1,\dots,t_k) \in [0,1]^k$*

$$c_n\nu\pi^{-1}_{t_1,\dots,t_k}(n\cdot) \to \mu\pi^{-1}_{t_1,\dots,t_k}(\cdot) \tag{4.4}$$

*in* $\mathbf{M}_0(\mathbf{R}^{dk})$ *as* $n \to \infty$*, and for each* $r > 0$ *and each* $\varepsilon > 0$

$$\sup_n c_n \nu(n[\mathbf{S} \setminus B_{0,r}]) < \infty, \tag{4.5}$$

$$\lim_{\delta \to 0} \sup_n c_n \nu(x : w_x(\delta) \geqslant n\varepsilon) = 0. \tag{4.6}$$

Recall that the regular variation statement $c_n\nu(n\cdot) \to \mu(\cdot)$ can be replaced by any of the equivalent statements in Theorem 3.1. For those statements there are corresponding versions of the conditions in Theorem 4.4. Regular variation on $\mathbf{C}$ can be used for studying extremal properties of stochastic processes. The mapping theorem can be used to determine the tail behavior of functionals of a heavy-tailed stochastic process with continuous sample paths. Another application is to characterize max-stable distributions in $\mathbf{C}$. For $f_1, \dots, f_n \in \mathbf{C}$ let $\bigvee_{i=1}^n f_i$ be the element in $\mathbf{C}$ given by $(\bigvee_{i=1}^n f_i)(t) = \max_{i=1,\dots,n} f_i(t)$. A random variable $X$ with values in $\mathbf{C}$ is said to be max-stable if, for each integer $n \geqslant 1$ there are functions $a_n(t) > 0$ and $b_n(t)$ such that $a_n^{-1}(\bigvee_{i=1}^n X_i - b_n) \stackrel{\mathrm{d}}{=} X$, where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution and $X_1, \dots, X_n$ are independent and identically distributed copies of $X$. The distribution of $X$ is called simple max-stable if one can choose $a_n(t) = n$ and $b_n(t) = 0$ for all $t$. Max-stable distributions appear as limiting distributions of pointwise maxima of independent and identically distributed stochastic processes. Their domain of attraction can be characterized in terms of regular variation. If $Y_1, Y_2, \dots$ are independent and identically distributed with values in $\mathbf{C}$, then $n^{-1} \bigvee_{i=1}^n Y_i \xrightarrow{d} X$ in $\mathbf{C}$ for some $X$ (where $X$ necessarily has a simple max-stable distribution) if and only if the distribution of $Y$ on $\mathbf{C}$ is regularly varying and satisfies statement (v) above (before Theorem 3.1) with $a_n = n$. The same characterization holds for random variables taking values in the space $\mathbf{D}$ studied below. See [**12**, Theorem 2.4] and [**11**] for more details. Theorem 4.4 provides necessary and sufficient conditions for a Borel (probability) measure on $\mathbf{C}$ to be regularly varying.

**4.3. The space $\mathbf{D}([0,1];\mathbf{R}^d)$.** Let $\mathbf{S}$ be the space $\mathbf{D} = \mathbf{D}([0,1]; R^d)$ of càdlàg functions equipped with the Skorohod $J_1$-topology. We refer to [**5**] for details on this space and the $J_1$-topology. In particular, elements in $\mathbf{D}$ are assumed to be left-continuous at 1. Notice that if $d$ is the $J_1$-metric then $d(x,0) = |x|_\infty$. Notice also that $\mathbf{D}$ is not complete under $d$ but there exists an equivalent metric under which $\mathbf{D}$ is complete (see Section 12 in [**5**]). Notice also that $\mathbf{D}$ is not a topological vector space since addition in $\mathbf{D}$ is in general not continuous (see e.g. [**23**]). For $T \subset [0,1]$ and $\delta > 0$, let

$$w_x(T) = \sup_{t_1,t_2 \in T} |x(t_2) - x(t_1)|,$$
$$w''_x(\delta) = \sup_{t,t_1,t_2} \{|x(t) - x(t_1)| \wedge |x(t_2) - x(t)|\},$$

where the supremum in the definition of $w''_x(\delta)$ is over all $(t, t_1, t_2)$ satisfying $0 \leqslant t_1 \leqslant t \leqslant t_2 \leqslant 1$ and $t_2 - t_1 \leqslant \delta$.

THEOREM 4.5. *A set $M \subset \mathbf{M}_0(\mathbf{D})$ is relatively compact if and only if for each $r > 0$ and $\varepsilon > 0$*

$$\sup_{\mu \in M} \mu(\mathbf{S} \setminus B_{0,r}) < \infty, \tag{4.7}$$

$$\lim_{R \to \infty} \sup_{\mu \in M} \mu(x : |x|_\infty > R) = 0, \tag{4.8}$$

$$\lim_{\delta \to 0} \sup_{\mu \in M} \mu(x : w''_x(\delta) \geqslant \varepsilon) = 0, \tag{4.9}$$

$$\lim_{\delta \to 0} \sup_{\mu \in M} \mu(x : w_x([0,\delta)) \geqslant \varepsilon) = 0, \tag{4.10}$$

$$\lim_{\delta \to 0} \sup_{\mu \in M} \mu(x : w_x([1-\delta,1)) \geqslant \varepsilon) = 0. \tag{4.11}$$

Similar to the space $\mathbf{C}$, regular variation for measures on $\mathbf{D}$ is typically proved by showing relative compactness in $\mathbf{M}_0(\mathbf{D})$ and convergence of finite dimensional projections (see [**13**, Theorem 10]).

THEOREM 4.6. *Let $\nu, \mu \in \mathbf{M}_0(\mathbf{D})$ be nonzero and let $\{c_n\}$ be a regularly varying sequence of positive numbers. Then $c_n\nu(n\cdot) \to \mu(\cdot)$ in $\mathbf{M}_0(\mathbf{D})$ as $n \to \infty$ if and only if there exists $T \subset [0,1]$ containing $0$, $1$ and all but at most countably many points of $[0,1]$ such that*

$$c_n \nu \pi^{-1}_{t_1,\dots,t_k}(n\cdot) \to \mu \pi^{-1}_{t_1,\dots,t_k}(\cdot) \tag{4.12}$$

*in $\mathbf{M}_0(\mathbf{R}^{dk})$ as $n \to \infty$ whenever $t_1, \dots, t_k \in T$, and for each $\varepsilon > 0$*

$$\lim_{\delta \to 0} \limsup_n c_n \nu(w''_x(\delta) \geqslant n\varepsilon) = 0 \tag{4.13}$$

$$\lim_{\delta \to 0} \limsup_n c_n \nu(w_x([0,\delta)) \geqslant n\varepsilon) = 0 \tag{4.14}$$

$$\lim_{\delta \to 0} \limsup_n c_n \nu(w_x([1-\delta,1)) \geqslant n\varepsilon) = 0. \tag{4.15}$$

Notice that the regular variation statement $c_n\nu(n\cdot) \to \mu(\cdot)$ can be replaced by any of the equivalent statements in Theorem 3.1. Notice also that the set $T$ in Theorem 4.6 appears because for $t \in (0,1)$ the map $\pi_t(x) = x(t)$ is continuous at $x \in \mathbf{D}$ if and only if $x$ is continuous at $t$. Regular variation on $\mathbf{D}$ can be used for studying extremal properties of stochastic processes, see [**13**] and [**15**]. In particular, the mapping theorem can be used to determine the tail behavior of functionals of a heavy-tailed stochastic process with càdlàg sample paths. Regular variation is also closely connected to max-stable distributions on $\mathbf{D}$, see [**12**].

Regularly varying sequences of measures in $\mathbf{M}_0(\mathbf{D})$ appear for instance when studying large deviations. Consider for instance the stochastic process $X^{(n)}_t = Z_1 + \cdots + Z_{[nt]}$, $t \in [0,1]$, where $\{Z_k\}$ is a sequence of independent and identically distributed $\mathbf{R}^d$-valued random variables whose common probability distribution $P(Z_1 \in \cdot)$ is regularly varying on $\mathbf{R}^d$ with $\alpha > 1$. Then the sequence $\nu_n(\cdot) = P(n^{-1}X^{(n)} \in \cdot)$ is regularly varying according to Definition 3.1 with $c_n = [nP(|Z_1| > n)]^{-1}$, see [**14**, Theorem 2.1].

## 5. Proofs

Proof of Theorem 2.1. Suppose $\mu_n \to \mu$ in $\mathbf{M}_0$ and take $f \in \mathscr{C}_0$. Given $\varepsilon > 0$ consider the neighborhood $N_{\varepsilon,f}(\mu) = \{\nu : |\int f\,d\nu - \int f\,d\mu| < \varepsilon\}$. By assumption there exists $n_0$ such that $n \geqslant n_0$ implies $\mu_n \in N_{\varepsilon,f}(\mu)$, i.e. $|\int f\,d\mu_n - \int f\,d\mu| < \varepsilon$. Hence $\int f\,d\mu_n \to \int f\,d\mu$.

Conversely, suppose that $\int f\,d\mu_n \to \int f\,d\mu$ for each $f \in \mathscr{C}_0$. Take $\varepsilon > 0$ and a neighborhood $N_{\varepsilon,f_1,\dots,f_k}(\mu) = \{\nu : |\int f_i\,d\nu - \int f_i\,d\mu| < \varepsilon, i = 1,\dots,k\}$. Let $n_i$ be an integer such that $n \geqslant n_i$ implies $|\int f_i\,d\mu_n - \int f_i\,d\mu| < \varepsilon$. Hence, $n \geqslant \max(n_1,\dots,n_k)$ implies $\mu_n \in N_{\varepsilon,f_1,\dots,f_k}(\mu)$. It follows that $\mu_n \to \mu$ in $\mathbf{M}_0$. □

Lemma 5.1. *Let $\mu \in \mathbf{M}_0$, let $r > 0$ be such that $\mu(\partial B_{0,r}) = 0$ and let $f \in \mathscr{C}_b(\mathbf{S} \setminus B_{0,r})$ be nonnegative. For each $\varepsilon > 0$ there exist nonnegative $f_1, f_2 \in \mathscr{C}_0$ such that $f_1 \leqslant f \leqslant f_2$ on $\mathbf{S} \setminus B_{0,r}$ and $|\int_{\mathbf{S}} f_2\,d\mu - \int_{\mathbf{S}} f_1\,d\mu| \leqslant \varepsilon$.*

Proof. For any $r' > r$ let $f_{1,r'}$ be a function on $\mathbf{S}_0$ given by

$$f_{1,r'} = \begin{cases} g_{r,r'}f & \text{on } \mathbf{S} \setminus B_{0,r}, \\ 0 & \text{on } B_{0,r} \setminus \{0\}, \end{cases}$$

where $g_{r,r'}(x) = \max\{\min\{d(x,s_0), r'\} - r, 0\}/(r' - r)$ for $x \in \mathbf{S}_0$. Then $f_{1,r'}$ is continuous, $f_{1,r'} \leqslant f$ on $\mathbf{S} \setminus B_{0,r}$ and $f_{1,r'}(x) \uparrow f(x)$ pointwise on $\mathbf{S} \setminus B^-_{0,r}$ as $r' \downarrow r$. We now consider an upper bound. By the Tietze extension theorem (Theorem 3.6.3 in [**9**]) there exists a nonnegative, bounded continuous extension $F$ of $f$ to $\mathbf{S}_0$ such that $F = f$ on $\mathbf{S} \setminus B_{0,r}$ and $\sup|F| = \sup|f|$. For $r'' < r$ let $f_{2,r''}$ be a function on $\mathbf{S}_0$ given by $f_{2,r''} = g_{r'',r}F$ on $\mathbf{S} \setminus B_{0,r}$ and 0 otherwise. Then $f_{2,r''}$ is continuous, $f_{2,r''} \geqslant f$ on $\mathbf{S} \setminus B_{0,r}$ and $f_{2,r''}(x) \downarrow f(x)$ pointwise on $\mathbf{S} \setminus B_{0,r}$ as $r'' \uparrow r$. In particular,

$$\Big|\int_{\mathbf{S}} f_{2,r''}\,d\mu - \int_{\mathbf{S}} f_{1,r'}\,d\mu\Big| \leqslant \sup|f|\,\mu(B_{0,r'} \setminus B_{0,r''}) \to \sup|f|\,\mu(\partial B_{0,r}) = 0$$

as $r' \downarrow r$ and $r'' \uparrow r$. Hence, for $r'$ and $r''$ sufficiently close to $r$ we may take $f_1 = f_{1,r'}$ and $f_2 = f_{2,r''}$. □

Proof of Theorem 2.2. (i) Let $R_\mu = \{r \in (0,\infty) : \mu(\partial B_{0,r}) = 0\}$ and notice that $(0,\infty) \setminus R_\mu$ is at most countable. Take $r \in R_\mu$ and, without loss of generality, a nonnegative $f \in \mathscr{C}_b(\mathbf{S} \setminus B_{0,r})$. Given $\varepsilon > 0$ there exist, by Lemma 5.1, nonnegative $f_1, f_2 \in \mathscr{C}_0(\mathbf{S})$ with $f_1 \leqslant f \leqslant f_2$ on $\mathbf{S} \setminus B_{0,r}$ such that $|\int f_2\,d\mu - \int f_1\,d\mu| \leqslant \varepsilon$. Hence, $\int f_1\,d\mu_n \leqslant \int f\,d\mu_n \leqslant \int f_2\,d\mu_n$ and by Theorem 2.1 $\mu_n \to \mu$ in $\mathbf{M}_0$ implies that

$$\int f_1\,d\mu \leqslant \liminf_n \int f\,d\mu_n \leqslant \limsup_n \int f\,d\mu_n \leqslant \int f_2\,d\mu.$$

Since $\varepsilon > 0$ was arbitrary it follows that $\int f\,d\mu_n \to \int f\,d\mu$.

(ii) Take $f \in \mathscr{C}_0(\mathbf{S})$; without loss of generality $f$ can be chosen nonnegative. The support of $f$ is contained in $\mathbf{S} \setminus B_{0,r_i}$ for some $r_i > 0$ such that $\mu_n^{(r_i)} \to \mu^{(r_i)}$ in $\mathbf{M}_b(\mathbf{S} \setminus B_{0,r_i})$. Hence $f \in \mathscr{C}_b(\mathbf{S} \setminus B_{0,r_i})$ and $\int f\,d\mu_n = \int f\,d\mu_n^{(r_i)} \to \int f\,d\mu^{(r_i)} = \int f\,d\mu$. □

Proof of Theorem 2.3. The proof consists of minor modifications of arguments that can be found in [**7**, pp. 628–630]. Here we change from $r$ to $1/r$. For the sake of completeness we have included a full proof.

We show that (i) $\mu_n \to \mu$ in $\mathbf{M}_0$ if and only if $d_{\mathbf{M}_0}(\mu_n, \mu) \to 0$, and (ii) $(\mathbf{M}_0, d_{\mathbf{M}_0})$ is complete and separable.

(i) Suppose that $d_{\mathbf{M}_0}(\mu_n, \mu) \to 0$. The integral expression in (2.1) can be written $d_{\mathbf{M}_0}(\mu_n, \mu) = \int_0^\infty e^{-r} g_n(r)\, dr$, so that for each $n$, $g_n(r)$ decreases with $r$ and is bounded by 1. Helly's selection theorem (p. 336 in [**4**]), applied to $1 - g_n$, implies that there exists a subsequence $\{n'\}$ and a non-increasing function $g$ such that $g_{n'}(r) \to g(r)$ for all continuity points of $g$. By dominated convergence, $\int_0^\infty e^{-r} g(r)\, dr = 0$ and since $g$ is monotone this implies that $g(r) = 0$ for all finite $r > 0$. Since this holds for all convergent subsequences $\{g_{n'}(r)\}$, it follows that $g_n(r) \to 0$ for all continuity points $r$ of $g$, and hence, for such $r$, $p_r(\mu_n^{(r)}, \mu^{(r)}) \to 0$ as $n \to \infty$. By Theorem 2.2, $\mu_n \to \mu$ in $\mathbf{M}_0$.

Suppose that $\mu_n \to \mu$ in $\mathbf{M}_0$. Then theorem 2.2 implies that $\mu_n^{(r)} \to \mu^{(r)}$ in $\mathbf{M}_b(\mathbf{S} \setminus B_{0,r})$ for all but at most countably many $r > 0$. Hence, for such $r$, $p_r(\mu_n^{(r)}, \mu^{(r)})[1 + p_r(\mu_n^{(r)}, \mu^{(r)})]^{-1} \to 0$, which by the dominated convergence theorem implies that $d_{\mathbf{M}_0}(\mu_n, \mu) \to 0$.

(ii) Separability: For $r > 0$ let $D_r$ be a countable dense set in $\mathbf{M}_b(\mathbf{S} \setminus B_{0,r})$ with the weak topology. Let $D$ be the union of $D_r$ for rational $r > 0$. Then $D$ is countable. Let us show $D$ is dense in $\mathbf{M}_0$. Given $\varepsilon > 0$ and $\mu \in \mathbf{M}_0$ pick $r' > 0$ such that $\int_0^{r'} e^{-r}\, dr < \varepsilon/2$. Take $\mu_{r'} \in D_{r'}$ such that $p_{r'}(\mu_{r'}, \mu^{(r')}) < \varepsilon/2$. Then $p_r(\mu_{r'}^{(r)}, \mu^{(r)}) < \varepsilon/2$ for all $r > r'$. In particular, $d_{\mathbf{M}_0}(\mu_{r'}, \mu) < \varepsilon$.

Completeness: Let $\{\mu_n\}$ be a Cauchy sequence for $d_{\mathbf{M}_0}$. Then $\{\mu_n^{(r)}\}$ is a Cauchy sequence for $p_r$ for all but at most countably many $r > 0$ and by completeness of $\mathbf{M}_b(\mathbf{S} \setminus B_{0,r})$ it has a limit $\mu_r$. These limits are consistent in the sense that $\mu_{r'}^{(r)} = \mu_r$ for $r' < r$. On $\mathscr{S}_0$ put $\mu(A) = \lim_{r \to 0} \mu_r(A \cap \mathbf{S} \setminus B_{0,r})$. Then $\mu$ is a measure. Clearly, $\mu \geqslant 0$ and $\mu(\emptyset) = 0$. Moreover, $\mu$ is countably additive: for disjoint $A_n \in \mathscr{S}_0$ the monotone convergence theorem implies that

$$\mu\Big(\bigcup_n A_n\Big) = \lim_{r \to 0} \mu_r\Big(\bigcup_n A_n \cap [\mathbf{S} \setminus B_{0,r}]\Big)$$
$$= \lim_{r \to 0} \sum_n \mu_r\big(A_n \cap [\mathbf{S} \setminus B_{0,r}]\big) = \sum_n \mu(A_n). \qquad \square$$

Proof of Theorem 2.4. We show that (i) $\Leftrightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii).

Suppose that (i) holds and take $A \in \mathscr{S}$ with $s_0 \notin A^-$ and $\mu(\partial A) = 0$. Since $s_0 \notin A^-$ there exists $r > 0$ with $\mu(\partial B_{0,r}) = 0$ such that $A \subset \mathbf{S} \setminus B_{0,r}$. By Theorem 2.2, $\mu_n^{(r)} \to \mu^{(r)}$ in $\mathbf{M}_b(\mathbf{S} \setminus B_{0,r})$. The Portmanteau theorem for weak convergence implies (ii).

Suppose that (ii) holds. The Portmanteau theorem for weak convergence implies that $\mu_n^{(r)} \to \mu^{(r)}$ in $\mathbf{M}_b(\mathbf{S} \setminus B_{0,r})$ for all $r > 0$ for which $\mu(\partial B_{0,r}) = 0$. Since $\mu(\partial B_{0,r}) = 0$ for all but at most countably many $r > 0$, Theorem 2.2 implies $\mu_n \to \mu$ in $\mathbf{M}_0$.

Suppose that (iii) holds and take $A \in \mathscr{S}$ with $s_0 \notin A^-$ and $\mu(\partial A) = 0$. Then,

$$\begin{aligned}\limsup_{n\to\infty} \mu_n(A) &\leqslant \limsup_{n\to\infty} \mu_n(A^-) \leqslant \mu(A^-) \\ &= \mu(A^\circ) \leqslant \liminf_{n\to\infty} \mu_n(A^\circ) \leqslant \liminf_{n\to\infty} \mu_n(A).\end{aligned}$$

Hence, $\lim_{n\to\infty} \mu_n(A) = \mu(A)$, so that (ii) holds.

Suppose that (ii) holds and take a closed $F \in \mathscr{S}$ with $s_0 \notin F$. Notice that for $F_\varepsilon = \{s : d(s,F) \leqslant \varepsilon\}$ it holds that for small $\varepsilon$, say $\varepsilon \in (0,c)$, $\partial F_\varepsilon = \{s : d(s,F) = \varepsilon\}$, $s_0 \notin F_\varepsilon$ and $\mu(\partial F_\varepsilon) = 0$ for all but at most countably many $\varepsilon \in (0,c)$. Hence, $\limsup_{n\to\infty} \mu_n(F) \leqslant \mu(F_\varepsilon)$. Since $F_\varepsilon \downarrow F$ as $\varepsilon \downarrow 0$ and since $F$ is closed, $\mu(F_\varepsilon) \downarrow \mu(F)$ as $\varepsilon \downarrow 0$. Hence, $\limsup_{n\to\infty} \mu_n(F) \leqslant \mu(F)$. Now take an open $G \in \mathscr{S}$ with $s_0 \notin G^-$ and an $r > 0$ such that $\mu(\partial B_{0,r}) = 0$ and $G \subset \mathbf{S} \setminus B_{0,r}$. Set $A = \mathbf{S} \setminus B_{0,r}^-$ and so that $F = A \setminus G$ is closed. Then

$$\liminf_{n\to\infty} \mu_n(G) = \liminf_{n\to\infty} (\mu_n(A) - \mu_n(F)) \geqslant \mu(A) - \mu(F) = \mu(G).$$

Hence, (ii) holds. $\square$

Proof of Lemma 2.1. (ii) $\Leftrightarrow$ (iii): Notice that $s_0' \notin A^-$ if and only if there exists $\varepsilon > 0$ such that $A \subset \mathbf{S}' \setminus B_{0',\varepsilon}$, and that $s_0 \notin h^{-1}(A)^-$ if and only if there exists $\delta > 0$ such that $h^{-1}(A) \subset \mathbf{S} \setminus B_{0,\delta}$. Hence, (ii) holds if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $h^{-1}(\mathbf{S}' \setminus B_{0',\varepsilon}) \subset \mathbf{S} \setminus B_{0,\delta}$. Taking complements shows that $h^{-1}(\mathbf{S}' \setminus B_{0',\varepsilon}) \subset \mathbf{S} \setminus B_{0,\delta}$ if and only if $B_{0,\delta} \subset h^{-1}(B_{0',\varepsilon})$.

(iii) $\Rightarrow$ (i):

(iii) implies that $h(B_{0,\delta}) \subset h(h^{-1}(B_{0',\varepsilon}))$. Since $h(h^{-1}(B_{0',\varepsilon})) \subset B_{0',\varepsilon}$ holds for any $h$, it follows that (i) holds.

(i) $\Rightarrow$ (iii): (i) implies that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $h(B_{0,\delta}) \subset B_{0',\varepsilon}$, which implies that $h^{-1}(h(B_{0,\delta})) \subset h^{-1}(B_{0',\varepsilon})$. Since $B_{0,\delta} \subset h^{-1}(h(B_{0,\delta}))$ holds for any $h$, it follows that (iii) holds. $\square$

Proof of Theorem 2.5. Take $A \in \mathscr{S}'$ with $s_0' \notin A^-$ and $\mu h^{-1}(\partial A) = 0$. Since $\partial h^{-1}(A) \subset h^{-1}(\partial A) \cup D_h$ (see e.g. (A2.3.2) in [**7**]), we have $\mu(\partial h^{-1}(A)) \leqslant \mu h^{-1}(\partial A) + \mu(D_h) = 0$. Since $\mu_n \to \mu$ in $\mathbf{M}_0(\mathbf{S})$, $\mu(\partial h^{-1}(A)) = 0$ and, by Lemma 2.1, $s_0 \notin h^{-1}(A)^-$, it follows by Theorem 2.4 (ii) that $\mu_n h^{-1}(A) \to \mu h^{-1}(A)$. Hence, $\mu_n h^{-1} \to \mu h^{-1}$ in $\mathbf{M}_0(\mathbf{S}')$. $\square$

Proof of Theorem 2.6. Suppose $M \subset \mathbf{M}_0(\mathbf{S})$ is relatively compact. Let $\{\mu_n\}$ be a subsequence in $M$. Then there exists a convergent subsequence $\mu_{n_k} \to \mu$ for some $\mu \in M^-$. By Theorem 2.2, there exists a sequence $\{r_i\}$ with $r_i \downarrow 0$ such that $\mu_{n_k}^{(r_i)} \to \mu^{(r_i)}$ in $\mathbf{M}_b(\mathbf{S} \setminus B_{0,r_i})$. Hence, $M^{(r_i)}$ is relatively compact in $\mathbf{M}_b(\mathbf{S} \setminus B_{0,r_i})$ for each such $r_i$.

Conversely, suppose there exists a sequence $\{r_i\}$ with $r_i \downarrow 0$ such that $M^{(r_i)} \subset \mathbf{M}_b(\mathbf{S} \setminus B_{0,r_i})$ is relatively compact for each $i$, and let $\{\mu_n\}$ be a sequence of elements in $M$. We use a diagonal argument to find a convergent subsequence. Since $M^{(r_1)}$ is relatively compact there exists a subsequence $\{\mu_{n_1(k)}\}$ of $\{\mu_n\}$ such that $\mu_{n_1(k)}^{(r_1)}$ converges to some $\mu_{r_1}$ in $\mathbf{M}_b(\mathbf{S} \setminus B_{0,r_1})$. Similarly since $M^{(r_2)}$ is relatively compact and $\{\mu_{n_1(k)}\} \subset M$ there exists a subsequence $\{\mu_{n_2(k)}\}$ of $\{\mu_{n_1(k)}\}$ such that $\mu^{(r_2)}_{n_2(k)}$ converges to some $\mu_{r_2}$ in $\mathbf{M}_b(\mathbf{S} \setminus B_{0,r_2})$. Continuing like this; for each $i \geqslant 3$ let $n_i(k)$ be a subsequence of $n_{i-1}(k)$ such that $\mu^{(r_i)}_{n_i(k)}$ converges to some $\mu_{r_i}$ in $\mathbf{M}_b(\mathbf{S} \setminus B_{0,r_i})$. Then the diagonal sequence $\{\mu_{n_k(k)}\}$ satisfies $\mu^{(r_i)}_{n_k(k)} \to \mu_{r_i}$ in $\mathbf{M}_b(\mathbf{S} \setminus B_{0,r_i})$ for each $i \geqslant 1$. Take $f \in \mathscr{C}_0(\mathbf{S})$. There exists some $i_0 \geqslant 1$ such that$f$ vanishes on $B_{0,r_i}$ for each $i \geqslant i_0$. In particular $f \in \mathscr{C}_b(\mathbf{S} \setminus B_{0,r_i})$ for each $i \geqslant i_0$ and

$$\int f \, d\mu_{r_i} = \lim_k \int f \, d\mu^{(r_i)}_{n_k(k)} = \lim_k \int f \, d\mu^{(r_{i_0})}_{n_k(k)} = \int f \, d\mu_{r_{i_0}}.$$

Hence, we can define $\mu' : \mathscr{C}_0(\mathbf{S}) \to [0,\infty]$ by $\mu'(f) = \lim_{i\to\infty} \int f \, d\mu_{r_i}$. This $\mu'$ induces a measure $\mu$ in $\mathbf{M}_0$. Indeed, for $A \in \mathscr{S}_0$ we can find a sequence $f_n \in \mathscr{C}_0(\mathbf{S})$ such that $0 \leqslant f_n \uparrow I_A$ and put $\mu(A) = \lim_n \mu'(f_n)$. If $A \cap B_{0,r} = \emptyset$ for some $r > 0$, then exists $f_n \in \mathscr{C}_0(\mathbf{S})$ such that $f_n \downarrow I_A$ and hence $\mu(A) \leqslant \mu'(f_n) < \infty$. Thus, $\mu$ is finite on sets $A$ with $s_0 \notin A^-$. To show $\mu$ is countably additive, let $A_1, A_2, \ldots$ be disjoint sets in $\mathscr{S}_0$ and $0 \leqslant f_{nk} \uparrow I_{A_k}$ for each $k$. Then $\sum_k f_{nk} \uparrow I_{\cup_k A_k}$ and we have by Fubini's theorem and the monotone convergence theorem that

$$\mu(\cup_k A_k) = \lim_n \mu'\biggl(\sum_k f_{nk}\biggr) = \sum_k \lim_n \mu'(f_{nk}) = \sum_k \mu(A_k).$$

By construction $\int f \, d\mu = \mu'(f)$ for each $f \in \mathscr{C}_0(\mathbf{S})$. Hence, $\int f \, d\mu_{n_k(k)} \to \int f \, d\mu$ for each $f \in \mathscr{C}_0(\mathbf{S})$ and we conclude that $M$ is relatively compact in $\mathbf{M}_0$. □

Proof of Theorem 2.7. Suppose $M \subset \mathbf{M}_0$ is relatively compact. By Theorem 2.6, there exists a sequence $\{r_i\}$ with $r_i \downarrow 0$ such that $M^{(r_i)} \subset \mathbf{M}_b(\mathbf{S} \setminus B_{0,r_i})$ is relatively compact for each $r_i$. Prohorov's theorem (Theorem A2.4.1 in [**7**]) implies that (2.2) and (2.3) hold.

Conversely, suppose there exists a sequence $\{r_i\}$ with $r_i \downarrow 0$ such that (2.2) and (2.3) hold. Then, by Prohorov's theorem, $M^{(r_i)} \subset \mathbf{M}_b(\mathbf{S} \setminus B_{0,r_i})$ is relatively compact for each $i$. By Theorem 2.6, $M \subset \mathbf{M}_0$ is relatively compact. □

Proof of Theorem 2.8. Take an open set $G \in \mathscr{S}_0$. For each $x \in G$ there exists a set $A_x \in \mathscr{A}$ such that $x \in A_x^\circ \subset A_x \subset G$. Since $\mathbf{S}$ is separable, there exists a countable subcollection $\{A^\circ_{x_i}\}$ of $\{A^\circ_x, x \in G\}$ that covers $G$. Hence $G \subset \cup_i A^\circ_{x_i}$. Since $A_x \subset G$ we also have $\cup_i A_{x_i} \subset G$. Hence $G = \cup_i A_{x_i}$. It follows that the $\pi$-system $\mathscr{A}$ generates the open sets and hence the Borel $\sigma$-algebra. By the uniqueness theorem $\mu = \nu$ on $\mathscr{S}_0$. □

For the proof of Theorem 3.1 the following simple observation will be helpful.

Lemma 5.2. *Let $\mu \in \mathbf{M}_0$. Let $E_1 = \mathbf{S} \setminus B_{0,1}$, $rE_1 = \{rx : x \in E_1\}$ for $r > 0$, and $R_\mu = \{r > 0 : \mu(\partial rE_1) = 0\}$. Then $(0,\infty) \setminus R_\mu$ is at most countable and for some $r_0 \in R_\mu$ it holds that $\mu(rE_1) > 0$ for $r \in (0, r_0)$.*

Proof. Since $\mu \in \mathbf{M}_0$ it holds that $\mu(rE_1) < \infty$ for all $r > 0$. In addition $\partial[rE_1] \cap \partial[r'E_1] = \emptyset$ for $r \neq r'$. Hence, $\mu(\partial[rE_1]) = 0$ for all but at most countably many $r > 0$. Since $\mu$ is nonzero $\mu(rE_1) > 0$ for $r \in (0, r_0)$ for some $r_0 > 0$ and we may choose $r_0 \in R_\mu$. □

PROOF OF THEOREM 3.1(A). We show this statement under the assumption that (iii) holds. This is sufficient since in the proof of statement (b), below, it is shown that the limiting measures are the same up to a constant factor.

Suppose that (iii) holds and set $E_1 = \mathbf{S} \setminus B_{0,1}$. By Lemma 5.2, $\mu(\partial[rE_1]) = 0$ for all $r \in R_\mu \subset (0,\infty)$, where $(0,\infty) \setminus R_\mu$ is at most countable. Moreover, $\mu(rE_1) > 0$ for $r \in (0, r_0)$ for some $r_0 \in R_\mu$. Hence, for $\lambda$ in a set of positive measure, e.g. $(1/2, 1)$,

$$\frac{\nu(t\lambda[r_0E_1])}{\nu(t[r_0E_1])} = \frac{\nu(tB)}{\nu(t[r_0E_1])}\frac{\nu(t\lambda[r_0E_1])}{\nu(tB)} \to \frac{\mu(\lambda[r_0E_1])}{\mu(r_0E_1)} \in (0,\infty)$$

as $t \to \infty$. Hence, by Theorem 1.4.1 in [**6**], $t \mapsto \nu(t[r_0E_1])$ is regularly varying and

$$\lim_{t\to\infty} \frac{\nu(t\lambda E_1)}{\nu(tE_1)} = \lim_{t\to\infty} \frac{\nu(t\lambda[r_0E_1])}{\nu(t[r_0E_1])} = \lambda^{-\alpha}$$

for some $\alpha > 0$. In particular, $\mu(\partial[\lambda E_1]) = 0$ for all $\lambda > 0$ and $\nu(tE_1)^{-1}\nu(t\cdot) \to \mu(E_1)^{-1}\mu(\cdot)$ in $\mathbf{M}_0$ as $t \to \infty$. Moreover, if $A \in \mathscr{S}$ with $0 \notin A^-$ and $\mu(\partial A) = 0$, then for any $\lambda > 0$,

$$\frac{\nu(t\lambda A)}{\nu(tE_1)} = \frac{\nu(t\lambda A)}{\nu(t\lambda E_1)}\frac{\nu(t\lambda E_1)}{\nu(tE_1)} \to \lambda^{-\alpha}\frac{\mu(A)}{\mu(E_1)}$$

as $t \to \infty$. Hence, $\mu(\lambda A) = \lambda^{-\alpha}\mu(A)$ for such set $A$ and all $\lambda > 0$. Since these sets $A$ form a $\pi$-system that generate the $\sigma$-algebra $\mathscr{S}_0$, $\mu(\lambda A) = \lambda^{-\alpha}\mu(A)$ for all $A \in \mathscr{S}_0$ and $\lambda > 0$. □

In the proof of Theorem 3.1(b) we will use a particular determining class. For $A \subset \mathbf{S}$, let $S(A) = \{sx : s \geqslant 1,\ x \in A\}$ and let $\widetilde{\mathscr{A}} = \{A \in \mathscr{S} : A = S(A),\ 0 \notin A^-\}$. Given a measure $\mu \in \mathbf{M}_0$ we write $\widetilde{\mathscr{A}_\mu}$ for the class of sets $\widetilde{A} \in \widetilde{\mathscr{A}}$ with $\mu(\partial\widetilde{A}) = 0$.

LEMMA 5.3. *Take $A \subset \mathbf{S}$ with $0 \notin A^-$. If $A$ is open* (*closed*), *then $S(A)$ is open* (*closed*).

PROOF. Assume that $A$ is open. Since $(\lambda, x) \mapsto \lambda x$ is continuous by assumption, the map $f_\lambda$ given by $f_\lambda(x) = \lambda x$ is continuous. Moreover, $S(A) = \cup_{\lambda\geqslant 1}\lambda A = \cup_{\lambda\geqslant 1} f_{1/\lambda}^{-1}(A)$. Hence, $S(A)$ is open.

Assume that $A$ is closed. Take $y_n \in S(A)$ with $y_n \to y$ for some $y \in \mathbf{S}$. Write $y_n = s_nx_n$, where $s_n \geqslant 1$ and $x_n \in A$. If $s_n \to \infty$, then $x_n \notin A$ for $n$ sufficiently large (recall that $0 \notin A$). Hence, $\{s_n\}$ has an accumulation point $s \in [1,\infty)$ so that $s_{n'} \to s$ for some subsequence $\{s_{n'}\}$. Hence, $x_{n'} \to x$ for some $x \in A$ and it follows that $y_{n'} \to y'$ for some $y' \in S(A)$. Since $y_n \to y$ we must have $y = y'$. □

LEMMA 5.4. *Let $\mu, \nu \in \mathbf{M}_0$. If $\mu(\widetilde{A}) = \nu(\widetilde{A})$ for each $\widetilde{A} \in \widetilde{\mathscr{A}_\mu}$, then $\mu = \nu$.*

PROOF. If $\mu$ and $\nu$ coincide on $\widetilde{\mathscr{A}_\mu}$ then they coincide on the $\pi$-system of finite differences of sets in $\widetilde{\mathscr{A}_\mu}$. By Theorem 2.8 it is sufficient to show that for each $x \in \mathbf{S}_0$ and $\varepsilon > 0$ there exists a set $D$ of the form $D = \widetilde{A}_1 \setminus \widetilde{A}_2$ with $\widetilde{A}_1, \widetilde{A}_2 \in \widetilde{\mathscr{A}_\mu}$ such that $x \in D^\circ \subset D \subset B_{x,\varepsilon}$. Take $\delta \in (0,\varepsilon]$ such that $0 \notin B_{x,\delta}^-$, $\mu(\partial B_{x,\delta}) = 0$, and

$\mu(\partial S(B_{x,\delta})) = 0$. Then $S(B_{x,\delta}) \setminus B_{x,\delta} \in \widetilde{\mathscr{A}_\mu}$ and hence we may choose $\widetilde{A}_1 = \widetilde{B}_{x,\delta}$ and $\widetilde{A}_2 = S(B_{x,\delta}) \setminus B_{x,\delta}$. $\square$

Proof of Theorem 3.1(b). We show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) and that (iv) $\Leftrightarrow$ (v).

Suppose that (i) holds and set $c(t) = c_{[t]}$. For each $\widetilde{A} \in \widetilde{\mathscr{A}}$ (recall the definition of $\widetilde{\mathscr{A}}$ before Lemma 5.4) and $t \geqslant 1$ it holds that

$$\frac{c_{[t]}}{c_{[t]+1}} c_{[t]+1}\nu(([t]+1)\widetilde{A}) \leqslant c(t)\nu(t\widetilde{A}) \leqslant c_{[t]}\nu([t]\widetilde{A}). \tag{5.1}$$

Since $\{c_n\}_{n\geqslant 1}$ is regularly varying it holds that $\lim_{n\to\infty} c_n/c_{n+1} = 1$. Hence, $\lim_{t\to\infty} c(t)\nu(t\widetilde{A}) = \mu(\widetilde{A})$ for all $\widetilde{A} \in \widetilde{\mathscr{A}_\mu}$. By Lemma 5.4 statement (ii) follows if we show that $\{c(t)\nu(t\cdot) : t > 0\}$ is relatively compact in $\mathbf{M}_0$. Indeed, Lemma 5.4 implies that all subsequential limits coincide. We know from (i) that $\{c_n\nu(n\cdot) : n \geqslant 1\}$ is relatively compact in $\mathbf{M}_0$ and by Theorem 2.7 there is a sequence $r_i \downarrow 0$ such that for each $i$, $\sup_n c(n)\nu(n[\mathbf{S} \setminus B_{0,r_i}]) < \infty$ and given $\eta > 0$ there is a compact set $C_i \subset \mathbf{S} \setminus B_{0,r_i}$ such that

$$\sup_n c(n)\nu(n[\mathbf{S} \setminus (B_{0,r_i} \cup C_i)]) \leqslant \eta. \tag{5.2}$$

Since $\mathbf{S} \setminus B_{0,r} \in \widetilde{\mathscr{A}}$, (5.1) implies that $\sup_t c(t)\nu(t[\mathbf{S} \setminus B_{0,r_i}]) < \infty$. Moreover, (5.2) holds for any compact set $C_i'$ with $C_i \subset C_i' \subset \mathbf{S} \setminus B_{0,r_i}$. We claim that we can choose this $C_i'$ such that $\mathbf{S} \setminus (B_{0,r_i} \cup C_i') \in \widetilde{\mathscr{A}}$. Then (5.1) implies that $\sup_t c(t)\nu(t[\mathbf{S} \setminus (B_{0,r} \cup C_i')]) \leqslant \eta$ and by Theorem 2.7 $\{c(t)\nu(t\cdot) : t > 0\}$ is relatively compact. We now show that it is possible to choose such a set $C_i'$. Take $C_i' = \{sx : s \in [0,1], x \in C_i\} \cap (\mathbf{S} \setminus B_{0,r_i})$. The set $\{sx : s \in [0,1], x \in C_i\}$ is compact because it is the image of the compact set $[0,1] \times C_i \subset \mathbf{R}_+ \times \mathbf{S}$ under the continuous map from $\mathbf{R}_+ \times \mathbf{S}$ to $\mathbf{S}$ given by $(\lambda, x) \mapsto \lambda x$. Since $C_i'$ is the intersection of a compact set and a closed set it is compact. Clearly, $C_i \subset C_i'$ and $\mathbf{S} \setminus (B_{0,r_i} \cup C_i') \in \widetilde{\mathscr{A}}$. This completes the proof of (ii).

Suppose that (ii) holds. By Lemma 5.2 there exists $r > 0$ such that $\mu(\partial[rE_1]) = 0$ and $\mu(rE_1) > 0$. For $t > 0$ and $A \in \mathscr{S}$ with $0 \notin A^-$ and $\mu(\partial A) = 0$,

$$\frac{\nu(tA)}{\nu(t[rE_1])} = \frac{c(t)\nu(tA)}{c(t)\nu(t[rE_1])} \to \frac{\mu(A)}{\mu(rE_1)}$$

as $t \to \infty$. Hence, by Theorem 2.4 (ii), (iii) holds.

The implication (iii) $\Rightarrow$ (iv) is shown above (proof of statement (a)).

Suppose that (iv) holds. Lemma 5.2 implies that for $r$ in a subset of $(0,\infty)$ of positive measure, $\lim_{t\to\infty} \nu(trE_1)/\nu(tE_1)$ exists and is positive. Hence, by Theorem 1.4.1 in [**6**], $c(t) = \nu(tE_1)^{-1}$ is regularly varying with some index $\alpha > 0$. Hence, the sequence $c_n = \nu(nE_1)^{-1}$ is regularly varying with index $\alpha > 0$ and $c_n\nu(n\cdot) \to \mu(\cdot)$ in $\mathbf{M}_0$, i.e. (i) holds.

Suppose that (iv) holds. As above, this implies that $t \mapsto \nu(tE_1)^{-1}$ is regularly varying with some index $\alpha > 0$. By Theorem 1.5.12 in [**6**] there exists a sequence $\{a_n\}$, which is regularly varying with index $1/\alpha$, such that $\lim_{n\to\infty} n/\nu(a_nE_1) = 1$. Hence, (v) holds.

Suppose that (v) holds. Take $r > 0$ such that $\mu(\partial[rE_1]) = 0$ and $\mu(rE_1) > 0$. For $t > a_1$, let $k = k(t)$ be the largest integer with $a_k \leqslant t$. Then $a_k \leqslant t < a_{k+1}$ and $k \to \infty$ as $t \to \infty$. Hence, for $\widetilde{A} \in \widetilde{\mathscr{A}}$,

$$\frac{k}{k+1}\frac{(k+1)\nu(a_{k+1}\widetilde{A})}{k\nu(a_k[rE_1])} \leqslant \frac{\nu(t\widetilde{A})}{\nu(t[rE_1])} \leqslant \frac{k+1}{k}\frac{k\nu(a_k\widetilde{A})}{(k+1)\nu(a_{k+1}[rE_1])}$$

from which it follows that $\lim_{t\to\infty} \nu(t\widetilde{A})/\nu(t[rE_1]) = \mu(\widetilde{A})/\mu(rE_1)$. To show that $\{\nu(t\cdot)/\nu(t[rE_1]) : t > 0\}$ is relatively compact we can apply the same argument as in the proof of the implication (i) $\Rightarrow$ (ii). Hence, (iii) holds. $\square$

Proof of Theorem 4.1. By Theorem 3.1(a), the measure $\mu$ has the scaling property $\mu(\lambda A) = \lambda^{-\alpha}\mu(A)$ for $\lambda > 0$ and $A \in \mathscr{S}_0$. It follows that $\mu(\partial[\mathbf{R}^d \setminus B_{0,u}]) = 0$ for each $u > 0$ and therefore $\lim_{n\to\infty} c_n\nu(n[\mathbf{R}^d \setminus B_{0,u}]) = \mu(\mathbf{R}^d \setminus B_{0,u})$ for each $u > 0$. Hence, $\{c_n\nu(n\cdot)\}$ is relatively compact in $\mathbf{M}_0$. By assumption all subsequential limits agree on sets $V_{u,S}$ with $\mu(\partial V_{u,S}) = 0$, and by Theorem 2.8 the finite differences of such sets form a determining class. Hence, all subsequential limits coincide and the proof is complete. $\square$

Proof of Theorem 4.2. Similar to the proof of Theorem 4.1. $\square$

Proof of Theorem 4.3. Let $r_i \downarrow 0$. If (4.1)–(4.3) hold for $M$ then they hold for each $M^{(r_i)}$. By [**5**, Theorem 7.3 p. 82] $M^{(r_i)}$ is relatively compact in $\mathbf{M}_b(\mathbf{C} \setminus B_{0,r_i})$ which by Theorem 2.6 implies that $M$ is relatively compact.

Conversely, suppose that $M$ is relatively compact in $\mathbf{M}_0(\mathbf{C})$. By Theorem 2.7 there exists a sequence $r_i \downarrow 0$ such that (4.1) holds for $r = r_i$ and given $\eta > 0$ there exists a compact set $C_i \subset \mathbf{C} \setminus B_{0,r_i}$ such that $\sup_{\mu\in M} \mu(\mathbf{C} \setminus (C_i \cup B_{0,r_i})) < \eta$. Choose such $C_i$. Given $\varepsilon > 0$, by the Arzelà–Ascoli theorem there exist $R_i$ and $\delta_i$ such that $C_i \subset \{x : |x(0)| \leqslant R_i\}$ and $C_i \subset \{x : w_x(\delta_i) \leqslant \varepsilon\}$. We may assume that $R_1 > r_1$ (otherwise $C_1 = \emptyset$) and since $\{x : |x(0)| > R_1\} \subset \{x : |x|_\infty > r_1\}$ we have

$$\sup_{\mu\in M} \mu(x : |x(0)| > R_1) = \sup_{\mu\in M} \mu(x : |x|_\infty > r_1, |x(0)| > R_1) < \eta.$$

Since $\eta$ was arbitrary(4.2) follows. Finally, let $i_0$ be an integer such that $r_{i_0} < \varepsilon/2$. Then $\{x : w_x(\delta_{i_0}) > \varepsilon\} \subset \{x : |x|_\infty > r_{i_0}\}$ and hence

$$\sup_{\mu\in M} \mu(x : w_x(\delta_{i_0}) > \varepsilon) = \sup_{\mu\in M} \mu(x : w_x(\delta_{i_0}) > \varepsilon, |x|_\infty > r_{i_0}) < \eta.$$

Since $\eta$ was arbitrary (4.3) holds. $\square$

Proof of Theorem 4.4. By (4.4) it follows in particular that $c_n\nu\pi_0^{-1}(n\cdot) \to \mu\pi_0^{-1}(\cdot)$ in $\mathbf{M}_0(\mathbf{R}^d)$. Hence $\{c_n\nu\pi_0^{-1}(n\cdot)\}$ is relatively compact and (4.2) holds. In addition, the conditions (4.5)–(4.6) imply by Theorem 4.3 that $\{c_n\nu(n\cdot)\}$ is relatively compact. Let $\mu$ and $\mu'$ be subsequential limits. By (4.4) $\mu = \mu'$ on the $\pi$-system of sets $\pi_{t_1,\ldots,t_k}^{-1}(H)$, for Borel sets $H \subset \mathbf{R}^{dk}$. Hence, $\mu = \mu'$ and $c_n\nu(n\cdot) \to \mu(\cdot)$ in $\mathbf{M}_0$.

Conversely, if $c_n\nu(n\cdot) \to \mu(\cdot)$ in $\mathbf{M}_0$, then (4.4) follows from the Mapping Theorem (Theorem 2.5). Since $\{c_n\nu(n\cdot)\}$ is relatively compact (4.5)–(4.6) hold by Theorem 4.3. $\square$

Proof of Theorem 4.5. Let $r_i \downarrow 0$. If (4.7)–(4.11) hold then they hold they hold for each $M^{(r_i)}$. By [**5**, Theorem 13.2 p. 139 and (13.8) p. 141] $M^{(r_i)}$ is relatively compact in $\mathbf{M}_b(\mathbf{D} \setminus B_{0,r_i})$ which by Theorem 2.6 implies that $M$ is relatively compact.

Conversely, suppose that $M$ is relatively compact in $\mathbf{M}_0(\mathbf{D})$. By Theorem 2.7 there exists a sequence $r_i \downarrow 0$ such that (4.7) holds and given $\eta > 0$ there exists a compact set $C_i \subset \mathbf{D} \setminus B_{0,r_i}$ such that $\sup_{\mu \in M} \mu(\mathbf{D} \setminus (C_i \cup B_{0,r_i})) < \eta$. Choose such $C_i$. As in the proof of Theorem 4.3 it follows that (4.8) holds. Given $\varepsilon > 0$, by Theorem 13.2 in [**5**, p. 139 and (13.8) p. 141] there exist $\delta_i$ such that $C_i \subset \{x : w''_x(\delta_i) \leqslant \varepsilon\}$, $C_i \subset \{x : w_x([0, \delta_i)) \leqslant \varepsilon\}$, and $C_i \subset \{x : w_x([1-\delta_i, 1)) \leqslant \varepsilon\}$. Let $i_0$ be an integer such that $r_{i_0} < \varepsilon/2$. Then

$$\begin{aligned} \{x : w''_x(\delta_{i_0}) > \varepsilon\} &\subset \{x : |x|_\infty > r_{i_0}\} \\ \{x : w_x([0, \delta_{i_0})) > \varepsilon\} &\subset \{x : |x|_\infty > r_{i_0}\} \\ \{x : w_x([1-\delta_i, 1)) > \varepsilon\} &\subset \{x : |x|_\infty > r_{i_0}\}. \end{aligned}$$

and hence

$$\begin{aligned} \sup_{\mu \in M} \mu(x : w''_x(\delta_{i_0}) > \varepsilon) &= \sup_{\mu \in M} \mu(x : w''_x(\delta_{i_0}) > \varepsilon, |x|_\infty > r_{i_0}) < \eta, \\ \sup_{\mu \in M} \mu(x : w_x([0, \delta_{i_0})) > \varepsilon) &= \sup_{\mu \in M} \mu(x : w_x([0, \delta_{i_0})) > \varepsilon, |x|_\infty > r_{i_0}) < \eta, \\ \sup_{\mu \in M} \mu(x : w_x((1-\delta_{i_0}, 1)) > \varepsilon) &= \sup_{\mu \in M} \mu(x : w_x((1-\delta_{i_0}, 1)) > \varepsilon, |x|_\infty > r_{i_0}) < \eta. \end{aligned}$$

Since $\eta$ was arbitrary (4.9)–(4.11) hold. □

Proof of Theorem 4.6. Suppose $\nu$ satisfies (4.12)–(4.15). We will use Theorem 4.5 to show that $\{c_n\nu(n\cdot)\}$ is relatively compact in $\mathbf{M}_0(\mathbf{D})$. For this we only need to check (4.8). Take $\eta > 0$. For any $R > 0$ and $0 = t_0 < t_1 < \cdots < t_k = 1$ in $T$ such that $t_i - t_{i-1} < \delta$ where $\delta > 0$

$$\begin{aligned} c_n\nu(x : |x|_\infty > nR) &\leqslant c_n\nu\Big(x : \max_{1 \leqslant i \leqslant k} |x(t_i)| > nR/2\Big) \\ &\quad + c_n\nu\Big(x : \max_{1 \leqslant i \leqslant k} |x(t_i)| \leqslant nR/2, \max_{1 \leqslant i \leqslant k} \sup_{t_{i-1} \leqslant t \leqslant t_i} |x(t)| > R\Big) \\ &\leqslant c_n\nu\Big(x : \max_{1 \leqslant i \leqslant k} |x(t_i)| > nR/2\Big) + c_n\nu\big(x : w''(x, \delta) > nR/2\big) \end{aligned}$$

By (4.12) $\{c_n\nu\pi^{-1}_{t_1,\ldots,t_k}(n\cdot)\}$ is relatively compact and hence we may choose $R$ big enough such that $\sup_n c_n\nu(x : \max_{1 \leqslant i \leqslant k} |x(t_i)| > nR/2) < \eta/2$. By (4.13) we may choose $\delta > 0$ such that $\sup_n c_n\nu(x : w''(x, \delta) > nR/2) < \eta/2$. Hence (4.8) holds. It follows that $\{c_n\nu(n\cdot)\}$ is relatively compact in $\mathbf{M}_0$.

Let $\mu$ and $\mu'$ be subsequential limits. We will show $\mu = \mu'$. Let $T_\mu$ ($T_{\mu'}$) consist of the points where the projection $\pi_t$ is continuous except at a set of $\mu$-measure ($\mu'$-measure) zero (see [**5**, p. 138]). By (4.12) we have $\mu\pi^{-1}_{t_1,\ldots,t_k} = \mu'\pi^{-1}_{t_1,\ldots,t_k}$ for $t_1, \ldots, t_k \in T \cap T_\mu \cap T_{\mu'}$. Since $0, 1 \in T \cap T_\mu \cap T_{\mu'}$ and this set is dense in $[0, 1]$ it follows by Theorem 13.1 in [**5**] that $\mu = \mu'$.

Conversely, suppose that $c_n\nu(n\cdot) \to \mu(\cdot)$ in $\mathbf{M}_0$. By the mapping theorem (Theorem 2.5) (4.12) holds for $t_1, \dots, t_k \in T_\mu$ and since $\{c_n\nu(n\cdot)\}$ is relatively compact (4.13)–(4.15) hold. $\square$


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## References


[1] A. Araujo and E. Giné, *The Central Limit Theorem for Real and Banach Valued Random Variables*, Wiley, New York, 1980.

[2] M. Barczy and G. Pap, *Portmanteau theorem for unbounded measures*, Statist. Probab. Lett. **76** (2006), 1831–1835.

[3] B. Basrak, *The Sample Autocorrelation Function of Non-Linear Time Series*, PhD Thesis, University of Groningen, Department of Mathematics, 2000.

[4] P. Billingsley, *Probability and Measures*, 3rd edition, Wiley, New York, 1995.

[5] P. Billingsley, *Convergence of Probability Measures*, 2nd edition, Wiley, New York, 1999.

[6] N. Bingham, C. Goldie and J. Teugels, *Regular Variation*, Cambridge University Press, Cambridge, 1987.

[7] D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes*, Springer-Verlag, New York, 1988.

[8] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. 2, 2nd edition, Wiley, New York, 1971.

[9] A. Friedman, *Foundations of Modern Analysis*, Dover Pulications, New York, 1982.

[10] E. Giné, *Central limit theorems in Banach spaces: A survey*, in *Lecture Notes in Math.* 860, 1981, 138–152.

[11] E. Giné, M. G. Hahn and P. Vatan, *Max-infinitely divisible and max-stable sample continuous processes*, Probab. Theory Related Fields. **87** (1990), 139–165.

[12] L. de Haan and T. Lin, *On convergence toward an extreme value distribution in* $C[0,1]$, Ann. Probab. **29** (2001), 467–483.

[13] H. Hult and F. Lindskog, *Extremal behavior of regularly varying stochastic processes*, Stochast. Process. Appl. **115** (2005), 249–274.

[14] H. Hult, F. Lindskog, T. Mikosch and G. Samorodnitsky, *Functional large deviations for multivariate regularly varying random walks*, Ann. Appl. Probab. **15** (2005), 2651–2680.

[15] H. Hult and F. Lindskog, *Extremal behavior of stochastic integrals driven by regularly varying Lévy processes*, to appear in Ann. Probab.

[16] H. Kesten, *Random difference equations and renewal theory for products of random matrices*, Acta Math. **131** (1973), 207–248.

[17] O. Kallenberg, *Random Measures*, 3rd edition, Akademie-Verlag, Berlin, 1983.

[18] M. M. Meerschaert, *Regular variation and domains of attraction in* $\mathbf{R}^k$, Stat. Probab. Lett. **4** (1986), 43–45.

[19] M. M. Meerschaert and H.-P. Scheffler, *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice*, Wiley, New York, 2001.

[20] S. Resnick, *Extreme Values, Regular Variation, and Point Processes*, Springer-Verlag, New York, 1987.

[21] S. Resnick, *On the foundations of multivariate heavy-tail analysis*, J. Appl. Probab. **41A** (2004), 191–212.

[22] E. L. Rvačeva, *On domains of attraction of multi-dimensional distributions*, Sel. Transl. Math. Stat. Probab. **2** (1962), 183–205.

[23] W. Whitt, *Some useful functions for functional limit theorems*, Math. Oper. Res. **5** (1980), 67–85.

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