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ON THE OPERATOR EQUATIONS $ABA = A^2$ AND $BAB = B^2$

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ABSTRACT. We generalize a result of I. Vidav concerning the operator equations $ABA = A^2$ and $BAB = B^2$.

1. Introduction

In [**7**] I. Vidav proved the following result:

THEOREM 1.1. *Let H be a complex Hilbert space and let A and B be bounded linear operators on H. Then the following assertions are equivalent:*

(a) *There is a uniquely determined bounded linear operator P on H such that $P^2 = P$ and $A = PP^*$ and $B = P^*P$.*
(b) *A and B are selfadjoint and satisfy the relations $ABA = A^2$ and $BAB = B^2$.*

Vidav gave two proofs of Theorem 1.1; the first proof is geometrical and the second one is algebraic. In [**6**] Rakočević gave another proof of Theorem 1.1.

The aim of this paper is to prove a result, which implies Theorem 1.1. Section 2 deals with Drazin invertible elements of rings. In Section 3 we consider bounded linear operators on Banach spaces. Operators on Hilbert spaces are considered in Section 4, where we will give a proof of Theorem 1.1. In the final section we investigate several special classes of operators.

2. Drazin inverses in rings

In this section $\mathcal{R}$ denotes an associative ring. An element $A \in \mathcal{R}$ is said to be *Drazin invertible* if there exists $C \in \mathcal{R}$ such that

(1) $A^m = A^{m+1}C$ for some integer $m \geqslant 0$,
(2) $C = AC^2$
(3) $AC = CA$.

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In this case C is called a *Drazin inverse* of A and the smallest integer $m \geqslant 0$ in (1) is called the *index* $i(A)$ of A.

If $\mathcal{R}$ has a neutral element I and if we define $A^0 = I$, then (1), (2) and (3) hold with $m = 0$ if and only if A is invertible.

PROPOSITION 2.1. *If $A \in \mathcal{R}$ is Drazin invertible, then A has a unique Drazin inverse.*

PROOF. [**4**]. $\square$

Our main result in this section is:

THEOREM 2.2. (a) *If $P, Q \in \mathcal{R}$, $P^2 = P$, $Q^2 = Q$, $A = PQ$ and $B = QP$, then $ABA = A^2$ and $BAB = B^2$.*

(b) *Suppose that $A, B \in \mathcal{R}$ are Drazin invertible, $i(A) = i(B) = 1$, $ABA = A^2$ and $BAB = B^2$. Then there are $P, Q \in \mathcal{R}$ such that $P^2 = P$, $Q^2 = Q$, $A = PQ$ and $B = QP$.*

PROOF. (a) We have $ABA = PQ^2P^2Q = (PQ)^2 = A^2$ and $BAB = QP^2Q^2P = (QP)^2 = B^2$.

(b) Since $i(A) = i(B) = 1$, there are $C, D \in \mathcal{R}$ with

$$\begin{gathered} ACA = A, \quad CAC = C, \quad AC = CA \\ BDB = B, \quad DBD = D, \quad BD = DB. \end{gathered}$$

Let $P := CAB$, $Q := BAC$ and $R := DBA$. Then

$$\begin{aligned} P^2 &= CABCAB = C(ABA)CB = CA^2CB = ACACB = ACB = CAB = P, \\ R^2 &= DBADBA = D(BAB)DA = DB^2DA = BDBDA = BDA = DBA = R, \\ Q^2 &= BACBAC = BC(ABA)C = BCA^2C = BCACA = BCA = BAC = Q. \end{aligned}$$

Furthermore we have

$$\begin{aligned} PQ &= CABBAC = CAB^2AC = CA(BAB)AC = C(ABA)BAC \\ &= CA^2BAC = ACABAC = ABAC = A^2C = ACA = A, \\ RP &= DB(ACA)B = DBAB = DB^2 = BDB = B. \end{aligned}$$

It follows that

$$\begin{aligned} QP &= BACCAB = B(ACA)CB = BACB = BCAB \\ &= BP = (RP)P = RP^2 = RP = B. \end{aligned}$$

$\square$

3. Bounded linear operators

In this section X denotes a complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X. If $A \in \mathcal{L}(X)$, then $\sigma(A)$, $\rho(A)$ and $r(A)$ denote the spectrum, the resolvent set and the spectral radius of A, respectively. We write $N(A)$ for the kernel of A and $A(X)$ for the range of A. Define $p(A)$ [resp. $q(A)$], the *ascent* [resp. the *descent*] of A, to be the smallest integer $n \geqslant 0$ such that $N(A^{n+1}) = N(A^n)$ [resp. $A^{n+1}(X) = A^n(X)$] or ∞ if no such n exists. It follows from [**5**, Satz 72.3] that if $p(A)$ and $q(A)$ are both finite, then they are equal and, if $p = p(A) = q(A) < \infty$, then $X = N(A^p) \oplus A^p(X)$.

A Drazin invertible operator $A \in \mathcal{L}(X)$ with $i(A) \leqslant 1$ is called *simply polar*.

The following proposition tells us exactly which operators are Drazin invertible.

PROPOSITION 3.1. *For $A \in \mathcal{L}(X)$ and $n \geqslant 1$ the following assertions are equivalent:*

(a) *A is Drazin invertible and $i(A) = n$.*
(b) *$p(A) = q(A) = n$.*
(c) *The resolvent $(\lambda I - A)^{-1}$ has a pole of order n at $\lambda = 0$.*

PROOF. [**2**, Theorem 5.2], [**5**, Satz 101.2]. □

As an immediate consequence of Proposition 3.1 and Theorem 2.2 we get the main result of this section:

THEOREM 3.2. *Suppose that $A, B \in \mathcal{L}(X)$, $p(A) = q(A) = 1$ and $p(B) = q(B) = 1$. Then the following assertions are equivalent:*

(a) *There are $P, Q \in \mathcal{L}(X)$ such that $P^2 = P$, $Q^2 = Q$, $A = PQ$ and $B = QP$.*
(b) *$ABA = A^2$ and $BAB = B^2$.*

We use $\sigma_p(A)$, $\sigma_{ap}(A)$, $\sigma_r(A)$ and $\sigma_c(A)$ to denote the point, approximate point, residual and continuous spectrum of $A \in \mathcal{L}(X)$, respectively.

COROLLARY 3.3. *Suppose that $A, B \in \mathcal{L}(X)$, $p(A) = q(A) = p(B) = q(B) = 1$, $ABA = A^2$ and that $BAB = B^2$. Then:*

(a) *$\sigma(A) = \sigma(B)$;*
(b) *$\sigma_p(A) = \sigma_p(B)$;*
(c) *$\sigma_{ap}(A) = \sigma_{ap}(B)$;*
(d) *$\sigma_r(A) = \sigma_r(B)$;*
(e) *$\sigma_c(A) = \sigma_c(B)$.*

PROOF. Recall that $\sigma_p(A)$, $\sigma_r(A)$ and $\sigma_c(A)$ are pairwise disjoint and that their union is $\sigma(A)$. Thus (a) follows from (b), (d) and (e).

(b) Since $p(A) = p(B) > 0$, $0 \in \sigma_p(A)$ and $0 \in \sigma_p(B)$. From [**1**, Theorem 3] and Theorem 3.2 we get

$$\sigma_p(A) \setminus \{0\} = \sigma_p(PQ) \setminus \{0\} = \sigma_p(QP) \setminus \{0\} = \sigma_p(B) \setminus \{0\}$$

hence $\sigma_p(A) = \sigma_p(B)$.

(c) Because of $\sigma_p(A) \subseteq \sigma_{ap}(A)$ and $\sigma_p(B) \subseteq \sigma_{ap}(B)$, it follows that $0 \in \sigma_{ap}(A)$ and $0 \in \sigma_{ap}(B)$. As in the proof of (b) we see with Theorem 3 in [**1**] that $\sigma_{ap}(A) = \sigma_{ap}(B)$.

(d) Since $0 \in \sigma_p(A)$ and $0 \in \sigma_p(B)$, $0 \notin \sigma_r(A)$ and $0 \notin \sigma_r(A)$. Proceed as in the proof of (b), to obtain $\sigma_r(A) = \sigma_r(B)$.

(e) Similar. □

An operator $A \in \mathcal{L}(X)$ is called a *Fredholm operator* if $\dim N(A) < \infty$ and $\operatorname{codim} A(X) < \infty$. In this case we set $\operatorname{ind}(A) = \dim N(A) - \operatorname{codim} A(X)$.

By $\mathcal{F}(X)$ we denote the ideal of all finite dimensional operators in $\mathcal{L}(X)$. Let $\widehat{\mathcal{L}}$ denote the quotient algebra $\mathcal{L}(X)/\mathcal{F}(X)$ and write $\widehat{A}$ for the coset $A+\mathcal{F}(X)$ of $A \in \mathcal{L}(X)$ in $\widehat{\mathcal{L}}$. From [**5**, Satz 81.1] we have

$$A \text{ is a Fredholm operator } \iff \widehat{A} \text{ is invertible in } \widehat{\mathcal{L}}.$$

Corollary 3.4. *Let A and B as in Corollary 3.3 and $\lambda \in \mathbb{C}$. Then:*

$$\lambda I - A \textit{ is a Fredholm operator } \iff \lambda I - B \textit{ is a Fredholm operator.}$$

In this case $\operatorname{ind}(\lambda I - A) = \operatorname{ind}(\lambda I - B)$.

Proof. We first consider the case $\lambda = 0$. Let A be a Fredholm operator, thus $\widehat{A}$ is invertible in $\widehat{\mathcal{L}}$. From $\widehat{A}\widehat{B}\widehat{A} = \widehat{A}^2$ we obtain $\widehat{B} = \widehat{I}$, hence B is a Fredholm operator. Since $\widehat{B}\widehat{A}\widehat{B} = \widehat{B}^2$, it follows that $\widehat{A} = \widehat{I}$. Hence there are $F_1, F_2 \in \mathcal{F}(X)$ such that $A = I + F_1$ and $B = I + F_2$. By [**5**, Satz 81.3],

$$\operatorname{ind}(A) = \operatorname{ind}(I + F_1) = \operatorname{ind}(I) = 0 = \operatorname{ind}(I + F_2) = \operatorname{ind}(B).$$

Now assume that $\lambda \neq 0$. Our statements follow directly from [**1**, Theorem 6] and Theorem 3.2. □

4. Operators on Hilbert spaces

In this section we will give a proof of Theorem 1.1. H denotes a complex Hilbert space. If $A \in \mathcal{L}(H)$ we write $\operatorname{iso}\sigma(A)$ for the set of all isolated points of $\sigma(A)$.

Proposition 4.1. *Let $A \in \mathcal{L}(H)$ be normal and $0 \in \operatorname{iso}\sigma(A)$.*

(a) *0 is simple pole of the resolvent $(\lambda I - A)^{-1}$.*
(b) *$p(A) = q(A) = 1$.*
(c) *A is Drazin invertible and $i(A) = 1$.*

Proof. (a) follows from [**5**, Satz 112.2], (b) and (c) follow from Proposition 3.1. □

Theorem 4.2. *Let $A, B \in \mathcal{L}(H)$ be selfadjoint, $ABA = A^2$ and $BAB = B^2$.*

(a) *$0 \in \rho(A)$ or 0 is a simple pole of $(\lambda I - A)^{-1}$.*
(b) *$\sigma(A) \subseteq \{0\} \cup [1, \infty)$ (hence $A \geqslant 0$).*
(c) *A is Drazin invertible and $i(A) \leqslant 1$.*
(d) *If C is the Drazin inverse of A, then $C = C^*$ and $0 \leqslant C \leqslant I$*
(e) *If $A \neq 0$, then $\|A\| \geqslant 1$.*
(f) *If $\|A\| = 1$, then $A^2 = A = B$.*

Proof. (a) and (b): From

$$\begin{aligned} A(B-I)^2 A &= A(B^2 - 2B + I)A = AB^2A - 2ABA + A^2 \\ &= ABABA - 2A^2 + A^2 = A^2BA - A^2 \\ &= A^3 - A^2 \end{aligned}$$

it follows that $A^3 - A^2 = A(B-I)\,(A(B-I))^* \geqslant 0$, therefore $\sigma(A^3 - A^2) \subseteq [0, \infty)$. Now take $\lambda \in \sigma(A) \setminus \{0\}$. The spectral mapping theorem gives $\lambda^2(\lambda - 1) = \lambda^3 - \lambda^2 \geqslant 0$, thus $\lambda \geqslant 1$. This shows (b) and $0 \in \operatorname{iso}\sigma(A)$ or $0 \in \rho(A)$. Now use Proposition 4.1 to derive (a).

(c) follows from (a), (b) and Proposition 4.1.

(d) Because of $ACA = A$, $CAC = C$ and $AC = CA$ it follows that $AC^*A = A$, $C^*AC^* = C^*$ and $AC^* = C^*A$, hence C^* is a Drazin inverse of A. By Proposition 2.1, $C = C^*$. If $0 \in \rho(A)$, then $A = I$, thus $C = I$, hence $\|C\| = 1$. Now let $0 \in \sigma(A)$. In [**2**, page 53] it is shown that $r(C)^{-1} = \operatorname{dist}(0, \sigma(A) \setminus \{0\})$.

Now we see from (b) that $r(C)^{-1} \geqslant 1$, hence, since $C = C^*$, $\|C\| = r(C) \leqslant 1$. We denote the inner product on H by $(\cdot|\cdot)$. Take $x \in H$ and let $y = Cx$. Then

$$(Cx|x) = (CACx|x) = (ACx|Cx) = (Ay|y) \geqslant 0,$$

since $A \geqslant 0$. Thus $C \geqslant 0$. From $\|C\| \leqslant 1$ we obtain $0 \leqslant C \leqslant I$.

(e) If $A \neq 0$, we have $\|A\| = r(A) \geqslant 1$, by (b).

(f) If $\|A\| = 1$, then $r(A) = 1$, thus we obtain from (b) that $\sigma(A) \subseteq \{0, 1\}$. By the spectral mapping theorem, $\sigma(A^2 - A) = \{0\}$, hence $\|A^2 - A\| = r(A^2 - A) = 0$, this gives $A^2 = A$. Since $\sigma(A) = \sigma(B)$ (Corollary 3.3), we see that $\|B\| = r(B) = r(A) = \|A\| = 1$. Hence, by the same arguments as above, $B^2 = B$. It follows that $ABA = A$ and $BAB = B$, hence $(AB)^2 = AB$, thus AB is a projection $\neq 0$, therefore $\|AB\| \geqslant 1$. But $\|AB\| \leqslant \|A\|, \|B\| \leqslant 1$. Consequently $\|AB\| = 1$. From [**8**, Satz V.5.9] we derive that $AB = (AB)^*$. Hence $AB = BA$. We conclude that $A = ABA = BA^2 = BA = B^2A = BAB = B$. □

Proof of Theorem 1.1. Theorem 2.2 (a) shows that (a) implies (b). Now suppose that (b) is valid. If $0 \in \rho(A)$, then $A = B = I$ and we are done. Therefore we can assume that $0 \in \sigma(A)$ and $0 \in \sigma(B)$. By Theorem 4.2, A and B are Drazin invertible and $i(A) = i(B) = 1$. Let P and Q as in the proof of Theorem 2.2(b). Hence $P = CAB$, $Q = BAC$, $PQ = A$, $QP = B$ and C is the Drazin inverse of A. From Theorem 4.2 we get $C = C^*$, thus $P^* = BAC = Q$.

It remains to show that P is uniquely determined. Suppose that $R^2 = R$, $PP^* = RR^*$ and $P^*P = R^*R$. Then $P^*P(I-R) = R^*R(I-R) = R^*R - R^*R = 0$, thus $P(I-R)(X) \subseteq N(P^*) = P(X)^{\perp}$, hence $P(I-R) = 0$. Therefore we have $PR = P$. A similar argument gives $R^*P^* = R^*$. Taking adjoints we obtain $R = PR = P$. □

5. Examples and remarks

In this section we give some examples of operators A which are Drazin invertible with $i(A) = 1$. X always denotes a complex Banach space.

An operator $A \in \mathcal{L}(X)$ is called *hermitian* if $\|\exp(itA)\| = 1$ for all $t \in \mathbb{R}$.

Example 5.1. If $A \in \mathcal{L}(X)$ is hermitian and if $0 \in \operatorname{iso}\sigma(A)$, then A is Drazin invertible and $i(A) = 1$.

Proof. Let P_0 be the spectral projection associated with $\{0\}$. Let $M_0 = P_0(X)$ and $A_0 = A_{|M_0}$. By [**5**, Satz 100.1] we have $A(M_0) \subseteq M_0$ and $\sigma(A_0) = \{0\}$. Since A_0 is hermitian operator on M_0 [**3**, Proposition 4.12], we have $\|A_0\| = r(A_0) = 0$ [**3**, Theorem 4.10]. It follows that $AP_0 = 0$. Now [**5**, (101.9)] shows that 0 is a simple pole of $(\lambda I - A)^{-1}$. Proposition 3.1 completes the proof. □

An operator $A \in \mathcal{L}(X)$ is said to be *paranormal* if $\|Ax\|^2 \leqslant \|A^2x\|\,\|x\|$ for all $x \in X$.

EXAMPLE 5.2. If $A \in \mathcal{L}(X)$ is paranormal and if $0 \in \operatorname{iso}\sigma(A)$, then A is Drazin invertible and $i(A) = 1$.

PROOF. Let P_0, M_0 and A_0 as in the proof of 5.1. From [**5**, page 500] we get $\|A_0\| = r(A_0) = 0$. Now proceed as in the proof of 5.1. $\square$

A bounded linear operator A on a Hilbert space H is called *hyponormal* if $\|A^*x\| \leqslant \|Ax\|$ for all $x \in H$. Since hyponormal operators are paranormal, we have by Example 5.2:

EXAMPLE 5.3. If $A \in \mathcal{L}(H)$ is hyponormal and if $0 \in \operatorname{iso}\sigma(A)$, then A is Drazin invertible and $i(A) = 1$.

REMARK 5.4. If $A, B \in \mathcal{L}(X)$, $ABA = A^2$, $BAB = B^2$, $AB = BA$, $p(A) \leqslant 1$ and $p(B) \leqslant 1$, then $A^2 = A = B$.

PROOF. From $A^2 = A^2B = ABAB = AB^2 = B^2$ it follows that $A^3 = AB^2 = A^2$, thus $A^2(A - I) = 0$. Since $p(A) \leqslant 1$, we get $A(A - I) = 0$, hence $A^2 = A$. In the same way we derive $B^2 = B$. Consequently

$$B = B^2 = B(AB) = B(AB^2) = BA^2 = BA = AB = A^2B = A^2 = A. \qquad \square$$

REMARK 5.5. Suppose that $A, B \in \mathcal{L}(X)$ are paranormal, $ABA = A^2$, $BAB = B^2$ and $AB = BA$; then $A^2 = A = B$.

PROOF. Since $\|Ax\|^2 \leqslant \|A^2x\|\,\|x\|$ for $x \in X$, it follows that $p(A) \leqslant 1$. Similarly $p(B) \leqslant 1$. Now use 5.4. $\square$

REMARK 5.6. Suppose that H is a complex Hilbert space, $A, B \in \mathcal{L}(H)$ are normal, $ABA = A^2$, $BAB = B^2$ and $AB = BA$. Then A is selfadjoint and $A^2 = A = B$.

PROOF. Since normal operators are paranormal, it follows from 5.5 that A and B are normal projections, hence they are selfadjoint. $\square$

REMARK 5.7. If $A \in \mathcal{L}(X)$ is hermitian, then $p(A) \leqslant 1$.

PROOF. Let $x \in N(A^2)$ and $\|x\| = 1$. Then for $t \in \mathbb{R}$,

$$\begin{aligned} 1 = \|x\| &= \|\exp(-itA)\exp(itA)x\| \leqslant \|\exp(-itA)\|\,\|\exp(itA)x\| \\ &= \|\exp(itA)x\| \leqslant \|\exp(itA)\|\,\|x\| = \|x\| = 1, \end{aligned}$$

thus, since $A^n x = 0$ for $n \geqslant 2$,

$$1 = \|\exp(itA)x\| = \|x + itAx\|.$$

Therefore $|t|\,\|Ax\| - 1 \leqslant 1$ for all $t \in \mathbb{R}$. This gives $x \in N(A)$. $\square$

REMARK 5.8. Suppose that $A, B \in \mathcal{L}(X)$ are hermitian, $ABA = A^2$, $BAB = B^2$ and that $AB = BA$; then $A^2 = A = B$.

PROOF. 5.7 and 5.4. $\square$

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