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GENERAL KERNEL CONVOLUTIONS WITH SLOWLY VARYING FUNCTIONS

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Abstract. We prove a theorem concerning asymptotic behavior of general complex-valued kernel convolutions with slowly varying functions in the sense of Karamata. In applications we showed that the content of some classical theorems can be naturally extended on some parts of complex z-plane.

1. Introduction

A real-valued function $L(x)$ is slowly varying in the sense of Karamata if it is defined for $x > 0$, positive, measurable and satisfying

$$L(\lambda x) \sim L(x) \quad (x \to \infty) \tag{1}$$

for each $\lambda > 0$. Some examples of slowly varying functions $L(\cdot)$ (for $x > x_0$) are

$$\log^b x, \quad \log^c(\log x), \quad \exp(\log^d x), \quad \exp\left(\frac{\log x}{\log\log x}\right); \quad b, c \in R, \quad 0 < d < 1.$$

For $0 < x \leqslant x_0$ we can take $L(x) := 1$.

There is a plenty of papers investigating convolutions with slowly varying functions and various real-valued kernels. As an example for later analysis we quote Karamata's Theorem for Laplace Transform

Theorem 1. *Assume $U(\cdot) \geqslant 0$, $\rho > -1$, $\hat{U}(s) := s\int_0^\infty e^{-sx}U(x)dx$ convergent for $s > 0$, and $\ell(\cdot)$ slowly varying. Then*

$$U(x) \sim x^\rho \ell(x) \ (x \to \infty) \quad \textit{implies} \quad \hat{U}(s) \sim \frac{\Gamma(\rho+1)}{s^\rho}\ell(1/s) \ (s \downarrow 0).$$

Converse statement also holds under some Tauberian conditions (see [2, p. 43]). We shall also mention here results of Karamata and Bojanić [4] on improper integrals, Aranđelović [1] on Mellin convolutions and a general one of Vuilleumier [3].

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For similar assertions in complex z-plane, it is possible to define slowly varying function as a holomorphic one in some angle of z-plane but with additional condition of almost uniform convergence in (1) (see [2, pp. 424–425]).

In this paper we obtain asymptotic behavior of general kernel convolutions with slowly varying functions in some domains D of complex z-plane, without altering its classical definition (1).

2. Results

We shall prove the following theorem.

THEOREM 2. *Define* $B_\rho(z) := \int_0^\infty t^\rho |K(t,z)|\,dt$ *for a complex-valued kernel* $K(t,z)$. *Suppose that for some positive constants* a, A, *there is a region* D *in complex* z*-plane such that* $B_\rho(z)$ *exists for* $-a \leqslant \rho \leqslant 1$ *and, for* $z \in D$:

$$\text{(i)}\quad \Big|\int_0^\infty K(t,z)\,dt\Big| > A; \qquad \text{(ii)}\quad B_0(z) \to 1;$$

$$\text{(iii)}\quad B_1(z) \to \infty; \qquad \text{(iv)}\quad B_{-a}(z) = O(B_1(z))^{-a}.$$

Then, for $L(\cdot)$ *slowly varying and* $z \in D$, *we have*

$$\int_0^\infty L(t)K(t,z)\,dt = L(B_1(z))\int_0^\infty K(t,z)\,dt \quad (1+o(1)).$$

For the proof we need the following two lemmas.

LEMMA 1. *For any slowly varying* $L(\cdot)$, *some* $\mu \in R^+$ *and* $y \to \infty$, *we have*

$$\text{(i)}\quad \sup_{0\leqslant x\leqslant y} (x^\mu L(x)) \sim y^\mu L(y); \qquad \text{(ii)}\quad \sup_{x\geqslant y}(x^{-\mu}L(x)) \sim y^{-\mu}L(y).$$

This is well-known assertion from the Theory of Regular Variation (see [2], [5]).

LEMMA 2. *Under the conditions of Theorem* 2, *for some* $\nu \in (0,1)$, *and* $z \in D$, *we have* $B_\nu(z) = O(B_1(z))^\nu$, *where the constant in* $O(1)$ *does not depend either on* z *or on* ν.

This is a consequence of Holder's inequality

$$\int fg \leqslant \Big(\int f^p\Big)^{1/p}\Big(\int g^q\Big)^{1/q}, \quad p>1, \quad \frac{1}{p}+\frac{1}{q}=1.$$

Putting there $f := \big(t|K(t,z)|\big)^\nu$, $g := |K(t,z)|^{1-\nu}$, $p := 1/\nu$, $q := 1/(1-\nu)$, and taking in account the condition (ii), we obtain the assertion of the lemma.

Now we can prove Theorem 2.

PROOF. For a slowly varying $L(\cdot)$ and $z \in D$, we shall estimate the expression

$$W := \Big|\frac{\int_0^\infty L(t)K(t,z)\,dt}{L(B_1(z))\int_0^\infty K(t,z)\,dt} - 1\Big| = \frac{1}{|\int_0^\infty K(t,z)\,dt|}\Big|\int_0^\infty K(t,z)\Big(\frac{L(t)}{L(B_1(z))} - 1\Big)\,dt\Big|.$$

Under the condition (i) and for some $\sigma \in (0,1)$ we have

$$W = O(1)\bigg(\Big|\int_0^{\sigma B_1(z)} (\cdot)\,dt\Big| + \Big|\int_{\sigma B_1(z)}^{B_1(z)/\sigma} (\cdot)\,dt\Big| + \Big|\int_{B_1(z)/\sigma}^\infty (\cdot)\,dt\Big|\bigg) = W_1 + W_2 + W_3.$$

Applying part (i) of Lemma 1 and the conditions (iii) and (iv) of Theorem 2, we get

$$\begin{aligned}
W_1 &= O(1)\Big|\int_0^{\sigma B_1(z)} K(t,z)\Big(\frac{L(t)}{L(B_1(z))}-1\Big)\,dt\Big| \\
&= O(1)\int_0^{\sigma B_1(z)} t^{-a}|K(t,z)|\Big|\frac{t^a L(t)}{L(B_1(z))}-t^a\Big|\,dt \\
&= \sup_{0\leqslant t\leqslant \sigma B_1(z)}\Big[\frac{t^a L(t)}{L(B_1(z))}+t^a\Big]O(B_{-a}(z)) \\
&= O(\sigma B_1(z))^a\cdot O(B_1(z))^{-a} = O(\sigma^a).
\end{aligned}$$

Similarly, using part (ii) of Lemma 1, and Lemma 2, we obtain the estimation of W_3,

$$W_3 = \sup_{t\geqslant B_1(z)/\sigma}\Big[\frac{t^{-\nu}L(t)}{L(B_1(z))}+t^{-\nu}\Big]\cdot O(B_\nu(z)) = O\Big(\frac{B_1(z)}{\sigma}\Big)^{-\nu}\cdot O(B_1(z))^{\nu}) = O(\sigma^\nu).$$

Finally, applying the condition (ii) of Theorem 2, we have

$$\begin{aligned}
W_2 &= O(1)\int_{\sigma B_1(z)}^{B_1(z)/\sigma}|K(t,z)|\Big|\frac{L(t)}{L(B_1(z))}-1\Big|\,dt \\
&= \sup_{\sigma B_1(z)\leqslant t\leqslant B_1(z)/\sigma}\Big|\frac{L(t)}{L(B_1(z))}-1\Big|\cdot O(B_0(z)) = o(1),
\end{aligned}$$

by the Uniform Convergence Theorem (cf. [2, pp. 6–11]). Hence

$$W = O(\sigma^a)+O(\sigma^\nu)+o(1);\quad z\in D.$$

Since a and ν are positive constants and σ can be taken arbitrarily small, we conclude that $W=o(1)$, $z\in D$, $z\to z_0$, i.e., Theorem 2 is proved. □

3. Applications

Theorem 2 is can be used to estimate asymptotic behaviours of convolutions with slowly varying functions and various complex-valued kernels. To illustrate this, we give some examples.

First one is an extension of Karamata's Theorem for Laplace Transform (see Introduction), on some angles in the right complex half-plane.

Proposition 1. *For $z\to 0$, $|\arg z|\leqslant \pi/2-\epsilon$; $\epsilon,\rho\in R^+$, we have*

$$\int_0^\infty e^{-zt}t^{\rho-1}L(t)\,dt = \frac{\Gamma(\rho)}{z^\rho}L\Big(\frac{1}{\operatorname{Re} z}\Big)(1+o(1)).$$

Proof. We shall apply Theorem 2 on the kernel $K(t,z)$ defined as

$$K(t,z) := \frac{(\operatorname{Re} z)^\rho}{\Gamma(\rho)}e^{-zt}t^{\rho-1},\quad \rho\in R^+.$$

Since z is in the right complex half-plane, it is obvious that $\operatorname{Re} z > 0$. Note also that $\operatorname{Re} z/|z| = \cos(\arg z) \geqslant \sin\epsilon$. Now

$$\Big|\int_0^\infty K(t,z)\,dt\Big| = \Big(\frac{\operatorname{Re} z}{|z|}\Big)^\rho \geqslant \sin^\rho \epsilon, \quad B_0(z) = 1;$$

and

$$B_1(z) = \frac{\rho}{\operatorname{Re} z} \to \infty; \quad B_{-\rho/2} = \frac{\Gamma(\rho/2)}{\Gamma(\rho)}(\operatorname{Re} z)^{\rho/2} = O\big((B_1(z))^{-\rho/2}\big).$$

Therefore the conditions of Theorem 2 are satisfied with $a = \rho/2$; $A = (\sin\epsilon)^\rho$, and the assertion of the proposition follows. □

If z approaches zero along a half-line $\arg z = \pi/4$, then $\operatorname{Re} z = \operatorname{Im} z := x > 0$, $z = x\sqrt{2}\exp(i\pi/4)$. Hence, applying Proposition 1 and taking separately real and imaginary parts, we obtain the following consequence.

Proposition 2. *For $L(\cdot)$ slowly varying and $x, \rho \in R^+$, we have*

$$\int_0^\infty e^{-xt}t^{\rho-1}L(t)\cos xt\,dt \sim \frac{\Gamma(\rho)}{2^{\rho/2}}\cos(\pi\rho/4)(1/x)^\rho L(1/x),$$
$$\int_0^\infty e^{-xt}t^{\rho-1}L(t)\sin xt\,dt \sim -\frac{\Gamma(\rho)}{2^{\rho/2}}\sin(\pi\rho/4)(1/x)^\rho L(1/x) \quad (x \downarrow 0).$$

The next example demonstrates the flexibility of definition of the domain D. Namely, complex parameter is fixed here in some strip in z-plane but real parameter tends to infinity.

Lemma 3. [**6**, p. 115] *For $-1 < \operatorname{Re} u < 1$, $c > 0$, we have*

$$\int_0^\infty \frac{t^u}{(t+c)^2}\,dt = \frac{\pi u}{\sin \pi u}c^{u-1}.$$

For $z = x + iy$, $-1 < x < 0$, $c > 0$, define $K(t,z) := \dfrac{t^z}{(t+c)^2}\dfrac{\sin\pi x}{\pi x}c^{1-x}$. Then, using Lemma 3, an easy calculation gives:

$$B_0(z) = 1; \quad B_1(z) = \frac{1-|x|}{|x|}c; \quad B_{-(1+x)/2}(z) = \frac{1+|x|}{|x|}\sin\Big(\frac{\pi|x|}{2}\Big)c^{-(1+x)/2},$$

and

$$\Big|\int_0^\infty K(t,z)\,dt\Big| = \frac{|z|}{|\sin\pi z|}\frac{\sin\pi|x|}{|x|} > \frac{\sin\pi|x|}{\cosh\pi y}.$$

Since z, defined as above is fixed, we see that the conditions of Theorem 2 are satisfied when $c \to \infty$. Therefore

Proposition 3. *For $-1 < \operatorname{Re} z < 0$, $c > 0$, we have*

$$I(z) := \int_0^\infty \frac{t^z}{(t+c)^2}L(t)\,dt = \frac{\pi z}{\sin\pi z}L(c)\,c^{z-1}(1+o(1)) \quad (c \to \infty).$$

In particular

$$\frac{\partial^m}{\partial z^m}I(z) = \int_0^\infty \frac{t^z}{(t+c)^2}\log^m t\,dt = \frac{\pi z\log^m c}{\sin\pi z}c^{z-1}(1+o(1)) \quad (c \to \infty).$$

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