PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 78(92) (2005), 1–33

COHERENCE OF PROOF-NET CATEGORIES

Kosta Došen and Zoran Petrić

Communicated by Žarko Mijajlović

ABSTRACT. The notion of proof-net category defined in this paper is closely related to graphs implicit in proof nets for the multiplicative fragment without constant propositions of linear logic. Analogous graphs occur in Kelly's and Mac Lane's coherence theorem for symmetric monoidal closed categories. A coherence theorem with respect to these graphs is proved for proof-net categories. Such a coherence theorem is also proved in the presence of arrows corresponding to the mix principle of linear logic. The notion of proof-net category catches the unit free fragment of the notion of star-autonomous category, a special kind of symmetric monoidal closed category.

1. Introduction

In this paper we introduce the notion of proof-net category, for which we will show that it is closely related to graphs implicit in proof nets for the multiplicative fragment without constant propositions of linear logic (see [**14**] and [**7**] for the notion of proof net). Analogous graphs occur in Kelly's and Mac Lane's coherence theorem for symmetric monoidal closed categories of [**17**].

The notion of proof-net category is based on the notion of symmetric net category of [**11**, Section 7.6]; these are categories with two multiplications, $\wedge$ and $\vee$, associative and commutative up to isomorphism, which have moreover arrows of the *dissociativity* type $A \wedge (B \vee C) \to (A \wedge B) \vee C$ (called *linear* or *weak* distribution in [**6**]). The symmetric net category freely generated by a set of objects is called **DS**. To obtain proof-net categories we add to symmetric net categories an operation on objects corresponding to negation, which is involutive up to isomorphism. With these operations come appropriate arrows. A number of equations between

2000 *Mathematics Subject Classification*: Primary 03F07, 03F52, 18D10, 18D15, 19D23.
Key words and phrases: generality of proofs, linear logic, mix, proof nets, linear distribution, dissociativity, categorial coherence, Kelly–Mac Lane graphs, Brauerian graphs, split equivalences, symmetric monoidal closed category, star-autonomous category.

arrows, of the kind called *coherence conditions* in category theory, are satisfied in proof-net categories.

A notion amounting to the notion of star-autonomous category of [**2**] is obtained in a similar manner in [**6**]. Star-autonomous categories, which stem from [**1**], are a special kind of symmetric monoidal closed categories. In contradistinction to symmetric net and proof-net categories they involve unit objects.

We introduce next a category *Br* whose arrows are called *Brauerian split equivalences* of finite ordinals. These equivalence relations, which stem from results in representation theory of [**3**], amount to the graphs used by Kelly and Mac Lane for their coherence theorem of symmetric monoidal categories mentioned above. Brauerian split equivalences express generality of proofs in linear logic (see [**9**], [**10**]).

For proof-net categories we prove a coherence theorem that says that there is a faithful functor from the proof-net category $\mathbf{PN}^{\neg}$ freely generated by a set of objects into *Br*. The coherence theorem for $\mathbf{PN}^{\neg}$ yields an elementary decision procedure for verifying whether a diagram of arrows commutes in $\mathbf{PN}^{\neg}$, and hence also in every proof-net category. This is a very useful result, which enables us in [**12**] to obtain other coherence results with respect to *Br*, in particular a coherence result for star-autonomous categories, involving the units. It is also shown in [**12**] with the help of coherence for $\mathbf{PN}^{\neg}$ that the notion of proof-net category catches the unit-free fragment of star-autonomous categories. (A different attempt to catch this fragment is made in [**18**] and [**15**].)

The coherence theorem for $\mathbf{PN}^{\neg}$ is proved by finding a category $\mathbf{PN}$, equivalent to $\mathbf{PN}^{\neg}$, in which negation can be applied only to the generating objects, and coherence is first established for $\mathbf{PN}$ by relying on coherence for symmetric net categories, previously established in [**11**, Chapter 7], and on an additional normalization procedure involving negation.

In the last two sections of the paper we consider proof-net categories that have *mix* arrows of the type $A \wedge B \vdash A \vee B$. We prove coherence with respect to *Br* for the appropriate notion of proof-net category with these arrows, which we call mix-proof-net category.

2. The category DS

The objects of the category $\mathbf{DS}$ are the formulae of the propositional language $\mathcal{L}_{\wedge,\vee}$, generated from a set $\mathcal{P}$ of propositional letters, which we call simply *letters*, with the binary connectives $\wedge$ and $\vee$. We use $p, q, r, \ldots$, sometimes with indices, for letters, and $A, B, C, \ldots$, sometimes with indices, for formulae. As usual, we omit the outermost parentheses of formulae and other expressions later on.

To define the arrows of $\mathbf{DS}$, we define first inductively a set of expressions called the *arrow terms* of $\mathbf{DS}$. Every arrow term of $\mathbf{DS}$ will have a *type*, which is an ordered pair of formulae of $\mathcal{L}_{\wedge,\vee}$. We write $f\colon A \vdash B$ when the arrow term f

is of type (A, B). (We use the turnstile $\vdash$ instead of the more usual $\rightarrow$, which we reserve for a connective and a biendofunctor.) We use $f, g, h, \ldots$, sometimes with indices, for arrow terms.

For all formulae A, B and C of $\mathcal{L}_{\wedge,\vee}$ the following *primitive arrow terms*:

$$\mathbf{1}_A \colon A \vdash A,$$

$$\hat{b}{}^{\rightarrow}_{A,B,C} \colon A \wedge (B \wedge C) \vdash (A \wedge B) \wedge C, \quad \overset{\vee}{b}{}^{\rightarrow}_{A,B,C} \colon A \vee (B \vee C) \vdash (A \vee B) \vee C,$$

$$\hat{b}{}^{\leftarrow}_{A,B,C} \colon (A \wedge B) \wedge C \vdash A \wedge (B \wedge C), \quad \overset{\vee}{b}{}^{\leftarrow}_{A,B,C} \colon (A \vee B) \vee C \vdash A \vee (B \vee C),$$

$$\hat{c}_{A,B} \colon A \wedge B \vdash B \wedge A, \qquad \overset{\vee}{c}_{A,B} \colon B \vee A \vdash A \vee B,$$

$$d_{A,B,C} \colon A \wedge (B \vee C) \vdash (A \wedge B) \vee C$$

are arrow terms of $\mathbf{DS}$. If $g \colon A \vdash B$ and $f \colon B \vdash C$ are arrow terms of $\mathbf{DS}$, then $f \circ g \colon A \vdash C$ is an arrow term of $\mathbf{DS}$; and if $f \colon A \vdash D$ and $g \colon B \vdash E$ are arrow terms of $\mathbf{DS}$, then $f \,\xi\, g \colon A \,\xi\, B \vdash D \,\xi\, E$, for $\xi \in \{\wedge, \vee\}$, is an arrow term of $\mathbf{DS}$. This concludes the definition of the arrow terms of $\mathbf{DS}$.

Next we define inductively the set of *equations* of $\mathbf{DS}$, which are expressions of the form $f = g$, where f and g are arrow terms of $\mathbf{DS}$ of the same type. We stipulate first that all instances of $f = f$ and of the following equations are equations of $\mathbf{DS}$:

$(cat\ 1)$ $\quad f \circ \mathbf{1}_A = \mathbf{1}_B \circ f = f \colon A \vdash B,$

$(cat\ 2)$ $\quad h \circ (g \circ f) = (h \circ g) \circ f,$

for $\xi \in \{\wedge, \vee\}$,

$(\xi\ 1)$ $\quad \mathbf{1}_A \,\xi\, \mathbf{1}_B = \mathbf{1}_{A \xi B},$

$(\xi\ 2)$ $\quad (g_1 \circ f_1) \,\xi\, (g_2 \circ f_2) = (g_1 \,\xi\, g_2) \circ (f_1 \,\xi\, f_2),$

for $f \colon A \vdash D$, $g \colon B \vdash E$ and $h \colon C \vdash F$,

$(\overset{\xi}{b}{}^{\rightarrow}\ nat)$ $\quad ((f \,\xi\, g) \,\xi\, h) \circ \overset{\xi}{b}{}^{\rightarrow}_{A,B,C} = \overset{\xi}{b}{}^{\rightarrow}_{D,E,F} \circ (f \,\xi\, (g \,\xi\, h)),$

$(\hat{c}\ nat)$ $\quad (g \wedge f) \circ \hat{c}_{A,B} = \hat{c}_{D,E} \circ (f \wedge g),$

$(\overset{\vee}{c}\ nat)$ $\quad (g \vee f) \circ \overset{\vee}{c}_{B,A} = \overset{\vee}{c}_{E,D} \circ (f \vee g),$

$(d\ nat)$ $\quad ((f \wedge g) \vee h) \circ d_{A,B,C} = d_{D,E,F} \circ (f \wedge (g \vee h)),$

$(\overset{\xi}{b}\overset{\xi}{b})$ $\quad \overset{\xi}{b}{}^{\leftarrow}_{A,B,C} \circ \overset{\xi}{b}{}^{\rightarrow}_{A,B,C} = \mathbf{1}_{A \xi (B \xi C)}, \qquad \overset{\xi}{b}{}^{\rightarrow}_{A,B,C} \circ \overset{\xi}{b}{}^{\leftarrow}_{A,B,C} = \mathbf{1}_{(A \xi B) \xi C},$

$(\overset{\xi}{b}5)$ $\quad \overset{\xi}{b}{}^{\leftarrow}_{A,B,C \xi D} \circ \overset{\xi}{b}{}^{\leftarrow}_{A \xi B,C,D} = (\mathbf{1}_A \,\xi\, \overset{\xi}{b}{}^{\leftarrow}_{B,C,D}) \circ \overset{\xi}{b}{}^{\leftarrow}_{A,B \xi C,D} \circ (\overset{\xi}{b}{}^{\leftarrow}_{A,B,C} \,\xi\, \mathbf{1}_D),$

$(\hat{c}\hat{c})$ $\quad \hat{c}_{B,A} \circ \hat{c}_{A,B} = \mathbf{1}_{A \wedge B},$

$(\overset{\vee}{c}\overset{\vee}{c})$ $\quad \overset{\vee}{c}_{A,B} \circ \overset{\vee}{c}_{B,A} = \mathbf{1}_{A \vee B},$

$(\hat{b}\hat{c})$ $\quad (\mathbf{1}_B \wedge \hat{c}_{C,A}) \circ \hat{b}^{\leftarrow}_{B,C,A} \circ \hat{c}_{A,B\wedge C} \circ \hat{b}^{\leftarrow}_{A,B,C} \circ (\hat{c}_{B,A} \wedge \mathbf{1}_C) = \hat{b}^{\leftarrow}_{B,A,C}$,

$(\check{b}\check{c})$ $\quad (\mathbf{1}_B \vee \check{c}_{A,C}) \circ \check{b}^{\leftarrow}_{B,C,A} \circ \check{c}_{B\vee C,A} \circ \check{b}^{\leftarrow}_{A,B,C} \circ (\check{c}_{A,B} \vee \mathbf{1}_C) = \check{b}^{\leftarrow}_{B,A,C}$,

$(d\wedge)$ $\quad (\hat{b}^{\leftarrow}_{A,B,C} \vee \mathbf{1}_D) \circ d_{A\wedge B,C,D} = d_{A,B\wedge C,D} \circ (\mathbf{1}_A \wedge d_{B,C,D}) \circ \hat{b}^{\leftarrow}_{A,B,C\vee D}$,

$(d\vee)$ $\quad d_{D,C,B\vee A} \circ (\mathbf{1}_D \wedge \check{b}^{\leftarrow}_{C,B,A}) = \check{b}^{\leftarrow}_{D\wedge C,B,A} \circ (d_{D,C,B} \vee \mathbf{1}_A) \circ d_{D,C\vee B,A}$,

for $d^R_{C,B,A} =_{\mathrm{df}} \check{c}_{C,B\wedge A} \circ (\hat{c}_{A,B} \vee \mathbf{1}_C) \circ d_{A,B,C} \circ (\mathbf{1}_A \wedge \check{c}_{B,C}) \circ \hat{c}_{C\vee B,A}$:

$$(C \vee B) \wedge A \vdash C \vee (B \wedge A),$$

$(d\hat{b})$ $\quad d^R_{A\wedge B,C,D} \circ (d_{A,B,C} \wedge \mathbf{1}_D) = d_{A,B,C\wedge D} \circ (\mathbf{1}_A \wedge d^R_{B,C,D}) \circ \hat{b}^{\leftarrow}_{A,B\vee C,D}$,

$(d\check{b})$ $\quad (\mathbf{1}_D \vee d_{C,B,A}) \circ d^R_{D,C,B\vee A} = \check{b}^{\leftarrow}_{D,C\wedge B,A} \circ (d^R_{D,C,B} \vee \mathbf{1}_A) \circ d_{D\vee C,B,A}$.

The set of equations of **DS** is closed under symmetry and transitivity of equality and under the rules

$$(cong\ \ \xi)\quad \frac{f = f_1 \qquad\qquad g = g_1}{f \,\xi\, g = f_1 \,\xi\, g_1}$$

where $\xi \in \{\circ, \wedge, \vee\}$, and if ξ is $\circ$, then $f \circ g$ is defined (namely, f and g have appropriate, composable, types).

On the arrow terms of **DS** we impose the equations of **DS**. This means that an arrow of **DS** is an equivalence class of arrow terms of **DS** defined with respect to the smallest equivalence relation such that the equations of **DS** are satisfied (see [**11**, Section 2.3]).

The equations $(\xi\ 1)$ and $(\xi\ 2)$ are called *bifunctorial* equations. They say that $\wedge$ and $\vee$ are biendofunctors (i.e. 2-endofunctors in the terminology of [**11**, Section 2.4]).

It is easy to show that for **DS** we have the equations

$(\overset{\xi}{b}{}^{\leftarrow}\ nat)$ $\quad (f \,\xi\, (g \,\xi\, h)) \circ \overset{\xi}{b}{}^{\leftarrow}_{A,B,C} = \overset{\xi}{b}{}^{\leftarrow}_{D,E,F} \circ ((f \,\xi\, g) \,\xi\, h)$,

$(d^R\ nat)$ $\quad (h \vee (g \wedge f)) \circ d^R_{C,B,A} = d^R_{F,E,D} \circ ((h \vee g) \wedge f)$.

We call these equations and other equations with "*nat*" in their names, like those in the list above, *naturality* equations. Such equations say that $\hat{b}^{\rightarrow}$, $\hat{b}^{\leftarrow}$, $\hat{c}$, etc. are natural transformations.

The equations $(d\wedge)$, $(d\vee)$, $(d\hat{b})$ and $(d\check{b})$ stem from [**6**, Section 2.1] (see [**5**, Section 2.1] for an announcement). The equation $(d\check{b})$ of [**11**, Section 7.2] amounts with $(\check{b}\check{b})$ to the present one.

3. The category $\mathbf{PN}^{\neg}$

The category $\mathbf{PN}^{\neg}$ is defined as **DS** save that we make the following changes and additions. Instead of $\mathcal{L}_{\wedge,\vee}$, we have the propositional language $\mathcal{L}_{\neg,\wedge,\vee}$, which has in addition to what we have for $\mathcal{L}_{\wedge,\vee}$ the unary connective $\neg$.

To define the arrow terms of $\mathbf{PN}^{\neg}$, in the inductive definition we had for the arrow terms of $\mathbf{DS}$ we assume in addition that for all formulae A and B of $\mathcal{L}_{\neg,\wedge,\vee}$ the following *primitive arrow terms*:

$$\overset{\wedge}{\Delta}_{B,A} \colon A \vdash A \wedge (\neg B \vee B),$$
$$\overset{\vee}{\Sigma}_{B,A} \colon (B \wedge \neg B) \vee A \vdash A,$$

are arrow terms of $\mathbf{PN}^{\neg}$. We call the index B, of $\overset{\wedge}{\Delta}_{B,A}$ and $\overset{\vee}{\Sigma}_{B,A}$ the *crown* index, and A the *stem* index. The *crown* of $\overset{\wedge}{\Delta}_{B,A}$ ic the right conjunct $\neg B \vee B$ in the target of $\overset{\wedge}{\Delta}_{B,A} \colon A \vdash A \wedge (\neg B \vee B)$, and the *crown* of $\overset{\vee}{\Sigma}_{B,A}$ is the left disjunct $B \wedge \neg B$ in the source of $\overset{\vee}{\Sigma}_{B,A} \colon (B \wedge \neg B) \vee A \vdash A$. We have analogous definitions of crown and stem indices, and crowns for $\overset{\wedge}{\Sigma}$, $\overset{\wedge}{\Delta}{}'$, $\overset{\wedge}{\Sigma}{}'$, $\overset{\vee}{\Delta}$, $\overset{\vee}{\Sigma}{}'$ and $\overset{\vee}{\Delta}{}'$, which will be defined below. (The symbol Δ should be associated with the Latin *d*exter, because in $\overset{\wedge}{\Delta}_{B,A}$, $\overset{\wedge}{\Delta}{}'_{B,A}$, $\overset{\vee}{\Delta}_{B,A}$ and $\overset{\vee}{\Delta}{}'_{B,A}$ the crown is on the right-hand side of the stem; analogously, Σ should be associated with *s*inister.)

To define the arrows of $\mathbf{PN}^{\neg}$, we assume in the inductive definition we had for the equations of $\mathbf{DS}$ the following additional equations, which we call the $\mathbf{PN}$ *equations* (and *n*ot $\mathbf{PN}^{\neg}$ equations):

$$(\overset{\wedge}{\Delta}\ \mathit{nat}) \quad (f \wedge \mathbf{1}_{\neg B \vee B}) \circ \overset{\wedge}{\Delta}_{B,A} = \overset{\wedge}{\Delta}_{B,D} \circ f,$$
$$(\overset{\vee}{\Sigma}\ \mathit{nat}) \quad f \circ \overset{\vee}{\Sigma}_{B,A} = \overset{\vee}{\Sigma}_{B,D} \circ (\mathbf{1}_{B \wedge \neg B} \vee f),$$
$$(\hat{b}\overset{\wedge}{\Delta}) \qquad \hat{b}{}^{\leftarrow}_{A,B,\neg C \vee C} \circ \overset{\wedge}{\Delta}_{C,A \wedge B} = \mathbf{1}_A \wedge \overset{\wedge}{\Delta}_{C,B},$$
$$(\overset{\vee}{b}\overset{\vee}{\Sigma}) \qquad \overset{\vee}{\Sigma}_{C,B \vee A} \circ \overset{\vee}{b}{}^{\leftarrow}_{C \wedge \neg C,B,A} = \overset{\vee}{\Sigma}_{C,B} \vee \mathbf{1}_A,$$

for $\overset{\wedge}{\Sigma}_{B,A} =_{\mathrm{df}} \hat{c}_{A,\neg B \vee B} \circ \overset{\wedge}{\Delta}_{B,A} : A \vdash (\neg B \vee B) \wedge A$,

$$(d\overset{\wedge}{\Sigma}) \qquad d_{\neg A \vee A,B,C} \circ \overset{\wedge}{\Sigma}_{A,B \vee C} = \overset{\wedge}{\Sigma}_{A,B} \vee \mathbf{1}_C,$$

for $\overset{\vee}{\Delta}_{B,A} =_{\mathrm{df}} \overset{\vee}{\Sigma}_{B,A} \circ \overset{\vee}{c}_{B \wedge \neg B,A} : A \vee (B \wedge \neg B) \vdash A$,

$$(d\overset{\vee}{\Delta}) \qquad \overset{\vee}{\Delta}_{A,C \wedge B} \circ d_{C,B,A \wedge \neg A} = \mathbf{1}_C \wedge \overset{\vee}{\Delta}_{A,B},$$
$$(\overset{\vee}{\Sigma}\overset{\wedge}{\Delta}) \qquad \overset{\vee}{\Sigma}_{A,A} \circ d_{A,\neg A,A} \circ \overset{\wedge}{\Delta}_{A,A} = \mathbf{1}_A,$$

for $\overset{\wedge}{\Delta}{}'_{B,A} =_{\mathrm{df}} (\mathbf{1}_A \wedge \overset{\vee}{c}_{B,\neg B}) \circ \overset{\wedge}{\Delta}_{B,A} : A \vdash A \wedge (B \vee \neg B)$ and
$\overset{\vee}{\Sigma}{}'_{B,A} =_{\mathrm{df}} \overset{\vee}{\Sigma}_{B,A} \circ (\hat{c}_{\neg B,B} \vee \mathbf{1}_A) : (\neg B \wedge B) \vee A \vdash A$,

$$(\overset{\vee}{\Sigma}{}'\overset{\wedge}{\Delta}{}') \qquad \overset{\vee}{\Sigma}{}'_{A,\neg A} \circ d_{\neg A,A,\neg A} \circ \overset{\wedge}{\Delta}{}'_{A,\neg A} = \mathbf{1}_{\neg A}.$$

It is easy to show that for $\mathbf{PN}^{\neg}$ we have the equations

$(\overset{\wedge}{\Sigma}\ nat)\qquad (\mathbf{1}_{\neg B\vee B}\wedge f)\circ\overset{\wedge}{\Sigma}_{B,A}=\overset{\wedge}{\Sigma}_{B,D}\circ f,$

$(\overset{\vee}{\Delta}\ nat)\qquad f\circ\overset{\vee}{\Delta}_{B,A}=\overset{\vee}{\Delta}_{B,D}\circ(f\vee\mathbf{1}_{B\wedge\neg B}).$

The naturality equations $(\overset{\wedge}{\Delta}\ nat)$ and $(\overset{\vee}{\Sigma}\ nat)$ together with these say that $\overset{\wedge}{\Delta}$, $\overset{\vee}{\Sigma}$, $\overset{\wedge}{\Sigma}$ and $\overset{\vee}{\Delta}$ are natural transformations in the stem index only, i.e. in the second index.

We also have the following abbreviations:

$$\overset{\wedge}{\Sigma}{}'_{B,A}=_{\mathrm{df}}\ \hat{c}_{A,B\vee\neg B}\circ\overset{\wedge}{\Delta}{}'_{B,A}:A\vdash(B\vee\neg B)\wedge A,$$

$$\overset{\vee}{\Delta}{}'_{B,A}=_{\mathrm{df}}\ \overset{\vee}{\Sigma}{}'_{B,A}\circ\overset{\vee}{c}_{\neg B\wedge B,A}:A\vee(\neg B\wedge B)\vdash A.$$

If Ξ stands for either Δ or Σ and $\xi\in\{\wedge,\vee\}$, then for every $(\overset{\xi}{\Xi}\ nat)$ equation we have in $\mathbf{PN}^{\neg}$ the equation $(\overset{\xi}{\Xi}{}'\ nat)$, which differs from $(\overset{\xi}{\Xi}\ nat)$ by replacing $\overset{\xi}{\Xi}$ by $\overset{\xi}{\Xi}{}'$, and the index of $\mathbf{1}$ by the appropriate index. For example, we have

$(\overset{\wedge}{\Delta}{}'\ nat)\qquad (f\wedge\mathbf{1}_{B\vee\neg B})\circ\overset{\wedge}{\Delta}{}'_{B,A}=\overset{\wedge}{\Delta}{}'_{B,D}\circ f.$

As alternative primitive arrow terms for defining $\mathbf{PN}^{\neg}$ we could take one of $\overset{\wedge}{\Xi}$ or $\overset{\wedge}{\Xi}{}'$ and one of $\overset{\vee}{\Xi}$ or $\overset{\vee}{\Xi}{}'$.

We can also derive for $\mathbf{PN}^{\neg}$ the following equations:

$(\hat{b}\overset{\wedge}{\Delta}\overset{\wedge}{\Sigma})\qquad \hat{b}{}^{\leftarrow}_{A,\neg B\vee B,C}\circ(\overset{\wedge}{\Delta}_{B,A}\wedge\mathbf{1}_C)=\mathbf{1}_A\wedge\overset{\wedge}{\Sigma}_{B,C},$

$(\hat{b}\overset{\wedge}{\Sigma})\qquad \hat{b}{}^{\rightarrow}_{\neg C\vee C,B,A}\circ\overset{\wedge}{\Sigma}_{C,B\wedge A}=\overset{\wedge}{\Sigma}_{C,B}\wedge\mathbf{1}_A.$

For the first equation, with indices omitted, we have

$$\begin{aligned}\hat{b}{}^{\leftarrow}\circ(\overset{\wedge}{\Delta}\wedge\mathbf{1})&=\hat{b}{}^{\leftarrow}\circ\hat{c}\circ(\mathbf{1}\wedge\overset{\wedge}{\Delta})\circ\hat{c},\quad\text{by }(\hat{c}\,\hat{c})\text{ and }(\hat{c}\ nat),\\&=\hat{b}{}^{\leftarrow}\circ\hat{c}\circ\hat{b}{}^{\leftarrow}\circ\overset{\wedge}{\Delta}\circ\hat{c},\quad\text{by }(\hat{b}\overset{\wedge}{\Delta}),\\&=(\mathbf{1}\wedge\hat{c})\circ\hat{b}{}^{\leftarrow}\circ\overset{\wedge}{\Delta},\quad\text{with }(\overset{\wedge}{\Delta}\ nat)\text{ and }(\hat{b}\,\hat{c}),\\&=\mathbf{1}\wedge\overset{\wedge}{\Sigma},\quad\text{by }(\hat{b}\overset{\wedge}{\Delta}),\end{aligned}$$

and for the second equation we have

$$\begin{aligned}\hat{b}{}^{\rightarrow}\circ\overset{\wedge}{\Sigma}&=\hat{b}{}^{\rightarrow}\circ\hat{c}\circ\hat{b}{}^{\rightarrow}\circ(\mathbf{1}\wedge\overset{\wedge}{\Delta}),\quad\text{with }(\hat{b}\overset{\wedge}{\Delta}),\\&=(\hat{c}\wedge\mathbf{1})\circ\hat{b}{}^{\rightarrow}\circ(\mathbf{1}\wedge\hat{c})\circ(\mathbf{1}\wedge\overset{\wedge}{\Delta}),\quad\text{by }(\hat{b}\,\hat{c}),\\&=\overset{\wedge}{\Sigma}\wedge\mathbf{1},\quad\text{with }(\hat{b}\overset{\wedge}{\Delta}\overset{\wedge}{\Sigma}).\end{aligned}$$

We derive analogously with the help of $(\overset{\vee}{b}\overset{\vee}{\Sigma})$ the equations

$(\overset{\vee}{b}\overset{\vee}{\Delta}\overset{\vee}{\Sigma})\qquad (\overset{\vee}{\Delta}_{B,A}\vee\mathbf{1}_C)\circ\overset{\vee}{b}{}^{\rightarrow}_{A,B\wedge\neg B,C}=\mathbf{1}_A\vee\overset{\vee}{\Sigma}_{B,C},$

$(\overset{\vee}{b}\overset{\vee}{\Delta})\qquad \overset{\vee}{\Delta}_{C,A\vee B}\circ\overset{\vee}{b}{}^{\rightarrow}_{A,B,C\wedge\neg C}=\mathbf{1}_A\vee\overset{\vee}{\Delta}_{C,B}.$

The arrows $\overset{\wedge}{\Delta}_{B,A}\colon A\vdash A\wedge(\neg B\vee B)$ and $\overset{\wedge}{\Sigma}_{B,A}\colon A\vdash(\neg B\vee B)\wedge A$ are analogous to the arrows of types $A\vdash A\wedge\top$ and $A\vdash\top\wedge A$ that one finds in monoidal categories. However, $\overset{\wedge}{\Delta}_{B,A}$ and $\overset{\wedge}{\Sigma}_{B,A}$ do not have inverses in $\mathbf{PN}^{\neg}$. The equations $(\hat{b}\overset{\wedge}{\Delta})$, $(\hat{b}\overset{\wedge}{\Delta}\overset{\wedge}{\Sigma})$, $(\hat{b}\overset{\wedge}{\Sigma})$ are analogous to equations that hold in monoidal categories (see [**19**, Section VII.1], [**11**, Section 4.6]). An analogous remark can be made for $\overset{\vee}{\Sigma}_{B,A}$ and $\overset{\vee}{\Delta}_{B,A}$.

We can also derive for $\mathbf{PN}^{\neg}$ the following equations by using essentially $(d\overset{\wedge}{\Sigma})$ and $(d\overset{\vee}{\Delta})$:

$$(d^R\overset{\wedge}{\Delta})\qquad d^R_{C,B,\neg A\vee A}\circ\overset{\wedge}{\Delta}_{A,C\vee B}=\mathbf{1}_C\vee\overset{\wedge}{\Delta}_{A,B},$$

$$(d^R\overset{\vee}{\Sigma})\qquad \overset{\vee}{\Sigma}_{A,B\wedge C}\circ d^R_{A\wedge\neg A,B,C}=\overset{\vee}{\Sigma}_{A,B}\wedge\mathbf{1}_C.$$

These two equations could replace $(d\overset{\wedge}{\Sigma})$ and $(d\overset{\vee}{\Delta})$ for defining $\mathbf{PN}^{\neg}$. The analogues of the equations $(d\overset{\wedge}{\Sigma})$, $(d\overset{\vee}{\Delta})$, $(d^R\overset{\wedge}{\Delta})$ and $(d^R\overset{\vee}{\Sigma})$ may be found in [**6**, Section 2.1], where they are assumed for linearly (alias weakly) distributive categories with negation (cf. [**11**, Section 7.9]).

It is easy to infer that in $\mathbf{PN}^{\neg}$ we have analogues of the equations $(\hat{b}\overset{\wedge}{\Delta})$, $(\hat{b}\overset{\wedge}{\Delta}\overset{\wedge}{\Sigma})$, $(\hat{b}\overset{\wedge}{\Sigma})$, $(\overset{\vee}{b}\overset{\vee}{\Sigma})$, $(\overset{\vee}{b}\overset{\vee}{\Delta}\overset{\vee}{\Sigma})$, $(\overset{\vee}{b}\overset{\vee}{\Delta})$, $(d\overset{\wedge}{\Sigma})$, $(d\overset{\vee}{\Delta})$, $(d^R\overset{\wedge}{\Delta})$ and $(d^R\overset{\vee}{\Sigma})$ obtained by replacing $\overset{\xi}{\Xi}$ by $\overset{\xi}{\Xi}{}'$, and the indices of the form $\neg B\vee B$ and $B\wedge\neg B$ by $B\vee\neg B$ and $\neg B\wedge B$ respectively. For example, we have

$$(\hat{b}\overset{\wedge}{\Delta}{}')\qquad \hat{b}{}^{\leftarrow}_{A,B,C\vee\neg C}\circ\overset{\wedge}{\Delta}{}'_{C,A\wedge B}=\mathbf{1}_A\wedge\overset{\wedge}{\Delta}{}'_{C,B}.$$

We can also derive for $\mathbf{PN}^{\neg}$ the following equations by using essentially $(\overset{\vee}{\Sigma}\overset{\wedge}{\Delta})$ and $(\overset{\vee}{\Sigma}{}'\overset{\wedge}{\Delta}{}')$:

$$(\overset{\vee}{\Delta}{}'\overset{\wedge}{\Sigma}{}')\qquad \overset{\vee}{\Delta}{}'_{A,A}\circ d^R_{A,\neg A,A}\circ\overset{\wedge}{\Sigma}{}'_{A,A}=\mathbf{1}_A,$$

$$(\overset{\vee}{\Delta}\overset{\wedge}{\Sigma})\qquad \overset{\vee}{\Delta}_{A,\neg A}\circ d^R_{\neg A,A,\neg A}\circ\overset{\wedge}{\Sigma}_{A,\neg A}=\mathbf{1}_{\neg A}.$$

These two equations could replace $(\overset{\vee}{\Sigma}\overset{\wedge}{\Delta})$ and $(\overset{\vee}{\Sigma}{}'\overset{\wedge}{\Delta}{}')$ for defining $\mathbf{PN}^{\neg}$. The equations $(\overset{\vee}{\Sigma}\overset{\wedge}{\Delta})$, $(\overset{\vee}{\Sigma}{}'\overset{\wedge}{\Delta}{}')$, $(\overset{\vee}{\Delta}{}'\overset{\wedge}{\Sigma}{}')$ and $(\overset{\vee}{\Delta}\overset{\wedge}{\Sigma})$ are related to the triangular equations of an adjunction (see [**19**, Section IV.1]; see also the next section). The analogues of these equations may be found in [**6**, Section 4].

A *proof-net* category is a category with two biendofunctors $\wedge$ and $\vee$, a unary operation $\neg$ on objects, and the natural transformations $\hat{b}{}^{\rightarrow}$, $\hat{b}{}^{\leftarrow}$, $\overset{\vee}{b}{}^{\rightarrow}$, $\overset{\vee}{b}{}^{\leftarrow}$, $\hat{c}$, $\overset{\vee}{c}$, d, $\overset{\wedge}{\Delta}$ and $\overset{\vee}{\Sigma}$ that satisfy the equations $(\overset{\xi}{b}5)$, $(\overset{\xi}{b}\overset{\xi}{b})$, ... , $(\overset{\vee}{\Sigma}{}'\overset{\wedge}{\Delta}{}')$ of $\mathbf{PN}^{\neg}$. The category $\mathbf{PN}^{\neg}$ is up to isomorphism the free proof-net category generated by the set of letters $\mathcal{P}$ (the set $\mathcal{P}$ may be understood as a discrete category).

If β is a primitive arrow term of $\mathbf{PN}^{\neg}$ except $\mathbf{1}_B$, then we call *β-terms* of $\mathbf{PN}^{\neg}$ the set of arrow terms defined inductively as follows: β is a β-term; if f is a β-term, then for every A in $\mathcal{L}_{\wedge,\vee}$ we have that $\mathbf{1}_A\,\xi\,f$ and $f\,\xi\,\mathbf{1}_A$, where $\xi\in\{\wedge,\vee\}$, are β-terms.

In a β-term the subterm β is called the *head* of this β-term. For example, the head of the $\hat{b}^{\rightarrow}_{B,C,D}$-term $\mathbf{1}_A \wedge (\hat{b}^{\rightarrow}_{B,C,D} \vee \mathbf{1}_E)$ is $\hat{b}^{\rightarrow}_{B,C,D}$.

We define **1**-*terms* as β-terms by replacing β in the definition above by $\mathbf{1}_B$. So **1**-terms are headless.

An arrow term of the form $f_n \circ \ldots \circ f_1$, where $n \geqslant 1$, with parentheses tied to $\circ$ associated arbitrarily, such that for every $i \in \{1, \ldots, n\}$ we have that f_i is composition-free is called *factorized*. In a factorized arrow term $f_n \circ \ldots \circ f_1$ the arrow terms f_i are called *factors*. A factor that is a β-term for some β is called a *headed* factor. A factorized arrow term is called *headed* when each of its factors is either headed or a **1**-term. A factorized arrow term $f_n \circ \ldots \circ f_1$ is called *developed* when f_1 is a **1**-term and if $n > 1$, then every factor of $f_n \circ \ldots \circ f_2$ is headed. It is sometimes useful to write the factors of a headed arrow term one above the other, as it is done for example in Figure 1 at the end of §6.

By using the categorial equations $(cat\ 1)$ and $(cat\ 2)$ and bifunctorial equations we can easily prove by induction on the length of f the following lemma.

DEVELOPMENT LEMMA. *For every arrow term f there is a developed arrow term f' such that $f = f'$ in* $\mathbf{PN}^{\neg}$.

Analogous definitions of β-term and developed arrow term can be given for **DS**, and an analogous Development Lemma can be proved for **DS**.

4. The category *Br*

We are now going to introduce a category called *Br*, which will serve to prove our main coherence result for proof-net categories. We will show that there is a faithful functor from $\mathbf{PN}^{\neg}$ to *Br*. The name of the category *Br* comes from "Brauerian". The arrows of this category correspond to graphs, or diagrams, that were introduced in [**3**] in connection with Brauer algebras. Analogous graphs were investigated in [**13**], and in [**17**] Kelly and Mac Lane relied on them to prove their coherence result for symmetric monoidal closed categories.

Let $\mathcal{M}$ be a set whose subsets are denoted by X, Y, Z, ... For $i \in \{s, t\}$ (where s stands for "source" and t for "target"), let $\mathcal{M}^i$ be a set in one-to-one correspondence with $\mathcal{M}$, and let $i \colon \mathcal{M} \to \mathcal{M}^i$ be a bijection. Let X^i be the subset of $\mathcal{M}^i$ that is the image of the subset X of $\mathcal{M}$ under i. If $u \in \mathcal{M}$, then we use u_i as an abbreviation for $i(u)$. We assume also that $\mathcal{M}$, $\mathcal{M}^s$ and $\mathcal{M}^t$ are mutually disjoint.

For $X, Y \subseteq \mathcal{M}$, let a *split relation* of $\mathcal{M}$ be a triple $\langle R, X, Y\rangle$ such that $R \subseteq (X^s \cup Y^t)^2$. The set $X^s \cup Y^t$ may be conceived as the disjoint union of X and Y. We denote a split relation $\langle R, X, Y\rangle$ more suggestively by $R \colon X \vdash Y$.

A split relation $R \colon X \vdash Y$ is a *split equivalence* when R is an equivalence relation. We denote by $\mathrm{part}(R)$ the partition of $X_s \cup Y_t$ corresponding to the split equivalence $R \colon X \vdash Y$.

A split equivalence $R\colon X \vdash Y$ is *Brauerian* when every member of part(R) is a two-element set. For $R\colon X \vdash Y$ a Brauerian split equivalence, every member of part(R) is either of the form $\{u_s, v_t\}$, in which case it is called a *transversal*, or of the form $\{u_s, v_s\}$, in which case it is called a *cup*, or, finally, of the form $\{u_t, v_t\}$, in which case it is called a *cap*.

For $X, Y, Z \in \mathcal{M}$, we want to define the composition $P * R\colon X \vdash Z$ of the split relations $R\colon X \vdash Y$ and $P\colon Y \vdash Z$ of $\mathcal{M}$. For that we need some auxiliary notions.

For $X, Y \subseteq \mathcal{M}$, let the function $\varphi^s\colon X \cup Y^t \to X^s \cup Y^t$ be defined by

$$\varphi^s(u) = \begin{cases} u_s & \text{if } u \in X \\ u & \text{if } u \in Y^t, \end{cases}$$

and let the function $\varphi^t\colon X^s \cup Y \to X^s \cup Y^t$ be defined by

$$\varphi^t(u) = \begin{cases} u & \text{if } u \in X^s \\ u_t & \text{if } u \in Y. \end{cases}$$

For a split relation $R\colon X \vdash Y$, let the two relations $R^{-s} \subseteq (X \cup Y^t)^2$ and $R^{-t} \subseteq (X^s \cup Y)^2$ be defined by

$$(u, v) \in R^{-i} \quad \text{iff} \quad (\varphi^i(u), \varphi^i(v)) \in R$$

for $i \in \{s, t\}$. Finally, for an arbitrary binary relation R, let $\mathrm{Tr}(R)$ be the transitive closure of R.

Then we define $P * R$ by

$$P * R =_{\mathrm{df}} \mathrm{Tr}(R^{-t} \cup P^{-s}) \cap (X^s \cup Z^t)^2.$$

It is easy to conclude that $P * R\colon X \vdash Z$ is a split relation of $\mathcal{M}$, and that if $R\colon X \vdash Y$ and $P\colon Y \vdash Z$ are (Brauerian) split equivalences, then $P * R$ is a (Brauerian) split equivalence.

We now define the category *Br*. The objects of *Br* are the members of the set of finite ordinals $\boldsymbol{N}$. (We have $0 = \emptyset$ and $n+1 = n \cup \{n\}$, while $\boldsymbol{N}$ is the ordinal ω.) The arrows of *Br* are the Brauerian split equivalences $R\colon m \vdash n$ of $\boldsymbol{N}$. The identity arrow $\mathbf{1}_n\colon n \vdash n$ of *Br* is the Brauerian split equivalence such that

$$\mathrm{part}(\mathbf{1}_n) = \{\{m_s, m_t\} \mid m < n\}.$$

Composition in *Br* is the operation $*$ defined above.

That *Br* is indeed a category (i.e. that $*$ is associative and that $\mathbf{1}_n$ is an identity arrow) is proved in [**9**] and [**10**]. This proof is obtained via an isomorphic representation of *Br* in the category *Rel*, whose objects are the finite ordinals and whose arrows are all the relations between these objects. Composition in *Rel* is the ordinary composition of relations. A direct formal proof would be more involved, though what we have to prove is rather clear if we represent Brauerian split equivalences geometrically (as this is done in [**3**], [**13**], and also in categories of tangles; see [**16**, Chapter 12] and references therein).

For example, for $R \subseteq (3^s \cup 9^t)^2$ and $P \subseteq (9^s \cup 1^t)^2$ such that

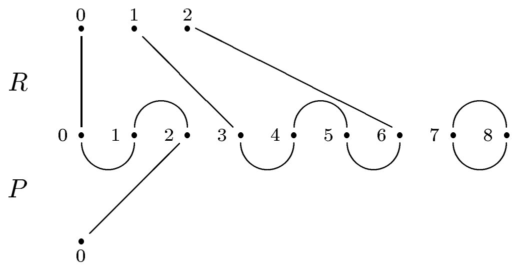
$$\mathrm{part}(R) = \{\{0_s, 0_t\}, \{1_s, 3_t\}, \{2_s, 6_t\}\} \cup \{\{n_t, (n+1)_t\} \mid n \in \{1, 4, 7\}\},$$

$$\mathrm{part}(P) = \{\{2_s, 0_t\}\} \cup \{\{n_s, (n+1)_s\} \mid n \in \{0, 3, 5, 7\}\},$$

the composition $P * R \subseteq (3^s \cup 1^t)^2$, for which we have

$$\mathrm{part}(P * R) = \{\{0_s, 0_t\}, \{1_s, 2_s\}\},$$

is obtained from the following diagram:



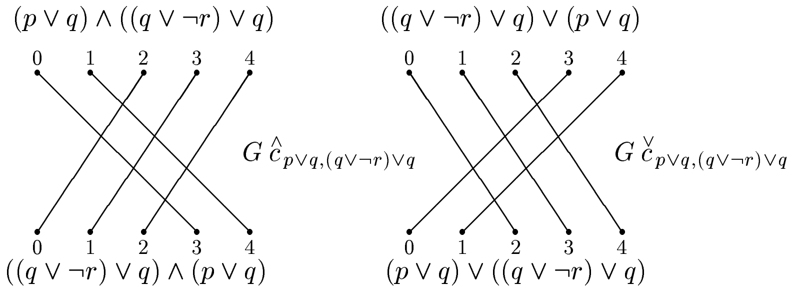
Every bijection f from X^s to Y^t corresponds to a Brauerian split equivalence $R\colon X \vdash Y$ such that the members of $\mathrm{part}(R)$ are of the form $\{u, f(u)\}$. The composition of such Brauerian split equivalences, which correspond to bijections, is then a simple matter: it amounts to composition of these bijections. If in *Br* we keep as arrows only such Brauerian split equivalences, then we obtain a subcategory of *Br* isomorphic to the category *Bij* whose objects are again the finite ordinals and whose arrows are the bijections between these objects. The category *Bij* is a subcategory of the category *Rel* (which played an important role in [**11**]), whose objects are the finite ordinals and whose arrows are all the relations between these objects. Composition in *Bij* and *Rel* is the ordinary composition of relations. The category *Rel* (which played an important role in [**11**]) is isomorphic to a subcategory of the category whose arrows are split relations of finite ordinals, of whom *Br* is also a subcategory.

We define a functor G from $\mathbf{PN}^{\neg}$ to *Br* in the following way. On objects, we stipulate that GA is the number of occurrences of letters in A. (If A has $n = \{0, 1, \ldots, n-1\}$ occurrences of letters, then the first occurrence corresponds to 0, the second to 1, etc.) On arrows, we have first that $G\alpha$ is an identity arrow of *Br* for α being $\mathbf{1}_A$, $\overset{\xi}{b}{}^{\rightarrow}_{A,B,C}$, $\overset{\xi}{b}{}^{\leftarrow}_{A,B,C}$ and $d_{A,B,C}$, where $\xi \in \{\wedge, \vee\}$.

Next, for $i, j \in \{s, t\}$, we have that $\{m_i, n_j\}$ belongs to $\mathrm{part}(G\,\hat{c}_{A,B})$ iff $\{n_i, m_j\}$ belongs to $\mathrm{part}(G\,\overset{\vee}{c}_{A,B})$, iff i is s and j is t, while $m, n < GA+GB$ and

$$(m-n-GA)(m-n+GB) = 0.$$

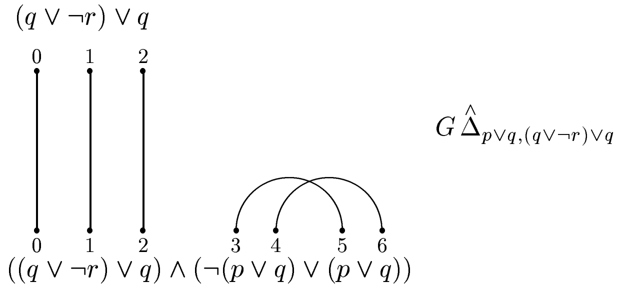
In the following example, we have $G(p \vee q) = 2 = \{0, 1\}$ and $G((q \vee \neg r) \vee q) = 3 = \{0, 1, 2\}$, and we have the diagrams



We have that $\{m_i, n_j\}$ belongs to $\mathrm{part}(G\,\hat{\Delta}_{B,A})$ iff either

> i is s and j is t, while $m, n < GA$ and $m = n$, or
>
> i and j are both t, while $m, n \in \{GA, \dots, GA+2GB-1\}$ and $|m-n| = GB$.

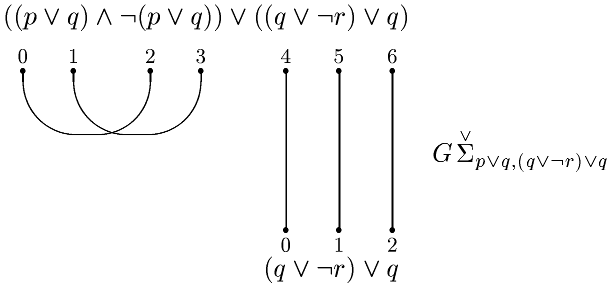
In the following example, for A being $(q \vee \neg r) \vee q$ and B being $p \vee q$, we have



We have that $\{m_i, n_j\}$ belongs to $\mathrm{part}(G\,\check{\Sigma}_{B,A})$ iff either

> i is s and j is t, while $m \in \{2GB, \dots, 2GB+GA-1\}$, $n < GA$ and $m-2GB = n$, or
>
> i and j are both s, while $m, n < 2GB$ and $|m-n| = GB$.

For A and B being as in the previous example, we have



Let $G(f \circ g) = Gf * Gg$. To define $G(f\ \xi\ g)$, for $\xi \in \{\wedge, \vee\}$, we need an auxiliary notion.

Suppose b_X is a bijection from X to X_1 and b_Y a bijection from Y to Y_1. Then for $R \subseteq (X^s \cup Y^t)^2$ we define $R^{b_X}_{b_Y} \subseteq (X_1^s \cup Y_1^t)^2$ by

$$(u_i, v_j) \in R^{b_X}_{b_Y} \quad \text{iff} \quad (i(b_U^{-1}(u)), j(b_V^{-1}(v))) \in R,$$

where $(i, U), (j, V) \in \{(s, X), (t, Y)\}$.

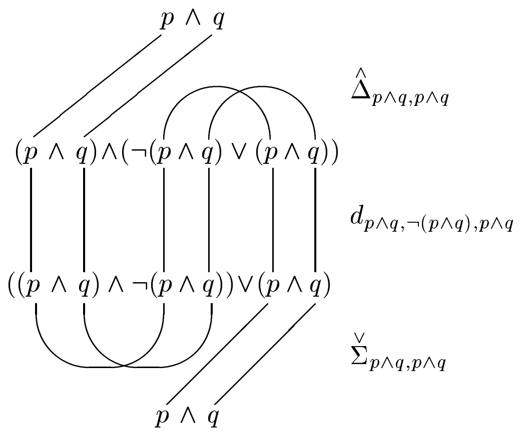
If $f\colon A \vdash D$ and $g\colon B \vdash E$, then for $\xi \in \{\wedge, \vee\}$ the set of ordered pairs $G(f \xi g)$ is

$$Gf \cup Gg^{+GA}_{+GD}$$

where $+GA$ is the bijection from GB to $\{n{+}GA \mid n \in GB\}$ that assigns $n{+}GA$ to n, and $+GD$ is the bijection from GE to $\{n{+}GD \mid n \in GE\}$ that assigns $n{+}GD$ to n.

It is not difficult to check that G so defined is indeed a functor from $\mathbf{PN}^{\neg}$ to *Br*. For that, we determine by induction on the length of derivation that for every equation $f = g$ of $\mathbf{PN}^{\neg}$ we have $Gf = Gg$ in *Br*.

Consider, for example, the following diagram, which illustrates an instance of $(\check{\Sigma}\hat{\Delta})$:



This diagram shows that the equation $(\check{\Sigma}\hat{\Delta})$, as well as the equation $(\check{\Sigma}'\hat{\Delta}')$, which is illustrated by analogous diagrams, is related to triangular equations of adjunctions (cf. [**8**, Section 4.10]). The triangular equations of adjunctions are essentially about "straightening a serpentine", and this straightening is based on planar ambient isotopies of knot theory (cf. [**4**, Section 1.A],).

We have shown by this induction that *Br* is a proof-net category, and the existence of a structure-preserving functor G from $\mathbf{PN}^{\neg}$ to *Br* follows from the freedom of $\mathbf{PN}^{\neg}$.

We can define analogously to G a functor, which we also call G, from the category $\mathbf{DS}$ to *Br*. We just omit from the definition of G above the clauses involving $\hat{\Delta}_{B,A}$ and $\check{\Sigma}_{B,A}$. The image of $\mathbf{DS}$ by G in *Br* is the subcategory of

Br isomorphic to Bij, which we mentioned above. The following is proved in [**11**, Section 7.6].

DS COHERENCE. *The functor G from* **DS** *to Br is faithful.*

It follows immediately from this coherence result that **DS** is isomorphic to a subcategory of $\mathbf{PN}^{\neg}$ (cf. [**11**, Section 14.4]).

Up to the end of §8 we will be occupied with proving the following.

$\mathbf{PN}^{\neg}$ COHERENCE. *The functor G from* $\mathbf{PN}^{\neg}$ *to Br is faithful.*

For this proof, we must deal first with some preliminary matters.

5. Some properties of DS

In this section we will prove some results about the category **DS**, which we will be use to ascertain that particular equations hold in $\mathbf{PN}^{\neg}$. We need these results also for the proof of $\mathbf{PN}^{\neg}$ Coherence.

First we introduce a definition. Suppose x is the n-th occurrence of a letter (counting from the left) in a formula A of $\mathcal{L}_{\neg,\wedge,\vee}$, and y is the m-th occurrence of the same letter in a formula B of $\mathcal{L}_{\neg,\wedge,\vee}$. Then we say that x and y are *linked* in an arrow $f\colon A \vdash B$ of $\mathbf{PN}^{\neg}$ when in the partition $\mathrm{part}(Gf)$ we have $\{(n-1)_s, (m-1)_t\}$ as a member. (Note that to find the n-th occurrence, we count starting from 1, but the ordinal $n > 0$ is $\{0, \ldots, n-1\}$.) We have an analogous definition of linked occurrences of the same letter for **DS**: we just replace $\mathcal{L}_{\neg,\wedge,\vee}$ by $\mathcal{L}_{\wedge,\vee}$ and $\mathbf{PN}^{\neg}$ by **DS**.

It is easy to established by induction on the complexity of f that for every arrow term $f\colon A \vdash B$ of **DS** we have $GA = GB$. Moreover, every occurrence of letter in A is linked to exactly one occurrence of the same letter in B, and vice versa. This is related to the fact that every arrow term $f\colon A \vdash B$ of **DS** may be obtained by substituting letters for letters out of an arrow term $f'\colon A' \vdash B'$ of **DS** such that every letter occurs in A' at most once, and the same for B' (see [**11**, Sections 3.3 and 7.6]).

Suppose for Lemmata 1D and 2D below that $f\colon A \vdash B$ is an arrow term of **DS** such that A has a subformula D in which $\wedge$ does not occur and B has a subformula D' in which $\wedge$ does not occur, and suppose that every occurrence of a letter in D is linked to an occurrence of a letter in D' and vice versa. Then we can prove the following.

LEMMA 1D. *The source A of f is D iff the target B of f is D'.*

This follows from the fact, noted above, that $GA = GB$. The arrow term f in this case can have as subterms that are primitive arrow terms only arrow terms of the forms $\mathbf{1}_E$, $\overset{\vee}{b}{}^{\rightarrow}_{E,F,G}$, $\overset{\vee}{b}{}^{\leftarrow}_{E,F,G}$ or $\overset{\vee}{c}_{E,F}$. We also have the following.

LEMMA 2D. *If $D \wedge A'$ or $A' \wedge D$ is a subformula of A, then $D' \wedge B'$ or $B' \wedge D'$ is a subformula of B for some B'.*

This is easily proved by induction on the complexity of the arrow term f, with the help of Lemma 1D.

Suppose for Lemmata 1C and 2C below that $f\colon A \vdash B$ is an arrow term of **DS** such that B has a subformula C in which $\vee$ does not occur and A has a subformula C' in which $\vee$ does not occur, and suppose that every occurrence of a letter in C is linked to an occurrence of a letter in C' and vice versa. Then we can prove the following duals of Lemmata 1D and 2D, in an analogous manner.

LEMMA 1C. *The target B of f is C iff the source A of f is C'.*

LEMMA 2C. *If $C \vee B'$ or $B' \vee C$ is a subformula of B, then $C' \vee A'$ or $A' \vee C'$ is a subformula of A for some A'.*

Suppose for the following lemma, which is a corollary of either Lemma 2D or Lemma 2C, that $f\colon A \vdash B$ is an arrow term of **DS** such that an occurrence x of a letter p in A is linked to an occurrence y of p in B.

LEMMA 2. *It is impossible that A has a subformula $x \wedge A'$ or $A' \wedge x$ and B has a subformula $y \vee B'$ or $B' \vee y$.*

Suppose for Lemmata 3D, 3C, 3 and 4 below that $f\colon A \vdash B$ is an arrow term of **DS**, and for $i \in \{1,2\}$ let x_i in A and y_i in B be occurrences of the letter p_i linked in f (here p_1 and p_2 may also be the same letter).

LEMMA 3D. *If in A we have a subformula $A_1 \vee A_2$ such that x_i occurs in A_i, then in B we have a subformula $B_1 \vee B_2$ or $B_2 \vee B_1$ such that y_i occurs in B_i.*

This is easily proved by induction on the complexity of the arrow term f. We prove analogously the following.

LEMMA 3C. *If in B we have a subformula $B_1 \wedge B_2$ such that y_i occurs in B_i, then in A we have a subformula $A_1 \wedge A_2$ or $A_2 \wedge A_1$ such that x_i occurs in A_i.*

As a corollary of either Lemma 3D or Lemma 3C we have the following.

LEMMA 3. *It is impossible that A has a subformula $x_1 \vee x_2$ or $x_2 \vee x_1$ and B has a subformula $y_1 \wedge y_2$ or $y_2 \wedge y_1$.*

The following lemma, dual to Lemma 3, is a corollary of Lemma 2.

LEMMA 4. *It is impossible that A has a subformula $x_1 \wedge x_2$ or $x_2 \wedge x_1$ and B has a subformula $y_1 \vee y_2$ or $y_2 \vee y_1$.*

Lemma 3 is related to the acyclicity condition of proof nets, while Lemma 4 is related to the connectedness condition (see [**7**]).

Next we can prove the following lemma.

p-q-r LEMMA. *Let $f\colon A \vdash B$ be an arrow of* **DS**, *let x_i for $i \in \{1,2,3\}$ be occurrences of the letters p, q and r, respectively, in A, and let y_i be occurrences of the letters p, q and r, respectively, in B, such that x_i and y_i are linked in f. Let, moreover, $x_2 \vee x_3$ be a subformula of A and $y_1 \wedge y_2$ a subformula of B. Then there is a $d_{p,q,r}$-term $h\colon A' \vdash B'$ such that x_i' are occurrences of the letters p, q and r, respectively, in the source $p \wedge (q \vee r)$ of the head of h and y_i' are occurrences of the letters p, q and r, respectively, in the target $(p \wedge q) \vee r$ of the head of h, such that for some arrows $f_x\colon A \vdash A'$ and $f_y\colon B' \vdash B$ of* **DS** *we have $f = f_y \circ h \circ f_x$ in* **DS**, *and x_i is linked to x_i' in f_x, while y_i' is linked to y_i in f_y.*

PROOF. The proof of this lemma, of which we give just a sketch, relies on a cut-elimination and related results of [**11**, Sections 7.7-8]. We first find in the category **GDS** introduced in [**11**, Section 7.7] a cut-free Gentzen term $f'\colon X \vdash Y$, which corresponds to f, by the relationship that exists between **DS** and **GDS**. According to the equations at the beginning of Section 7.8 of [**11**], which are used for the proof of the Invertibility Lemmata in the same section, in **GDS** we have the equation $f' = f''$ for a Gentzen term f'' that has as a subterm either $\wedge_{p,q}(\mathbf{1}_p, \vee_{q,r}(\mathbf{1}_q, \mathbf{1}_r))$ or $\vee_{q,r}(\wedge_{p,q}(\mathbf{1}_p, \mathbf{1}_q), \mathbf{1}_r)$ both of type $p \wedge (q \vee r) \vdash (p \wedge q) \vee r$. By the relationship that exists between **DS** and **GDS**, we can find starting from f'' an arrow term $f_y \circ h \circ f_x$ equal to f in **DS**, which satisfies the conditions of the lemma. $\dashv$

The full force of the Cut-Elimination Theorem of Section 7.7 of [**11**] is not essential for this proof, but applying this theorem simplifies the proof.

6. The category PN

We now introduce a category called **PN**, which is equivalent to $\mathbf{PN}^{\neg}$. In the objects of **PN**, the negation connective $\neg$ will be prefixed only to letters, and hence $\overset{\wedge}{\Delta}_{B,A}$ and $\overset{\vee}{\Sigma}_{B,A}$ will be primitive only for the crown index B being a letter. Here is the formal definition of **PN**.

For $\mathcal{P}$ being the set of letters that we used to generate $\mathcal{L}_{\wedge,\vee}$ and $\mathcal{L}_{\neg,\wedge,\vee}$ in §§2-3, let $\mathcal{P}^{\neg}$ be the set $\{\neg p \mid p \in \mathcal{P}\}$. The objects of **PN** are the formulae of the propositional language $\mathcal{L}^{\neg p}_{\wedge,\vee}$ generated from $\mathcal{P} \cup \mathcal{P}^{\neg}$ with the binary connectives $\wedge$ and $\vee$. To define the arrow terms of **PN**, in the inductive definition we had for the arrow terms of **DS** we assume in addition that for every formula A of $\mathcal{L}^{\neg p}_{\wedge,\vee}$ and every letter p

$$\overset{\wedge}{\Delta}_{p,A}\colon A \vdash A \wedge (\neg p \vee p),$$

$$\overset{\vee}{\Sigma}_{p,A}\colon (p \wedge \neg p) \vee A \vdash A,$$

are primitive arrow terms of **PN**.

To define the arrows of **PN**, we assume as additional equations in the inductive definition we had for the equations of **DS** the **PN** equations of §3 restricted to the arrow terms $\overset{\wedge}{\Delta}_{p,A}$ and $\overset{\vee}{\Sigma}_{p,A}$. This means that in ($\overset{\wedge}{\Delta}$ *nat*) and ($\overset{\vee}{\Sigma}$ *nat*) the crown index B will be p, in $(\hat{b}\hat{\Delta})$ and $(\check{b}\check{\Sigma})$ the crown index C will be p, and in $(d\hat{\Sigma})$, $(d\check{\Delta})$, $(\check{\Sigma}\hat{\Delta})$ and $(\check{\Sigma}'\hat{\Delta}')$ the crown index A will be p. We define $\hat{\Sigma}_{p,A}$, $\check{\Delta}_{p,A}$, $\hat{\Delta}'_{p,A}$, $\check{\Sigma}'_{p,A}$, $\hat{\Sigma}'_{p,A}$ and $\check{\Delta}'_{p,A}$ for **PN** as they were defined in $\mathbf{PN}^{\neg}$ in terms of $\hat{\Delta}_{p,A}$ and $\check{\Sigma}_{p,A}$.

The following equations of **PN**, and hence also of $\mathbf{PN}^{\neg}$, which we call *stem-increasing* equations, enable us to have in developed arrow terms only $\hat{\Delta}_{A,B}$-terms and $\check{\Sigma}_{A,B}$-terms that coincide with their heads:

$$
\begin{array}{lll}
(\mathbf{1}\wedge\hat{\Delta}) & \mathbf{1}_A\wedge\hat{\Delta}_{p,B} = \hat{b}^{\leftarrow}_{A,B,\neg p\vee p}\circ\hat{\Delta}_{p,A\wedge B}, & \text{by } (\hat{b}\hat{\Delta}),\\
(\hat{\Delta}\wedge\mathbf{1}) & \hat{\Delta}_{p,B}\wedge\mathbf{1}_A = \hat{c}_{A,B\wedge(\neg p\vee p)}\circ\hat{b}^{\leftarrow}_{A,B,\neg p\vee p}\circ(\hat{c}_{B,A}\wedge\mathbf{1}_{\neg p\vee p})\circ\hat{\Delta}_{p,B\wedge A}, & \\
& \qquad \text{by } (\hat{c}\hat{c}),\ (\hat{c}\ nat),\ (\mathbf{1}\wedge\hat{\Delta}) \text{ and } (\hat{\Delta}\ nat), & \\
(\mathbf{1}\vee\hat{\Delta}) & \mathbf{1}_A\vee\hat{\Delta}_{p,B} = d^R_{A,B,\neg p\vee p}\circ\hat{\Delta}_{p,A\vee B}, & \text{by } (d^R\hat{\Delta}),\\
(\hat{\Delta}\vee\mathbf{1}) & \hat{\Delta}_{p,B}\vee\mathbf{1}_A = \check{c}_{B\wedge(\neg p\vee p),A}\circ d^R_{A,B,\neg p\vee p}\circ(\check{c}_{A,B}\wedge\mathbf{1}_{\neg p\vee p})\circ\hat{\Delta}_{p,B\vee A}, & \\
& \qquad \text{by } (\check{c}\check{c}),\ (\check{c}\ nat),\ (\mathbf{1}\vee\hat{\Delta}) \text{ and } (\hat{\Delta}\ nat), & \\
(\check{\Sigma}\vee\mathbf{1}) & \check{\Sigma}_{p,B}\vee\mathbf{1}_A = \check{\Sigma}_{p,B\vee A}\circ\check{b}^{\leftarrow}_{p\wedge\neg p,B,A}, & \text{by } (\check{b}\check{\Sigma}),\\
(\mathbf{1}\vee\check{\Sigma}) & \mathbf{1}_A\vee\check{\Sigma}_{p,B} = \check{\Sigma}_{p,A\vee B}\circ(\mathbf{1}_{p\wedge\neg p}\vee\check{c}_{A,B})\circ\check{b}^{\leftarrow}_{p\wedge\neg p,B,A}\circ\check{c}_{(p\wedge\neg p)\vee B,A}, & \\
& \qquad \text{by } (\check{c}\check{c}),\ (\check{c}\ nat),\ (\check{\Sigma}\vee\mathbf{1}) \text{ and } (\check{\Sigma}\ nat), & \\
(\check{\Sigma}\wedge\mathbf{1}) & \check{\Sigma}_{p,B}\wedge\mathbf{1}_A = \check{\Sigma}_{p,B\wedge A}\circ d^R_{p\wedge\neg p,B,A}, & \text{by } (d^R\check{\Sigma}),\\
(\mathbf{1}\wedge\check{\Sigma}) & \mathbf{1}_A\wedge\check{\Sigma}_{p,B} = \check{\Sigma}_{p,A\wedge B}\circ(\mathbf{1}_{p\wedge\neg p}\vee\hat{c}_{B,A})\circ d^R_{p\wedge\neg p,B,A}\circ\hat{c}_{A,(p\wedge\neg p)\vee B}, & \\
& \qquad \text{by } (\hat{c}\hat{c}),\ (\hat{c}\ nat),\ (\check{\Sigma}\wedge\mathbf{1}) \text{ and } (\check{\Sigma}\ nat). &
\end{array}
$$

Note that in the stem-increasing equations the stem index B of $\hat{\Delta}$ and $\check{\Sigma}$ becomes more complex on the right-hand sides, whereas the crown index p does not change. We have analogous stem-increasing equations for $\hat{\Sigma}$, $\hat{\Delta}'$, $\hat{\Sigma}'$, $\check{\Delta}$, $\check{\Sigma}'$ and $\check{\Delta}'$.

We will next prove several lemmata concerning **PN**, which we will find useful for calculations later on. For these lemmata we need the following.

Let $\mathbf{DS}^{\neg p}$ be the category defined as **DS** save that it is generated not by $\mathcal{P}$, but by $\mathcal{P}\cup\mathcal{P}^{\neg}$. So the objects of $\mathbf{DS}^{\neg p}$ are formulae of $\mathcal{L}^{\neg p}_{\wedge,\vee}$, i.e. the objects of **PN**. For A and B formulae of $\mathcal{L}^{\neg p}_{\wedge,\vee}$, we define when an occurrence of p in A is linked to an occurrence of p in B in an arrow $f\colon A\vdash B$ of $\mathbf{DS}^{\neg p}$ analogously to what we had at the beginning of the preceding section.

Let $\overset{\xi}{\Xi}$ for $\xi\in\{\wedge,\vee\}$ stand for either $\overset{\xi}{\Delta}$, or $\overset{\xi}{\Delta}{}'$, or $\overset{\xi}{\Sigma}$, or $\overset{\xi}{\Sigma}{}'$, and let a $\overset{\xi}{\Xi}_{B,A}$-*term* be defined as a β-term in §3, save that β is replaced by $\overset{\xi}{\Xi}_{B,A}$. We use also Θ as a variable alternative to Ξ. Then we have the following.

$\hat{\Xi}$-PERMUTATION LEMMA. *Let* $g\colon C \vdash D$ *be a* $\hat{\Xi}_{p,B}$*-term of* $\mathbf{PN}$ *such that* x_1 *and* $\neg x_2$ *are respectively the occurrences within* D *of* p *and* $\neg p$ *in the crown of the head* $\hat{\Xi}_{p,B}$ *of* g*, and let* $f\colon D \vdash E$ *be an arrow term of* $\mathbf{DS}^{\neg p}$ *such that we have an occurrence* y_1 *of* p *and an occurrence* $\neg y_2$ *of* $\neg p$ *within a subformula of* E *of the form* $y_1 \vee \neg y_2$ *or* $\neg y_2 \vee y_1$*, and* x_i *is linked to* y_i *for* $i \in \{1,2\}$ *in* f*. Then there is a* $\hat{\Theta}_{p,B'}$*-term* $g'\colon D' \vdash E$ *of* $\mathbf{PN}$ *the crown of whose head is* $y_1 \vee \neg y_2$ *or* $\neg y_2 \vee y_1$*, and there is an arrow term* $f'\colon C \vdash D'$ *of* $\mathbf{DS}^{\neg p}$ *such that in* $\mathbf{PN}$ *we have* $f \circ g = g' \circ f'$*.*

PROOF. By the Development Lemma we can assume that f is a developed arrow term, and then it is enough to consider the case when f is either a β-term for β a primitive arrow term of $\mathbf{DS}^{\neg p}$ or f is $\mathbf{1}_E$. Note that in the developed arrow term $f_n \circ \ldots \circ f_1$, which is equal to f, we have that f_1 is $\mathbf{1}_D$, and that f_2, if it exists, cannot be a $d_{B,p,\neg p}$-term or a $d_{B,\neg p,p}$-term such that x_1 and $\neg x_2$ are the occurrences of p and $\neg p$ in the right conjunct of the source $B \wedge (\neg p \vee p)$ or $B \wedge (p \vee \neg p)$ of the head of f_2. Otherwise, in the target of the head of f_2 we would obtain as the left disjunct $B \wedge \neg p$ or $B \wedge p$, which together with Lemma 2 would contradict the conditions put on f, and hence also on $f_n \circ \ldots \circ f_1$, in the formulation of the $\hat{\Xi}$-Permutation Lemma.

The case when f is $\mathbf{1}_E$ is trivial, and there are also many easy cases settled by bifunctorial and naturality equations. The remaining, more interesting, cases are settled by the following equations of $\mathbf{PN}$:

$$
\begin{array}{ll}
\hat{b}^{\rightarrow}_{A,B,\neg p\vee p} \circ (\mathbf{1}_A \wedge \hat{\Delta}_{p,B}) = \hat{\Delta}_{p,A\wedge B}, & \text{by } (\hat{b}\hat{\Delta}),\\
\hat{b}^{\leftarrow}_{B_1,B_2,\neg p\vee p} \circ \hat{\Delta}_{p,B_1\wedge B_2} = \mathbf{1}_{B_1} \wedge \hat{\Delta}_{p,B_2}, & \text{by } (\hat{b}\hat{\Delta}),\\
\hat{b}^{\rightarrow}_{A,\neg p\vee p,B} \circ (\mathbf{1}_A \wedge \hat{\Sigma}_{p,B}) = \hat{\Delta}_{p,A} \wedge \mathbf{1}_B, & \text{by } (\hat{b}\hat{\Delta}\hat{\Sigma}),\\
\hat{b}^{\leftarrow}_{B,\neg p\vee p,A} \circ (\hat{\Delta}_{p,B} \wedge \mathbf{1}_A) = \mathbf{1}_B \wedge \hat{\Sigma}_{p,A}, & \text{by } (\hat{b}\hat{\Delta}\hat{\Sigma}),\\
\hat{b}^{\rightarrow}_{\neg p\vee p,B_1,B_2} \circ \hat{\Sigma}_{p,B_1\wedge B_2} = \hat{\Sigma}_{p,B_1} \wedge \mathbf{1}_{B_2}, & \text{by } (\hat{b}\hat{\Sigma}),\\
\hat{b}^{\leftarrow}_{\neg p\vee p,B,A} \circ (\hat{\Sigma}_{p,B} \wedge \mathbf{1}_A) = \hat{\Sigma}_{p,B\wedge A}, & \text{by } (\hat{b}\hat{\Sigma}),\\
\hat{c}_{B,\neg p\vee p} \circ \hat{\Delta}_{p,B} = \hat{\Sigma}_{p,B}, & \text{by definition},\\
\hat{c}_{\neg p\vee p,B} \circ \hat{\Sigma}_{p,B} = \hat{\Delta}_{p,B}, & \text{by definition and } (\hat{c}\hat{c}),\\
(\mathbf{1}_B \wedge \check{c}_{p,\neg p}) \circ \hat{\Delta}_{p,B} = \hat{\Delta}'_{p,B}, & \text{by definition},\\
(\check{c}_{p,\neg p} \wedge \mathbf{1}_B) \circ \hat{\Sigma}_{p,B} = \hat{\Sigma}'_{p,B}, & \text{by definition and } (\hat{c}\ nat),\\
d_{\neg p\vee p,B_1,B_2} \circ \hat{\Sigma}_{p,B_1\vee B_2} = \hat{\Sigma}_{p,B_1} \vee \mathbf{1}_{B_2}, & \text{by } (d\hat{\Sigma}).
\end{array}
$$

Besides these equations, we have analogous equations where $\neg p \vee p$ is replaced by $p \vee \neg p$, while $\hat{\Delta}$ and $\hat{\Sigma}$ are replaced by $\hat{\Delta}'$ and $\hat{\Sigma}'$ respectively, and vice versa. $\dashv$

We prove analogously the following dual of the preceding lemma.

$\overset{\vee}{\Xi}$-PERMUTATION LEMMA. *Let $g\colon D\vdash C$ be a $\overset{\vee}{\Xi}_{p,B}$-term of* **PN** *such that x_1 and $\neg x_2$ are respectively the occurrences within D of p and $\neg p$ in the crown of the head $\overset{\vee}{\Xi}_{p,B}$ of g, and let $f\colon E\vdash D$ be an arrow term of* $\mathbf{DS}^{\neg p}$ *such that we have an occurrence y_1 of p and an occurrence $\neg y_2$ of $\neg p$ within a subformula of E of the form $y_1\wedge\neg y_2$ or $\neg y_2\wedge y_1$, and y_i is linked to x_i for $i\in\{1,2\}$ in f. Then there is a $\overset{\vee}{\Theta}_{p,B'}$-term $g'\colon E\vdash D'$ of* **PN** *the crown of whose head is $y_1\wedge\neg y_2$ or $\neg y_2\wedge y_1$, and there is an arrow term $f'\colon D'\vdash C$ of* $\mathbf{DS}^{\neg p}$ *such that in* **PN** *we have $g\circ f=f'\circ g'$.*

Next we prove the following lemma, which involves the p-q-r Lemma of the preceding section.

p-$\neg p$-p LEMMA. *Let x_1, $\neg x_2$ and x_3 be occurrences of p, $\neg p$ and p, respectively, in a formula A of $\mathcal{L}^{\neg p}_{\wedge,\vee}$, and let y_1, $\neg y_2$ and y_3 be occurrences of p, $\neg p$ and p, respectively in a formula B of $\mathcal{L}^{\neg p}_{\wedge,\vee}$. Let $\neg x_2\vee x_3$ or $x_3\vee\neg x_2$ be a subformula of A and $y_1\wedge\neg y_2$ or $\neg y_2\wedge y_1$ a subformula of B. Let $g_1\colon A'\vdash A$ be a $\overset{\wedge}{\Xi}_{p,C}$-term of* **PN** *such that $\neg x_2\vee x_3$ or $x_3\vee\neg x_2$ is the crown of the head of g_1, let $g_2\colon B\vdash B'$ be a $\overset{\vee}{\Theta}_{p,D}$-term of* **PN** *such that $y_1\wedge\neg y_2$ or $\neg y_2\wedge y_1$ is the crown of the head of g_2, and let $f\colon A\vdash B$ be an arrow term of* $\mathbf{DS}^{\neg p}$ *such that x_i and y_i are linked in f for $i\in\{1,2,3\}$. Then $g_2\circ f\circ g_1$ is equal in* **PN** *to an arrow term of* $\mathbf{DS}^{\neg p}$.

PROOF. By the p-q-r Lemma, $f\colon A\vdash B$ is equal in $\mathbf{DS}^{\neg p}$, and hence also in **PN**, to an arrow term of the form $f_y\circ h\circ f_x$, where h is a $d_{p,\neg p,p}$-term, and the other conditions of the p-q-r Lemma are satisfied. So in **PN** we have

$$g_2\circ f\circ g_1=g_2\circ f_y\circ h\circ f_x\circ g_1=f'_y\circ g'_2\circ h\circ g'_1\circ f'_x,$$

by the $\overset{\xi}{\Xi}$-Permutation Lemmata above. Here the head of g'_1 must be $\overset{\wedge}{\Delta}_{p,p}$: $p\vdash p\wedge(\neg p\vee p)$, the head of h is $d_{p,\neg p,p}\colon p\wedge(\neg p\vee p)\vdash(p\wedge\neg p)\vee p$, and the head of g'_2 must be $\overset{\vee}{\Sigma}_{p,p}$: $(p\wedge\neg p)\vee p\vdash p$. By applying $(\overset{\vee}{\Sigma}\overset{\wedge}{\Delta})$, and perhaps bifunctorial equations, we obtain that $g'_2\circ h\circ g'_1$ is equal in **PN** to an arrow term of the form $\mathbf{1}_A$, and hence we have $g_2\circ f\circ g_1=f'_y\circ f'_x$ in **PN**, which proves the lemma. ⊣

To give an example of the application of the p-$\neg p$-p Lemma, consider the diagram in Figure 1. This diagram corresponds to $G(\overset{\vee}{\Sigma}_{q,p\wedge q}\circ h\circ\overset{\wedge}{\Delta}_{q,p\wedge q})$ for an arrow term h of **PN**, which is of the form $g_2\circ f\circ g_1$ for the arrow term g_1 being $\mathbf{1}_{p\wedge q}\wedge(\mathbf{1}_{\neg q}\vee\overset{\wedge}{\Sigma}_{p,q})$, the arrow term g_2 being $(\mathbf{1}_q\wedge\overset{\vee}{\Sigma}_{p,\neg q})\vee\mathbf{1}_{p\wedge q}$ and f an arrow term of $\mathbf{DS}^{\neg p}$. Then by applying the p-$\neg p$-p Lemma we obtain an arrow term f' of $\mathbf{DS}^{\neg p}$ equal to $g_2\circ f\circ g_1$ in **PN**, and next by applying the p-$\neg p$-p Lemma (as a matter of fact, the q-$\neg q$-q Lemma), we obtain an arrow term h' of $\mathbf{DS}^{\neg p}$ equal to $\overset{\vee}{\Sigma}_{q,p\wedge q}\circ f'\circ\overset{\wedge}{\Delta}_{q,p\wedge q}$ in **PN**. By **DS** Coherence of §4, we may conclude that h', and hence also $\overset{\vee}{\Sigma}_{q,p\wedge q}\circ h\circ\overset{\wedge}{\Delta}_{q,p\wedge q}$, is equal to $\mathbf{1}_{p\wedge q}$ in **PN**.

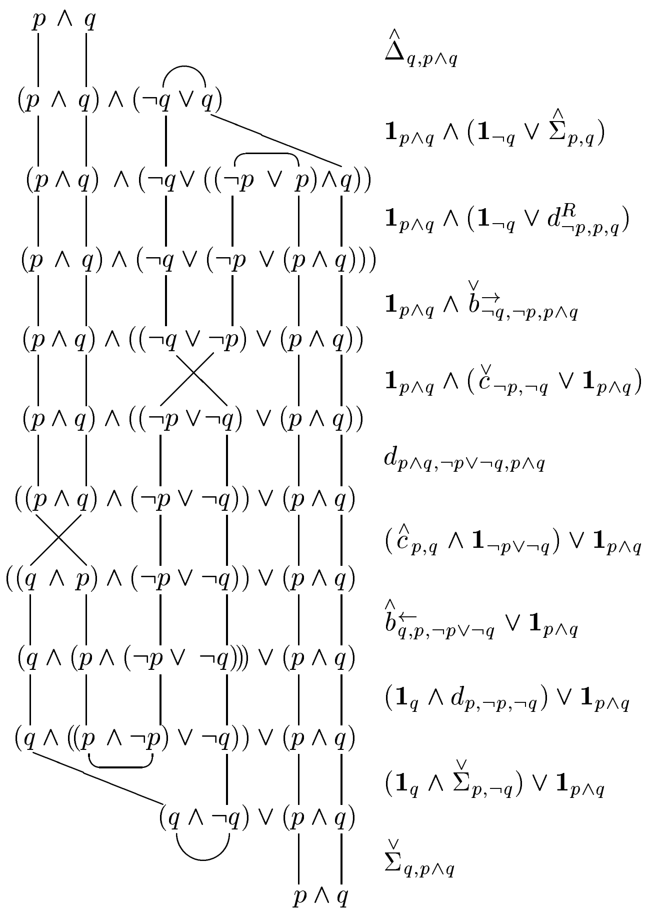


FIGURE 1

Here is a lemma analogous to the p-$\neg p$-p Lemma.

$\neg p$-p-$\neg p$ LEMMA. *Let $\neg x_1$, x_2 and $\neg x_3$ be occurrences of $\neg p$, p and $\neg p$, respectively, in a formula A of $\mathcal{L}^{\neg p}_{\wedge,\vee}$, and let $\neg y_1$, y_2 and $\neg y_3$ be occurrences of $\neg p$, p and $\neg p$, respectively, in a formula B of $\mathcal{L}^{\neg p}_{\wedge,\vee}$. Let $g_1 : A' \vdash A$ be a $\hat{\Xi}_{p,C}$-term of* **PN** *such that $x_2 \vee \neg x_3$ or $\neg x_3 \vee x_2$ is the crown of the head of g_1, let $g_2 : B \vdash B'$ be a $\check{\Theta}_{p,D}$-term of* **PN** *such that $\neg y_1 \wedge y_2$ or $y_2 \wedge \neg y_1$ is the crown of the head of g_2, and let $f : A \vdash B$ be an arrow term of $\mathbf{DS}^{\neg p}$ such that x_i and y_i are linked in f for $i \in \{1, 2, 3\}$. Then $g_2 \circ f \circ g_1$ is equal in* **PN** *to an arrow term of $\mathbf{DS}^{\neg p}$.*

To prove this lemma we proceed as for the p-$\neg p$-p Lemma, relying on the equation $(\check{\Sigma}'\hat{\Delta}')$ of **PN**.

7. The equivalence of $\mathbf{PN}^{\neg}$ and $\mathbf{PN}$

In this section we show that the categories $\mathbf{PN}^{\neg}$ and $\mathbf{PN}$ are equivalent categories. We define inductively a functor F from the category $\mathbf{PN}^{\neg}$ to $\mathbf{PN}$ in the following manner. On objects we have

$$Fp = p, \quad \text{for } p \text{ a letter},$$
$$F(A\,\xi\,B) = FA\,\xi\,FB, \quad \text{for } \xi \in \{\wedge, \vee\},$$
$$F\neg p = \neg p, \quad \text{for } p \text{ a letter},$$
$$F\neg\neg A = FA,$$
$$F\neg(A \wedge B) = F\neg A \vee F\neg B,$$
$$F\neg(A \vee B) = F\neg A \wedge F\neg B.$$

On arrows we have

$$F\alpha_{A_1,\ldots,A_n} = \alpha_{FA_1,\ldots,FA_n},$$

for $\alpha_{A_1,\ldots,A_n}$ being $\mathbf{1}_A$, $\overset{\xi}{b}{}^{\rightarrow}_{A,B,C}$, $\overset{\xi}{b}{}^{\leftarrow}_{A,B,C}$, $\overset{\xi}{c}_{A,B}$ or $d_{A,B,C}$ where $\xi \in \{\wedge, \vee\}$,

$$F\overset{\wedge}{\Delta}_{p,A} = \overset{\wedge}{\Delta}_{p,FA} \colon FA \vdash FA \wedge (\neg p \vee p),$$
$$F\overset{\vee}{\Sigma}_{p,A} = \overset{\vee}{\Sigma}_{p,FA} \colon (p \wedge \neg p) \vee FA \vdash FA,$$

$$F\overset{\wedge}{\Delta}_{\neg B,A} = (\mathbf{1}_{FA} \wedge \overset{\vee}{c}_{FB,F\neg B}) \circ F\overset{\wedge}{\Delta}_{B,A} \colon FA \vdash FA \wedge (FB \vee F\neg B),$$
$$F\overset{\vee}{\Sigma}_{\neg B,A} = F\overset{\vee}{\Sigma}_{B,A} \circ (\hat{c}_{F\neg B,FB} \vee \mathbf{1}_{FA}) \colon (F\neg B \wedge FB) \vee FA \vdash FA,$$

$$\begin{aligned} F\overset{\wedge}{\Delta}_{B\wedge C,A} = {} & (\mathbf{1}_{FA} \wedge ((\overset{\vee}{c}_{F\neg B,F\neg C} \vee \mathbf{1}_{FB\wedge FC}) \circ \overset{\vee}{b}{}^{\rightarrow}_{F\neg C,F\neg B,FB\wedge FC} \circ \\ & \circ (\mathbf{1}_{F\neg C} \vee (d^{R}_{F\neg B,FB,FC} \circ \hat{c}_{FC,F\neg B\vee FB} \circ F\overset{\wedge}{\Delta}_{B,C})))) \circ F\overset{\wedge}{\Delta}_{C,A} \colon \\ & FA \vdash FA \wedge ((F\neg B \vee F\neg C) \vee (FB \wedge FC)), \end{aligned}$$

$$\begin{aligned} F\overset{\vee}{\Sigma}_{B\wedge C,A} = {} & F\overset{\vee}{\Sigma}_{C,A} \circ ((\mathbf{1}_{FC} \wedge (F\overset{\vee}{\Sigma}_{B,\neg C} \circ d_{FB,F\neg B,F\neg C})) \circ \\ & \circ \hat{b}{}^{\leftarrow}_{FC,FB,F\neg B\vee F\neg C} \circ (\hat{c}_{FB,FC} \wedge \mathbf{1}_{F\neg B\vee F\neg C})) \vee \mathbf{1}_{FA}) \colon \\ & ((FB \wedge FC) \wedge (F\neg B \vee F\neg C)) \vee FA \vdash FA, \end{aligned}$$

$$\begin{aligned} F\overset{\wedge}{\Delta}_{B\vee C,A} = {} & (\mathbf{1}_{FA} \wedge ((\hat{c}_{F\neg C,F\neg B} \vee \mathbf{1}_{FB\vee FC}) \circ \overset{\vee}{b}{}^{\leftarrow}_{F\neg C\wedge F\neg B,FB,FC} \circ \\ & \circ ((d_{F\neg C,F\neg B,FB} \circ F\overset{\wedge}{\Delta}_{B,\neg C}) \vee \mathbf{1}_{FC}))) \circ F\overset{\wedge}{\Delta}_{C,A} \colon \\ & FA \vdash FA \wedge ((F\neg B \wedge F\neg C) \vee (FB \vee FC)), \end{aligned}$$

$$\begin{aligned} F\overset{\vee}{\Sigma}_{B\vee C,A} = {} & F\overset{\vee}{\Sigma}_{C,A} \circ (((F\overset{\vee}{\Sigma}_{B,C} \circ \overset{\vee}{c}_{FB\wedge F\neg B,FC} \circ d^{R}_{FC,FB,F\neg B}) \wedge \mathbf{1}_{F\neg C}) \circ \\ & \circ \hat{b}{}^{\rightarrow}_{FC\vee FB,F\neg B,F\neg C} \circ (\overset{\vee}{c}_{FC,FB} \wedge \mathbf{1}_{F\neg B\wedge F\neg C})) \vee \mathbf{1}_{FA}) \colon \\ & ((FB \vee FC) \wedge (F\neg B \wedge F\neg C)) \vee FA \vdash FA, \end{aligned}$$

$$F(f \circ g) = Ff \circ Fg,$$
$$F(f\,\xi\,g) = Ff\,\xi\,Fg, \quad \text{for } \xi \in \{\wedge, \vee\}.$$

It is easy to infer

$$
\begin{array}{ll}
F\overset{\wedge}{\Delta}_{\neg B,A} = F\overset{\wedge}{\Delta}{}'_{B,A}, & F\overset{\vee}{\Sigma}_{\neg B,A} = F\overset{\vee}{\Sigma}{}'_{B,A},\\
F\overset{\wedge}{\Delta}{}'_{\neg B,A} = F\overset{\wedge}{\Delta}_{B,A}, & F\overset{\vee}{\Sigma}{}'_{\neg B,A} = F\overset{\vee}{\Sigma}_{B,A},\\
F\overset{\wedge}{\Delta}_{B,A} \;= F\overset{\wedge}{\Delta}_{B,FA}, & F\overset{\vee}{\Sigma}_{B,A} \;= F\overset{\vee}{\Sigma}_{B,FA}.
\end{array}
$$

To ascertain that F so defined is indeed a functor, we have to verify that if $f=g$ is an instance of one of the **PN** equations, then $Ff=Fg$ holds in **PN**. This is done by induction on the number od occurrences of connectives in the crown indices occurring in these equations.

For $(\overset{\wedge}{\Delta}\; nat)$ and $(\overset{\vee}{\Sigma}\; nat)$ this is a very easy matter. For $(\hat{b}\overset{\wedge}{\Delta})$, $(\overset{\vee}{b}\overset{\vee}{\Sigma})$, $(d\overset{\wedge}{\Sigma})$ and $(d\overset{\vee}{\Delta})$ we use essentially naturality equations. (In that context, it might be easier to rely on the equations $(d^R\overset{\wedge}{\Delta})$ and $(d^R\overset{\vee}{\Sigma})$, which are alternative to $(d\overset{\wedge}{\Sigma})$ and $(d\overset{\vee}{\Delta})$.)

To verify $(\overset{\vee}{\Sigma}\overset{\wedge}{\Delta})$ in cases where A is of the form $B\wedge C$ or $B\vee C$, we rely on the induction hypothesis that if $f=g$ is an instance of a **PN** equation such that the crown indices are B and C, then we have $Ff=Fg$ in **PN**. This induction hypothesis entails that we can proceed as in the proof of the p-$\neg p$-p Lemma in the preceding section, first for p replaced by B, and then for p replaced by C. Finally, we apply **DS** Coherence (see the example at the end of the preceding section). To verify $(\overset{\vee}{\Sigma}\overset{\wedge}{\Delta})$ in case A is of the form $\neg B$, we rely on the induction hypothesis for the equation $(\overset{\vee}{\Sigma}{}'\overset{\wedge}{\Delta}{}')$.

To verify $(\overset{\vee}{\Sigma}{}'\overset{\wedge}{\Delta}{}')$ we proceed analogously. In case A is $B\wedge C$ or $B\vee C$, we rely on the proof of the $\neg p$-p-$\neg p$ Lemma in the preceding section, and in case A is $\neg B$ we rely on the induction hypothesis for the equation $(\overset{\vee}{\Sigma}\overset{\wedge}{\Delta})$. This concludes the verification that F is a functor from $\mathbf{PN}^{\neg}$ to **PN**.

In the definition of F, there is some freedom in choosing the clauses for $F\overset{\xi}{\Xi}_{B\psi C,A}$, where $\Xi\in\{\Delta,\Sigma\}$ and $\xi,\psi\in\{\wedge,\vee\}$. Ours enable us to apply easily the p-$\neg p$-p and $\neg p$-p-$\neg p$ Lemmata in verifying that F is a functor.

We define a functor $F^{\neg}$ from **PN** to $\mathbf{PN}^{\neg}$ by stipulating that $F^{\neg}A=A$ and $F^{\neg}f=f$. It is clear that if $f=g$ in **PN**, then $F^{\neg}f=F^{\neg}g$ in $\mathbf{PN}^{\neg}$; so $F^{\neg}$ is indeed a functor.

Our purpose is to show that $\mathbf{PN}^{\neg}$ and **PN** are equivalent categories via the functors F and $F^{\neg}$. It is clear that $FF^{\neg}A=A$ and $FF^{\neg}f=f$. Since $F^{\neg}FA=FA$, we have to define in $\mathbf{PN}^{\neg}$ an isomorphism $i_A\colon A\vdash FA$. For that we need the following auxiliary definitions in $\mathbf{PN}^{\neg}$:

$$
\begin{aligned}
n^{\rightarrow}_A &=_{\mathrm{df}} \overset{\vee}{\Sigma}{}'_{\neg A,A}\circ d_{\neg\neg A,\neg A,A}\circ \overset{\wedge}{\Delta}_{A,\neg\neg A} : \neg\neg A\vdash A,\\
n^{\leftarrow}_A &=_{\mathrm{df}} \overset{\vee}{\Sigma}_{A,\neg\neg A}\circ d_{A,\neg A,\neg\neg A}\circ \overset{\wedge}{\Delta}{}'_{\neg A,A} : A\vdash \neg\neg A,
\end{aligned}
$$

$$\hat{r}{}^{\rightarrow}_{A,B} =_{\mathrm{df}} \overset{\vee}{\Sigma}{}'_{A\wedge B,\neg A\vee\neg B} \circ d_{\neg(A\wedge B),A\wedge B,\neg A\vee\neg B} \circ (\mathbf{1}_{\neg(A\wedge B)} \wedge ((\mathbf{1}_{A\wedge B} \vee \overset{\vee}{c}_{\neg A,\neg B}) \circ \overset{\vee}{b}{}^{\leftarrow}_{A\wedge B,\neg B,\neg A} \circ ((d_{A,B,\neg B} \circ \overset{\wedge}{\Delta}{}'_{B,A}) \vee \mathbf{1}_{\neg A}))) \circ \overset{\wedge}{\Delta}{}'_{A,\neg(A\wedge B)} : \neg(A\wedge B) \vdash \neg A \vee \neg B,$$

$$\hat{r}{}^{\leftarrow}_{A,B} =_{\mathrm{df}} \overset{\vee}{\Sigma}{}'_{A,\neg(A\wedge B)} \circ ((((\overset{\vee}{\Delta}{}'_{B,\neg A} \circ d^{R}_{\neg A,\neg B,B}) \wedge \mathbf{1}_A) \circ \hat{b}{}^{\rightarrow}_{\neg A\vee\neg B,B,A} \circ (\mathbf{1}_{\neg A\vee \neg B} \wedge \hat{c}_{A,B})) \vee \mathbf{1}_{\neg(A\wedge B)}) \circ d_{\neg A\vee\neg B,A\wedge B,\neg(A\wedge B)} \circ \overset{\wedge}{\Delta}{}'_{A\wedge B,\neg A\vee\neg B} : \neg A\vee\neg B \vdash \neg(A\wedge B),$$

$$\overset{\vee}{r}{}^{\rightarrow}_{A,B} =_{\mathrm{df}} \overset{\vee}{\Sigma}{}'_{A\vee B,\neg A\wedge\neg B} \circ d_{\neg(A\vee B),A\vee B,\neg A\wedge\neg B} \circ (\mathbf{1}_{\neg(A\vee B)} \wedge ((\overset{\vee}{c}_{A,B} \vee \mathbf{1}_{\neg A\wedge\neg B}) \circ \overset{\vee}{b}{}^{\rightarrow}_{B,A,\neg A\wedge\neg B} \circ (\mathbf{1}_B \vee (d^{R}_{A,\neg A,\neg B} \circ \overset{\wedge}{\Sigma}{}'_{A,\neg B})))) \circ \overset{\wedge}{\Delta}{}'_{B,\neg(A\vee B)} : \neg(A\vee B) \vdash \neg A\wedge\neg B,$$

$$\overset{\vee}{r}{}^{\leftarrow}_{A,B} =_{\mathrm{df}} \overset{\vee}{\Sigma}{}'_{B,\neg(A\vee B)} \circ (((\mathbf{1}_{\neg B} \wedge (\overset{\vee}{\Sigma}{}'_{A,B} \circ d_{\neg A,A,B})) \circ \hat{b}{}^{\leftarrow}_{\neg B,\neg A,A\vee B} \circ (\hat{c}_{\neg A,\neg B} \wedge \mathbf{1}_{A\vee B})) \vee \mathbf{1}_{\neg(A\vee B)}) \circ d_{\neg A\wedge\neg B,A\vee B,\neg(A\vee B)} \circ \overset{\wedge}{\Delta}{}'_{A\vee B,\neg A\wedge\neg B} : \neg A\wedge\neg B\vdash\neg(A\vee B).$$

It can be shown that in $\mathbf{PN}^{\neg}$ we have the following equations:

$$\begin{array}{ll} n^{\rightarrow}_{A} \circ n^{\leftarrow}_{A} = \mathbf{1}_A, & n^{\leftarrow}_{A} \circ n^{\rightarrow}_{A} = \mathbf{1}_{\neg\neg A}, \\ \hat{r}{}^{\rightarrow}_{A,B} \circ \hat{r}{}^{\leftarrow}_{A,B} = \mathbf{1}_{\neg A\vee\neg B}, & \hat{r}{}^{\leftarrow}_{A,B} \circ \hat{r}{}^{\rightarrow}_{A,B} = \mathbf{1}_{\neg(A\wedge B)}, \\ \overset{\vee}{r}{}^{\rightarrow}_{A,B} \circ \overset{\vee}{r}{}^{\leftarrow}_{A,B} = \mathbf{1}_{\neg A\wedge\neg B}, & \overset{\vee}{r}{}^{\leftarrow}_{A,B} \circ \overset{\vee}{r}{}^{\rightarrow}_{A,B} = \mathbf{1}_{\neg(A\vee B)}, \end{array}$$

which means that $n^{\rightarrow}$ and $n^{\leftarrow}$, as well as $\overset{\xi}{r}{}^{\rightarrow}$ and $\overset{\xi}{r}{}^{\leftarrow}$ are inverses of each other. To derive these equations in $\mathbf{PN}^{\neg}$, we use essentially ($\overset{\wedge}{\Delta}$ *nat*), ($\overset{\vee}{\Sigma}$ *nat*), the p-$\neg p$-p and $\neg p$-p-$\neg p$ Lemmata, and **DS** Coherence. (If an equation holds in **PN**, then every substitution instance of it obtained by replacing letters uniformly by formulae of $\mathcal{L}_{\neg,\wedge,\vee}$ holds in $\mathbf{PN}^{\neg}$; this enables us to apply the p-$\neg p$-p and $\neg p$-p-$\neg p$ Lemmata.) The definitions of $n^{\rightarrow}$, $n^{\leftarrow}$, $\overset{\xi}{r}{}^{\rightarrow}$ and $\overset{\xi}{r}{}^{\leftarrow}$, for $\xi \in \{\wedge,\vee\}$, are such that they enable an easy application of the p-$\neg p$-p and $\neg p$-p-$\neg p$ Lemmata.

Then we define $i_A : A \vdash FA$ and its inverse $i_A^{-1} : FA \vdash A$ by induction on the complexity of the formula A of $\mathcal{L}_{\neg,\wedge,\vee}$ (cf. [**11**, Section 14.1]):

$$i_A = i_A^{-1} = \mathbf{1}_A, \quad \text{if } A \text{ is } p \text{ or } \neg p, \text{ for } p \text{ a letter},$$

$$\begin{array}{lll} i_{A_1 \xi A_2} = i_{A_1} \,\xi\, i_{A_2}, & i^{-1}_{A_1 \xi A_2} = i^{-1}_{A_1} \,\xi\, i^{-1}_{A_2}, & \text{for } \xi \in \{\wedge,\vee\}, \\ i_{\neg\neg B} = i_B \circ n^{\rightarrow}_{B}, & i^{-1}_{\neg\neg B} = n^{\leftarrow}_{B} \circ i^{-1}_{B}, & \\ i_{\neg(A_1\wedge A_2)} = (i_{\neg A_1} \vee i_{\neg A_2}) \circ \hat{r}{}^{\rightarrow}_{A_1,A_2}, & i^{-1}_{\neg(A_1\wedge A_2)} = \hat{r}{}^{\leftarrow}_{A_1,A_2} \circ (i^{-1}_{\neg A_1} \vee i^{-1}_{\neg A_2}), & \\ i_{\neg(A_1\vee A_2)} = (i_{\neg A_1} \wedge i_{\neg A_2}) \circ \overset{\vee}{r}{}^{\rightarrow}_{A_1,A_2}, & i^{-1}_{\neg(A_1\vee A_2)} = \overset{\vee}{r}{}^{\leftarrow}_{A_1,A_2} \circ (i^{-1}_{\neg A_1} \wedge i^{-1}_{\neg A_2}). & \end{array}$$

We can then prove the following (cf. [**11**, Section 14.1]).

AUXILIARY LEMMA. *For every arrow term $f\colon A \vdash B$ of $\mathbf{PN}^{\neg}$ we have $f = i_B^{-1} \circ Ff \circ i_A$ in $\mathbf{PN}^{\neg}$.*

PROOF. We proceed by induction on the complexity of the arrow term f. If f is a primitive arrow term $\mathbf{1}_A$, $\stackrel{\xi}{b}{}^{\rightarrow}_{A,B,C}$, $\stackrel{\xi}{b}{}^{\leftarrow}_{A,B,C}$, $\stackrel{\xi}{c}_{A,B}$ or $d_{A,B,C}$, for $\xi \in \{\wedge, \vee\}$, then we use naturality equations, and the fact that i_D is an isomorphism.

If f is $\hat{\Delta}_{D,A}$, then we proceed by induction on the complexity of D. (This is an auxiliary induction in the basis of the main induction.) If D is p, then we use ($\hat{\Delta}$ *nat*) and the fact that i_A is an isomorphism.

If D is $\neg B$, then we rely on the following equation of $\mathbf{PN}^{\neg}$:

$$(\hat{\Delta} n) \qquad \hat{\Delta}_{\neg B,A} = (\mathbf{1}_A \wedge (n^{\leftarrow}_B \vee \mathbf{1}_{\neg B})) \circ \hat{\Delta}'_{B,A},$$

together with the induction hypothesis. To derive $(\hat{\Delta} n)$ we have

$$\begin{aligned}
(\mathbf{1}_A \wedge (n^{\leftarrow}_B \vee \mathbf{1}_{\neg B})) \circ \hat{\Delta}'_{B,A} \\
&= (\mathbf{1}_A \wedge (\check{\Sigma}_{B,\neg\neg B} \vee \mathbf{1}_{\neg B})) \circ (\mathbf{1}_A \wedge (d_{B,\neg B,\neg\neg B} \vee \mathbf{1}_{\neg B})) \circ \\
&\qquad \circ (\mathbf{1}_A \wedge (\hat{\Delta}'_{\neg B,B} \vee \mathbf{1}_{\neg B})) \circ \hat{\Delta}'_{B,A}, \quad \text{by bifunctorial equations,} \\
&= (\mathbf{1}_A \wedge (\check{\Sigma}_{B,\neg\neg B} \vee \mathbf{1}_{\neg B})) \circ (\mathbf{1}_A \wedge ((d_{B,\neg B,\neg\neg B} \vee \mathbf{1}_{\neg B}) \circ \\
&\qquad \circ \check{c}_{B\wedge(\neg B\vee\neg\neg B),\neg B} \circ d^R_{\neg B,B,\neg B\vee\neg\neg B} \circ (\check{c}_{\neg B,B} \wedge \mathbf{1}_{\neg B\vee\neg\neg B}))) \circ \\
&\qquad \circ \hat{b}^{\leftarrow}_{A,\neg B,B\vee\neg B} \circ (\hat{\Delta}'_{B,A} \wedge \mathbf{1}_{\neg B\vee\neg\neg B}) \circ (\mathbf{1}_A \wedge \check{c}_{\neg B,\neg\neg B}) \circ \hat{\Delta}_{\neg B,A},
\end{aligned}$$

by stem-increasing equations involving $\hat{\Delta}'$ analogous to $(\mathbf{1} \vee \hat{\Delta})$ and $(\mathbf{1} \wedge \hat{\Delta})$ of the preceding section, and also ($\hat{\Delta}'$ *nat*). The equation $(\hat{\Delta} n)$ follows by applying the $\neg p$-p-$\neg p$ Lemma (with p replaced by A), and **DS** Coherence.

If D is $B \wedge C$, then we rely on the following equation of $\mathbf{PN}^{\neg}$:

$$\begin{aligned}
(\hat{\Delta} r) \qquad \hat{\Delta}_{B\wedge C,A} = (\mathbf{1}_A \wedge (((\hat{r}^{\leftarrow}_{B,C} \circ \check{c}_{\neg B,\neg C}) \vee \mathbf{1}_{B\wedge C}) \circ \check{b}^{\rightarrow}_{\neg C,\neg B,B\wedge C} \circ \\
\circ (\mathbf{1}_{\neg C} \vee (d^R_{\neg B,B,C} \circ \hat{\Sigma}_{B,C})))) \circ \hat{\Delta}_{C,A},
\end{aligned}$$

together with the induction hypothesis. To show that $(\hat{\Delta} r)$ holds in $\mathbf{PN}^{\neg}$ we proceed as above, by applying essentially stem-increasing equations together with the p-$\neg p$-p Lemma. We proceed analogously when D is $B \vee C$.

The cases we have if f is $\check{\Sigma}_{D,A}$ are dual to those we had above for f being $\hat{\Delta}_{D,A}$. In all these cases we proceed in an analogous manner. This concludes the basis of the induction.

If f is $f_2 \circ f_1$, then by the induction hypothesis we have

$$f_2 \circ f_1 = i_B^{-1} \circ Ff_2 \circ i_C \circ i_C^{-1} \circ Ff_1 \circ i_A$$

which yields $f = i_B^{-1} \circ Ff \circ i_A$, by the fact that i_C is an isomorphism and by the functoriality of F.

If f is $f_1 \,\xi\, f_2$, for $\xi \in \{\wedge, \vee\}$, then $i_{A_1 \xi A_2}$ is $i_{A_1} \,\xi\, i_{A_2}$ and $i^{-1}_{B_1 \xi B_2}$ is $i^{-1}_{B_1} \,\xi\, i^{-1}_{B_2}$; we obtain $f = i_B^{-1} \circ Ff \circ i_A$ by using bifunctorial equations. ⊣

The Auxiliary Lemma shows that i_A is an isomorphism natural in A, and so we may conclude that $\mathbf{PN}^{\neg}$ and $\mathbf{PN}$ are equivalent categories.

8. PN Coherence

We define a functor G from $\mathbf{PN}$ to Br as we defined it from $\mathbf{PN}^{\neg}$ to Br. In the clauses for $\overset{\wedge}{\Delta}_{B,A}$ and $\overset{\vee}{\Sigma}_{B,A}$ we just restrict B to a letter p. For f an arrow term of $\mathbf{PN}^{\neg}$ we have that GFf coincides with Gf where F is the functor from $\mathbf{PN}^{\neg}$ to $\mathbf{PN}$ of the preceding section, G in GFf is the functor G from $\mathbf{PN}$ to Br and G in Gf is the functor G from $\mathbf{PN}^{\neg}$ to Br. To show that, it is essential to check that $GF\overset{\wedge}{\Delta}_{B,A}$ and $GF\overset{\vee}{\Sigma}_{B,A}$ coincide with $G\overset{\wedge}{\Delta}_{B,A}$ and $G\overset{\vee}{\Sigma}_{B,A}$ respectively.

In this section we will prove that G from $\mathbf{PN}$ to Br is faithful. This will imply that G from $\mathbf{PN}^{\neg}$ to Br is faithful too.

Analogously to what we had at the beginning of §5, we define when an occurrence x of a letter p in A is *linked* to an occurrence y of the same letter p in B in an arrow $f\colon A \vdash B$ of $\mathbf{PN}$. We say that x and y are *directly linked* in a headed factorized arrow term $f_n \circ \ldots \circ f_1$ of $\mathbf{PN}$ when x and y are linked in the arrow $f_n \circ \ldots \circ f_1$, and for every $i \in \{2, \ldots, n\}$ if f_i is a $\overset{\vee}{\Sigma}_{p,C}$-term and z is one of the two occurrences of p in the crown $p \wedge \neg p$ of the head of f_i, then x and z are not linked in the arrow $f_{i-1} \circ \ldots \circ f_1$ (see the end of §3 for the definition of headed factorized arrow term).

An alternative definition of directly linked x and y in a headed factorized arrow term $f_1 \circ \ldots \circ f_n$ of $\mathbf{PN}$ is obtained by stipulating that x and y are linked in the arrow $f_1 \circ \ldots \circ f_n$, and for every $i \in \{2, \ldots, n\}$ if f_i is a $\overset{\wedge}{\Delta}_{p,D}$-term and z is one of the two occurrences of p in the crown $\neg p \vee p$ of the head of f_i, then z and y are not linked in the arrow $f_1 \circ \ldots \circ f_{i-1}$.

For example, the occurrence of q in the source $p \wedge q$ and the occurrence of q in the target $q \wedge p$ of

$$\hat{c}_{p,q} \circ (\overset{\vee}{\Sigma}_{p,p} \wedge \mathbf{1}_q) \circ (d_{p,\neg p,p} \wedge \mathbf{1}_q) \circ (\overset{\wedge}{\Delta}_{p,p} \wedge \mathbf{1}_q)$$

are directly linked in this headed factorized arrow term of $\mathbf{PN}$, while the two occurrences of p in its source and target are not directly linked.

Take a headed factorized arrow term of $\mathbf{PN}$ of the form $g_2 \circ f \circ g_1$ where g_1 is a $\overset{\wedge}{\Delta}_{p,D}$-term and g_2 is $\overset{\vee}{\Sigma}_{p,C}$-term. Let $\neg x_1 \vee x_2$ be the crown of the head of g_1 (so x_1 and x_2 are both occurrences of p) and let $y_2 \wedge \neg y_1$ be the crown of the head of g_2 (so y_1 and y_2 are also occurrences of the same letter p). We say that g_1 and g_2

are *confronted* through f when x_i and y_i are directly linked for some $i \in \{1,2\}$ in the arrow term f.

Let a $\hat{\Delta}_{p,A}$-term that is a factor of a factorized arrow term f be called a $\hat{\Delta}$-*factor*. We have an analogous definition of $\check{\Sigma}$-*factor* obtained by replacing $\hat{\Delta}$ by $\check{\Sigma}$. We can then prove the following lemma.

CONFRONTATION LEMMA. *For every headed factorized arrow term $g_2 \circ f \circ g_1$ of* **PN** *such that g_1 and g_2 are confronted through f there is a headed factorized arrow term h of* **PN** *with a subterm of the form $g_2' \circ f' \circ g_1'$ such that g_1' is a $\hat{\Delta}$-factor, g_2' is a $\check{\Sigma}$-factor, g_1' and g_2' are confronted through f', and, moreover,*

(1) *f' is an arrow term of* $\mathbf{DS}^{\neg p}$*,*

(2) *$g_2 \circ f \circ g_1 = h$ in* **PN***,*

(3) *the number of $\hat{\Delta}$-factors is equal in $g_2 \circ f \circ g_1$ and h, and the same for the number of $\check{\Sigma}$-factors.*

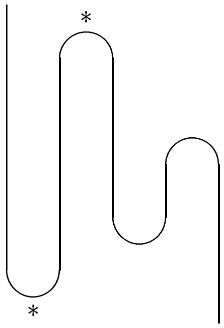
PROOF. We proceed by induction on the number n of factors of f that are $\hat{\Delta}$-factors or $\check{\Sigma}$-factors. If $n = 0$, then the arrow term f' coincides with the arrow term f.

If $n > 0$, then let $g_2 \circ f \circ g_1$ be of the form $f_2 \circ g \circ f_1$ for g a $\hat{\Delta}_{q,E}$-term (we proceed analogously when g is a $\check{\Sigma}_{q,E}$-term). According to the stem-increasing equations of §6, we may assume that g coincides with its head $\hat{\Delta}_{q,E}$. Then by ($\hat{\Delta}$ *nat*) we obtain in **PN**

$$g_2 \circ f \circ g_1 = f_2 \circ (f_1 \wedge \mathbf{1}_{\neg q \vee q}) \circ \hat{\Delta}_{q,E'} .$$

After $f_1 \wedge \mathbf{1}_{\neg q \vee q}$ in $f_2 \circ (f_1 \wedge \mathbf{1}_{\neg q \vee q})$ is replaced by a headed factorized arrow term $g_2 \circ f'' \circ (g_1 \wedge \mathbf{1}_{\neg q \vee q})$, we may apply the induction hypothesis to this arrow term, because it can easily be seen that $g_1 \wedge \mathbf{1}_{\neg q \vee q}$ and g_2 are confronted through f'', and f'' has one $\hat{\Delta}$-factor less than f. ⊣

A headed factorized arrow term of **PN** that has no subterm of the form $g_2 \circ f \circ g_1$ with g_1 and g_2 confronted through f is called *pure*. For a pure arrow term f there is a one-to-one correspondence, which we call the $\hat{\Delta}$-*cap bijection*, between the $\hat{\Delta}$-factors of f and the caps of the partition part(Gf). In this bijection, a cap ties, in an obvious sense, the occurrences of p in the crown $\neg p \vee p$ of the head of the corresponding $\hat{\Delta}$-factor. There is an analogous one-to-one correspondence, which we call the $\check{\Sigma}$-*cup bijection*, between the $\check{\Sigma}$-factors of f and the cups of part(Gf) (see §4 for the notions of cup and cap). Intuitively speaking, this follows from the fact that in a sequence of cups and caps tied to each other as in the following example:



cups and caps must alternate. For a pair made of a cap and a cup that is its immediate neighbour, like those marked with $*$ in the picture, we can find a subterm $g_2 \circ f \circ g_1$ such that g_1 and g_2 are confronted through f.

We can then prove the following.

PURIFICATION LEMMA. *Every arrow term of* **PN** *is equal in* **PN** *to a pure arrow term of* **PN**.

PROOF. We apply first the Development Lemma of §3. If in the resulting developed arrow term h we have a subterm $g_2 \circ f \circ g_1$ with g_1 and g_2 confronted through f, then we apply first the Confrontation Lemma to obtain a developed arrow term h' with a subterm of the form $g_2' \circ f' \circ g_1'$ where g_1' and g_2' are confronted through f', and f' is an arrow term of $\mathbf{DS}^{\neg p}$.

Suppose that $\neg x_2 \vee x_3$ is the crown of the head of g_1', and $y_1 \wedge \neg y_2$ is the crown of the head of g_2'. Suppose x_2 is linked to y_2 in f'. Then, by Lemma 3 of §5, it is impossible that x_3 is linked to y_1, and so there must be an occurrence x_1 of p different from x_3 in the source of f' such that x_1 is linked to y_1 in f', and there must be an occurrence y_3 of p different from y_1 in the target of f' such that x_3 is linked to y_3 in f'. Next we apply the p-$\neg p$-p Lemma of §6 to conclude that $g_2' \circ f' \circ g_1'$ is equal to an arrow term h'' of $\mathbf{DS}^{\neg p}$.

After replacing $g_2' \circ f' \circ g_1'$ in h' by h'', we obtain a headed factorized arrow term in which there is one $\overset{\wedge}{\Delta}$-factor and one $\overset{\vee}{\Sigma}$-factor less than in h', and hence also than in h, by clause (3) of the Confrontation Lemma.

If x_3 is linked to y_1, then we reason analogously by applying Lemma 3 of §5 and the $\neg p$-p-$\neg p$ Lemma of §6.

We can iterate this procedure, which must terminate, because the number of $\overset{\wedge}{\Delta}$-factors and $\overset{\vee}{\Sigma}$-factors in h is finite. $\dashv$

We can then prove the following.

PN COHERENCE. *The functor G from* **PN** *to Br is faithful.*

PROOF. Suppose for f and g arrow terms of **PN** of the same type $A \vdash B$ we have $Gf = Gg$. By the Purification Lemma, we can assume that f and g are pure arrow terms. Since $Gf = Gg$, by the $\stackrel{\wedge}{\Delta}$-cap and $\stackrel{\vee}{\Sigma}$-cup bijections we must have the same number $n \geqslant 0$ of $\stackrel{\wedge}{\Delta}$-factors in f and g and the same number $m \geqslant 0$ of $\stackrel{\vee}{\Sigma}$-factors in f and g. We proceed by induction on $n+m$.

If $n+m = 0$, then we just apply **DS** Coherence. Suppose now $n > 0$. So there is a $\stackrel{\wedge}{\Delta}$-factor in f and a $\stackrel{\wedge}{\Delta}$-factor in g that correspond by the $\stackrel{\wedge}{\Delta}$-cap bijections to the same cap of $\mathrm{part}(Gf)$, which is equal to $\mathrm{part}(Gg)$. By using the head increasing equations of §6, together with ($\stackrel{\wedge}{\Delta}$ *nat*), we obtain in **PN**

$$f = f' \circ \stackrel{\wedge}{\Delta}_{p,A}, \qquad\qquad g = g' \circ \stackrel{\wedge}{\Delta}_{p,A}$$

for f' and g' pure arrow terms of the same type $A \wedge (\neg p \vee p) \vdash B$, and such that the number of $\stackrel{\wedge}{\Delta}$-factors in f' and g' is $n-1$ in each, and the number of $\stackrel{\vee}{\Sigma}$-factors in f' and g' is m in each, the same number we had for the $\stackrel{\vee}{\Sigma}$-factors in f and g. So we have

$$G(f' \circ \stackrel{\wedge}{\Delta}_{p,A}) = Gf = Gg = G(g' \circ \stackrel{\wedge}{\Delta}_{p,A}).$$

We can show that $Gf' = Gg'$. We obtain Gf' out of $G(f' \circ \stackrel{\wedge}{\Delta}_{p,A})$ in the following manner. We first remove from the partition $\mathrm{part}(G(f' \circ \stackrel{\wedge}{\Delta}_{p,A}))$ a cap $\{k_t, l_t\}$, where the $k+1$-th occurrence of letter in B is an occurrence of p in a subformula $\neg p$ of B, and the $l+1$-th occurrence of letter in B is an occurrence of p that is not in a subformula $\neg p$ of B (here we have either $k < l$ or $l < k$). After this removal, we add two new transversals:

$$\{GA_s, k_t\}, \quad \{(GA+1)_s, l_t\},$$

and this yields $\mathrm{part}(Gf')$. Since Gg' is obtained from $G(g' \circ \stackrel{\wedge}{\Delta}_{p,A})$, which is equal to $G(f' \circ \stackrel{\wedge}{\Delta}_{p,A})$ in exactly the same manner, we obtain that $Gf' = Gg'$.

Then, by the induction hypothesis, we have that $f' = g'$ in **PN**, which implies that $f = g$ in **PN**. We proceed analogously in the induction step when $m > 0$, via $\stackrel{\vee}{\Sigma}$-factors. ⊣

From **PN** Coherence and the equivalence between the categories $\mathbf{PN}^{\neg}$ and **PN**, proved in the preceding section, we may conclude in the following manner that the functor G from $\mathbf{PN}^{\neg}$ to Br is faithful.

Proof of $\mathbf{PN}^{\neg}$ Coherence. Suppose that for f and g arrows of $\mathbf{PN}^{\neg}$ of the same type we have $Gf = Gg$. Then, as we noted at the beginning of this section, we have $GFf = GFg$, and hence $Ff = Fg$ in **PN** by **PN** Coherence. It follows that $f = g$ in $\mathbf{PN}^{\neg}$ by the equivalence of the categories $\mathbf{PN}^{\neg}$ and **PN**. ⊣

So we have proved $\mathbf{PN}^{\neg}$ Coherence, announced at the end of §4.

9. The category MDS

In this and in the next section we add *mix* arrows of the type $A \wedge B \vdash A \vee B$ to proof-net categories, together with appropriate conditions that will enable us to prove coherence with respect to *Br* for the resulting categories, which we call mix-proof-net categories. The mix arrows, which underly the mix principle of linear logic, were treated extensively in [**11**, Chapters 8, 10, 11, 13]. The proof of coherence for mix-proof-net categories is an adaptation of the proof of coherence for proof-net categories given in the preceding sections.

The category **MDS** is defined as the category **DS** in §2 save that we have the additional primitive arrow terms $m_{A,B} : A \wedge B \vdash A \vee B$ for all objects, i.e. for all formulae, A and B of $\mathcal{L}_{\wedge,\vee}$, and we assume the following additional equations:

$(m\ nat)$ $\quad (f \vee g) \circ m_{A,B} = m_{D,E} \circ (f \wedge g), \quad$ for $f : A \vdash D$ and $g : B \vdash E$,

$(\hat{b} m)$ $\quad m_{A \wedge B, C} \circ \hat{b}{}^{\rightarrow}_{A,B,C} = d_{A,B,C} \circ (\mathbf{1}_A \wedge m_{B,C})$,

$(\overset{\vee}{b} m)$ $\quad \overset{\vee}{b}{}^{\rightarrow}_{C,B,A} \circ m_{C, B \vee A} = (m_{C,B} \vee \mathbf{1}_A) \circ d_{C,B,A}$,

(cm) $\quad m_{B,A} \circ \hat{c}_{A,B} = \overset{\vee}{c}_{B,A} \circ m_{A,B}$.

The proof-theoretical principle underlying $m_{A,B}$ is called *mix* (see [**11**, Section 8.1] and references therein).

To obtain the functor G from **MDS** to *Br*, we extend the definition of the functor G from **DS** to *Br* (see §4) by adding the clause that says that $Gm_{A,B}$ is the identity arrow $\mathbf{1}_{GA+GB}$ of *Br*. We have the following result of [**11**, Section 8.4].

MDS COHERENCE. *The functor G from* **MDS** *to Br is faithful.*

In the remainder of this section we will prove some lemmata concerning **MDS**, which we will use for the proof of coherence in the next section. For that we need some preliminaries.

For x a particular proper subformula of a formula A of $\mathcal{L}_{\wedge,\vee}$, and $\xi \in \{\wedge, \vee\}$, we define A^{-x} inductively as follows:

$$(B \,\xi\, x)^{-x} = (x \,\xi\, B)^{-x} = B,$$

for x a proper subformula of C,

$$(B \,\xi\, C)^{-x} = B \,\xi\, C^{-x},$$
$$(C \,\xi\, B)^{-x} = C^{-x} \,\xi\, B.$$

For $i \in \{1,2\}$, let A_i be a formula of $\mathcal{L}_{\wedge,\vee}$ with a proper subformula x_i, which is an occurrence of a letter q, and let x_i be the n_i-th occurrence of letter counting from the left. We define the following functions $\mu_i : \boldsymbol{N} - \{n_i - 1\} \rightarrow \boldsymbol{N}$:

$$\mu_i(n) =_{\text{df}} \begin{cases} n & \text{if } n < n_i - 1 \\ n-1 & \text{if } n > n_i - 1. \end{cases}$$

The definition of *linked* occurrence of a letter in an arrow of **MDS** is analogous to what we had in §5. Then we can prove the following.

LEMMA 1. *For every arrow term $f\colon A_1 \vdash A_2$ of* **MDS** *such that x_1 and x_2 are linked in the arrow f, there is an arrow term $f^{-q}\colon A_1^{-x_1} \vdash A_2^{-x_2}$ of* **MDS** *such that the members of* $\mathrm{part}(Gf^{-q})$ *are* $\{s(\mu_1(m_1)), t(\mu_2(m_2))\}$ *for each* $\{s(m_1), t(m_2)\}$ *in* $\mathrm{part}(Gf)$, *provided* $m_i \neq n_i - 1$.

PROOF. We proceed by induction on the complexity of the arrow term f. If f is a primitive arrow term $\alpha_{B_1,\ldots,B_m}$, then for some $j \in \{1,\ldots,m\}$ we have that x_i occurs in a subformula B_j of A_i. If x_i is a proper subformula of this subformula B_j, then $B_j^{-x_i}$ is defined, and f^{-q} is

$$\alpha_{B_1,\ldots,B_{j-1},B_j^{-x_i},B_{j+1},\ldots,B_m}$$

(note that $B_j^{-x_1}$ and $B_j^{-x_2}$ are the same formula). If x_i is not a proper subformula of the subformula B_j, then f^{-q} is $\mathbf{1}_{A_i^{-x_i}}$.

If f is $g \circ h$, then f^{-q} is $g^{-q} \circ h^{-q}$, and if f is $g \,\xi\, h$ for $\xi \in \{\wedge, \vee\}$, then f^{-q} is either $g^{-q} \,\xi\, h$, or $g \,\xi\, h^{-q}$, or g when h is $\mathbf{1}_{x_1}$, or h when g is $\mathbf{1}_{x_1}$. $\dashv$

Here is an example of the application of Lemma 1. If $f\colon A_1 \vdash A_2$ is

$$((m_{q,p\wedge q} \circ (\mathbf{1}_q \wedge \hat{c}_{q,p}) \circ \hat{c}_{q\wedge p,q}) \vee \mathbf{1}_p) \circ d_{q\wedge p,q,p} \circ \hat{b}^{\rightarrow}_{q,p,q\vee p} \circ \hat{c}_{p\wedge(q\vee p),q}\colon$$
$$(p \wedge (q \vee p)) \wedge q \vdash (q \vee (p \wedge q)) \vee p,$$

where x_1 is the second (rightmost) occurrence of q in $(p \wedge (q \vee p)) \wedge q$, while x_2 is the second occurrence of q in $(q \vee (p \wedge q)) \vee p$, then $f^{-q}\colon A_1^{-q} \vdash A_2^{-q}$ is

$$((m_{q,p} \circ (\mathbf{1}_q \wedge \mathbf{1}_p) \circ \hat{c}_{p,q}) \vee \mathbf{1}_p) \circ d_{p,q,p} \circ \mathbf{1}_{p\wedge(q\vee p)} \circ \mathbf{1}_{p\wedge(q\vee p)} : p \wedge (q \vee p) \vdash (q \vee p) \vee p,$$

which is equal to $((m_{q,p} \circ \hat{c}_{p,q}) \vee \mathbf{1}_p) \circ d_{p,q,p}$.

We define inductively a notion we call a *context*:

- $\Box$ is a context;
- if Z is a context and A a formula of $\mathcal{L}_{\wedge,\vee}$, then $Z \,\xi\, A$ and $A \,\xi\, Z$ are contexts for $\xi \in \{\wedge, \vee\}$.

Next we define inductively what it means for a context Z to be applied to an object B of **MDS**, which we write $Z(B)$, or to an arrow term f of **MDS**, which we write $Z(f)$:

$$\Box(B) = B, \qquad \Box(f) = f,$$
$$(Z \,\xi\, A)(B) = Z(B) \,\xi\, A, \qquad (Z \,\xi\, A)(f) = Z(f) \,\xi\, \mathbf{1}_A,$$
$$(A \,\xi\, Z)(B) = A \,\xi\, Z(B); \qquad (A \,\xi\, Z)(f) = \mathbf{1}_A \,\xi\, Z(f).$$

We use $X, Y, Z, \ldots$ for contexts.

For $f\colon A \vdash C$ an arrow of **MDS**, we say that an occurrence x of a formula B as a subformula of A and an occurrence y of the same formula B as a subformula of C are *linked* in f when the n-th letter in x is linked in f to the n-th letter in y.

Let $f\colon X(p) \wedge B \vdash Y(p \wedge B)$ be an arrow term of **MDS** such that the displayed occurrences of p in the source and target, and also the displayed occurrences of B, are linked in the arrow f. Then, by successive applications of Lemma 1, for each occurrence of a letter in B, we obtain the arrow term $f^{-B}\colon X(p)\vdash Y(p)$ of **MDS**, and the displayed occurrences of p in $X(p)$ and $Y(p)$ are linked in the arrow f^{-B}.

Let $f^{\dagger}\colon X(p\wedge B)\vdash Y(p\wedge B)$ be the arrow term of **MDS** obtained from f^{-B} by replacing the occurrences of p that correspond to those displayed in $X(p)$ and $Y(p)$ by occurrences of $p\wedge B$. This replacement is made in the indices of primitive arrow terms that occur in f^{-B}, and it need not involve all the occurrences of p in these indices. For example, if X is $\Box\wedge(q\vee p)$ and Y is $(q\vee\Box)\vee p$, while f^{-B} is

$$((m_{q,p}\circ \hat{c}_{p,q})\vee \mathbf{1}_p)\circ d_{p,q,p}\colon p\wedge(q\vee p)\vdash (q\vee p)\vee p,$$

then $f^{\dagger}$ is

$$((m_{q,p\wedge B}\circ \hat{c}_{p\wedge B,q})\vee \mathbf{1}_p)\circ d_{p\wedge B,q,p}\colon (p\wedge B)\wedge(q\vee p)\vdash (q\vee (p\wedge B))\vee p.$$

Then we can prove the following.

LEMMA 2$\wedge$. *Let $f\colon X(p)\wedge B\vdash Y(p\wedge B)$ and $f^{\dagger}\colon X(p\wedge B)\vdash Y(p\wedge B)$ be as above. Then there is an arrow term $h_X\colon X(p)\wedge B\vdash X(p\wedge B)$ of* **DS** *such that $f=f^{\dagger}\circ h_X$ in* **MDS**.

PROOF. We construct the arrow term h_X of **DS** by induction on the complexity of the context X. For the basis we have that $h_{\Box}$ is $\mathbf{1}_{p\wedge B}$. In the induction step we have

$$\begin{aligned}
h_{Z\wedge A} &= (h_Z\wedge \mathbf{1}_A)\circ \hat{c}_{A,Z(p)\wedge B}\circ \hat{b}{}^{\leftarrow}_{A,Z(p),B}\circ(\hat{c}_{Z(p),A}\wedge \mathbf{1}_B),\\
h_{Z\vee A} &= (h_Z\vee \mathbf{1}_A)\circ \overset{\vee}{c}_{Z(p)\wedge B,A}\circ d^R_{A,Z(p),B}\circ(\overset{\vee}{c}_{A,Z(p)}\wedge \mathbf{1}_B),\\
h_{A\wedge Z} &= (\mathbf{1}_A\wedge h_Z)\circ \hat{b}{}^{\leftarrow}_{A,Z(p),B},\\
h_{A\vee Z} &= (\mathbf{1}_A\vee h_Z)\circ d^R_{A,Z(p),B}.
\end{aligned}$$

It is easy to see that $Gf=G(f^{\dagger}\circ h_X)$, and then the lemma follows by applying **MDS** Coherence. $\dashv$

Let $f\colon Y(B\vee p)\vdash B\vee X(p)$ be an arrow term of **MDS** such that the displayed occurrences of p in the source and target, and also the displayed occurrences of B, are linked in the arrow f. Then, as above by Lemma 1, we obtain the arrow term $f^{-B}\colon Y(p)\vdash X(p)$ of **MDS**, and the displayed occurrences of p in $Y(p)$ and $X(p)$ are linked in the arrow f^{-B}.

Let $f^{\dagger}\colon Y(B\vee p)\vdash X(B\vee p)$ be the arrow term of **MDS** obtained from f^{-B} by replacing the occurrences of p that correspond to those displayed in $Y(p)$ and $X(p)$ by occurrences of $B\vee p$ (cf. the example above). Then we can prove the following, analogously to Lemma 2$\wedge$.

LEMMA 2$\vee$. *Let $f\colon Y(B\vee p)\vdash B\vee X(p)$ and $f^{\dagger}\colon Y(B\vee p)\vdash X(B\vee p)$ be as above. Then there is an arrow term $h_X\colon X(B\vee p)\vdash B\vee X(p)$ of* **DS** *such that $f=h_X\circ f^{\dagger}$ in* **MDS**.

10. $\mathbf{MPN}^{\neg}$ Coherence

The category $\mathbf{MPN}^{\neg}$ is defined as the category $\mathbf{PN}^{\neg}$ in §3 save that we have the additional primitive arrow terms $m_{A,B}\colon A\wedge B\vdash A\vee B$ for all objects A and B of $\mathbf{PN}^{\neg}$, and we assume as additional equations $(m\ nat)$, $(\stackrel{\wedge}{b}m)$, $(\stackrel{\vee}{b}m)$ and (cm) of the preceding section. To obtain the functor G from $\mathbf{MPN}^{\neg}$ to Br, we extend the definition of the functor G from $\mathbf{PN}^{\neg}$ to Br by adding the clause that says that $Gm_{A,B}$ is the identity arrow $\mathbf{1}_{GA+GB}$ of Br.

A *mix-proof-net* category is defined as a proof-net category (see §3) that has in addition a natural transformation m satisfying the equations $(\stackrel{\wedge}{b}m)$, $(\stackrel{\vee}{b}m)$ and (cm). The category $\mathbf{MPN}^{\neg}$ is up to isomorphism the free mix-proof-net category generated by $\mathcal{P}$.

The category $\mathbf{MPN}$ is defined as the category $\mathbf{PN}$ in §6 save that we have the additional primitive arrow terms $m_{A,B}$ for all objects of $\mathbf{PN}$, and we assume as additional equations $(m\ nat)$, $(\stackrel{\wedge}{b}m)$, $(\stackrel{\vee}{b}m)$ and (cm). We can prove that $\mathbf{MPN}^{\neg}$ and $\mathbf{MPN}$ are equivalent categories as in §7. (We have an additional case involving $m_{A,B}$ in the proof of the analogue of the Auxiliary Lemma of §7, and similar trivial additions elsewhere; otherwise the proof is quite analogous.)

We have a functor G from $\mathbf{MPN}$ to Br defined by restricting the definition of the functor G from $\mathbf{MPN}^{\neg}$ to Br (cf. the beginning of §8), and we will prove the following.

MPN COHERENCE. *The functor G from* **MPN** *to Br is faithful.*

The proof of this coherence proceeds as the proof of **PN** Coherence in §8. The only difference is in the $\stackrel{\wedge}{\Xi}$-Permutation and $\stackrel{\vee}{\Xi}$-Permutation Lemmata of §6.

The formulation of the $\stackrel{\wedge}{\Xi}$-Permutation Lemma is modified by replacing **PN** and $\mathbf{DS}^{\neg p}$ by respectively **MPN** and $\mathbf{MDS}^{\neg p}$, where the category $\mathbf{MDS}^{\neg p}$ is defined as **MDS** save that it is generated not by $\mathcal{P}$, but by $\mathcal{P}\cup\mathcal{P}^{\neg}$ (cf. §6); moreover, we assume that y_1 and $\neg y_2$ occur in E within a subformula of the form $p\wedge(\neg y_2\vee y_1)$ or $\neg p\wedge(y_1\vee\neg y_2)$. We modify the proof of this lemma as follows.

If in E we have $p\wedge(\neg y_2\vee y_1)$, then by the stem-increasing equations of §6 we have that the $\stackrel{\wedge}{\Xi}_{p,B}$-term $g\colon C\vdash D$ is equal to $f''\circ\stackrel{\wedge}{\Delta}_{p,C}$ for $f''\colon C\wedge(\neg p\vee p)\vdash D$ an arrow term of $\mathbf{DS}^{\neg p}$, and so for $f\colon D\vdash E$ an arrow term of $\mathbf{MDS}^{\neg p}$ satisfying the conditions of the lemma we have in **MPN**

$$f\circ g=f\circ f''\circ\stackrel{\wedge}{\Delta}_{p,C}\,.$$

Then we apply Lemma 2$\wedge$ of the preceding section to

$$f\circ f''\colon C\wedge(\neg p\vee p)\vdash E,$$

where C is $X(p)$, $\neg p\vee p$ is B and E is $Y(p\wedge(\neg p\vee p))$. So for

$$h_X\colon X(p)\wedge(\neg p\vee p)\vdash X(p\wedge(\neg p\vee p))$$

an arrow term of $\mathbf{DS}^{\neg p}$, and

$$(f\circ f'')^\dagger\colon X(p\wedge(\neg p\vee p))\vdash Y(p\wedge(\neg p\vee p))$$

we have

$$f\circ f'' = (f\circ f'')^\dagger\circ h_X.$$

By the $\stackrel{\wedge}{\Xi}$-Permutation Lemma of §6 we have

$$h_X\circ\stackrel{\wedge}{\Delta}_{p,C} = g'\circ f'$$

where g' is the $\stackrel{\wedge}{\Delta}_{p,p}$-term $X(\stackrel{\wedge}{\Delta}_{p,p})$, and by bifunctorial and naturality equations we have

$$(f\circ f'')^\dagger\circ X(\stackrel{\wedge}{\Delta}_{p,p}) = Y(\stackrel{\wedge}{\Delta}_{p,p})\circ(f\circ f'')^{-(\neg p\vee p)}.$$

Note that $(f\circ f'')^\dagger$ is obtained from $(f\circ f'')^{-(\neg p\vee p)}\colon X(p)\vdash Y(p)$ by replacement of p.

So we have in **MPN**

$$\begin{aligned} f\circ g &= f\circ f''\circ\stackrel{\wedge}{\Delta}_{p,C}\\ &= (f\circ f'')^\dagger\circ h_X\circ\stackrel{\wedge}{\Delta}_{p,C}\\ &= (f\circ f'')^\dagger\circ X(\stackrel{\wedge}{\Delta}_{p,p})\circ f'\\ &= Y(\stackrel{\wedge}{\Delta}_{p,p})\circ f''' \end{aligned}$$

for f''', which is $(f\circ f'')^{-(\neg p\vee p)}\circ f'$, an arrow term of $\mathbf{MDS}^{\neg p}$.

We proceed analogously if in E we have $\neg p\wedge(y_1\vee\neg y_2)$; instead of $\stackrel{\wedge}{\Delta}_{p,p}$ we then have $\stackrel{\wedge}{\Delta}'_{p,p}$. We have an analogous reformulation of the $\stackrel{\vee}{\Xi}$-Permutation Lemma of §6, with a proof based on Lemma 2$\vee$ of the preceding section.

Instead of Lemma 2$\wedge$ of the preceding section, we could have proved, with more difficulty, an analogous lemma where f is of type

$$Z(X_1(p)\wedge X_2(B))\vdash Y(p\wedge B),$$

and $f^\dagger$ is of one of the following types:

$$Z(X_1(p\wedge B)\wedge(X_2(B))^{-B})\vdash Y(p\wedge B),$$

$$Z(X_1(p\wedge B))\vdash Y(p\wedge B).$$

Then in the proof of the $\stackrel{\wedge}{\Xi}$-Permutation Lemma modified for **MPN** we would not need to pass from g to $f''\circ\stackrel{\wedge}{\Delta}_{p,C}$ via stem-increasing equations, but this alternative approach is altogether less clear.

Note that we have no analogue of Lemma 2 of §5 for **MDS**. The lack of this lemma, on which we relied in §6 for the proof of the $\stackrel{\wedge}{\Xi}$-Permutation and $\stackrel{\vee}{\Xi}$-Permutation Lemmata, is tied to the modifications we made for these lemmata with **MPN**. We have also no analogue of Lemma 4 of §5, but the analogue of Lemma 3 of §5 does hold.

From **MPN** Coherence and the equivalence of the categories $\mathbf{MPN}^{\neg}$ and **MPN** we can then infer the following.

$\mathbf{MPN}^{\neg}$ COHERENCE. *The functor* G *from* $\mathbf{MPN}^{\neg}$ *to* Br *is faithful.*

References

[1] M. Barr, *∗-Autonomous Categories*, Lecture Notes in Mathematics 752, Springer-Verlag, Berlin, 1979

[2] ———, *∗-Autonomous categories and linear logic*, Mathematical Structures in Computer Science 1 (1991), 159–178

[3] R. Brauer, *On algebras which are connected with the semisimple continuous groups*, Annals of Mathematics 38 (1937), 857–872

[4] G. Burde and H. Zieschang, *Knots*, de Gruyter, Berlin, 1985

[5] J. R. B. Cockett and R. A. G. Seely, *Weakly distributive categories*, in: M. P. Fourman et al., (editors) *Applications of Categories in Computer Science*, Cambridge University Press, Cambridge, 1992, pp. 45-65

[6] ———, *Weakly distributive categories*, J. Pure Appl. Algebra 114 (1997), 133–173 (updated version available at: http://www.math.mcgill.ca/rags)

[7] V. Danos and L. Regnier, *The structure of multiplicatives*, Archive for Mathematical Logic 28 (1989), 181–203

[8] K. Došen, *Cut Elimination in Categories*, Kluwer, Dordrecht, 1999

[9] K. Došen and Z. Petrić, *Generality of proofs and its Brauerian representation*, J. Symbolic Logic 68 (2003), 740–750 (available at: http://arXiv.org/math.LO/0211090)

[10] ———, *A Brauerian representation of split preorders*, Math. Logic Quart. 49 (2003), 579–586 (misprints corrected in text available at: http://arXiv.org/math.LO/0211277)

[11] ———, *Proof-Theoretical Coherence*, KCL Publications, London, 2004

[12] ———, *Proof-Net Categories*, preprint, 2005

[13] S. Eilenberg and G. M. Kelly, *A generalization of the functorial calculus*, J. Algebra 3 (1966), 366–375

[14] J.-Y. Girard, *Linear logic*, Theoret. Comput. Sci. 50 (1987), 1–101

[15] R. Houston, D. Hughes and A. Schalk, *Modelling linear logic without units (Preliminary results)*, preprint, 2005 (available at: http://arXiv.org/math.CT/0504037)

[16] C. Kassel, *Quantum Groups*, Springer-Verlag, Berlin, 1995

[17] G. M. Kelly and S. Mac Lane, *Coherence in closed categories*, J. Pure Appl. Algebra 1 (1971), 97–140, 219

[18] F. Lamarche and L. Straßburger, *Constructing free Boolean categories*, preprint, 2005

[19] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, Berlin, 1971 (expanded second edition, 1998)

Matematički institut SANU
Kneza Mihaila 35
11000 Beograd, p.p. 367
Serbia and Montenegro
{kosta, zpetric}@mi.sanu.ac.yu

(Received 16 03 2005)
(Revised 05 09 2005)